The differential equation for the vertical component w of the perturbation velocity for internal gravity waves in the ocean or atmosphere, including baroclinicity, vorticity, and rate of strain *

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*feb05b.tex, November 29, 2005, 1347

Abstract. The linearized differential equation for the vertical component w of the perturbation velocity for internal gravity waves in the ocean or atmosphere is generalized to include baroclinicity, vorticity, and the symmetric rate-of-strain tensor. The operator-ordering terms are shown explicitly, both in general and for geostrophic flow. Comparison with other work in special cases is included.

index terms: 4544: Internal and inertial waves; 3384: Waves and tides; 350: Pressure, density, and temperature;

keywords: internal gravity waves, dispersion relation, wave propagation

1. The differential equation for $w(x, y, z, t)$

We derive the differential equation for vertical velocity $w(x, y, z, t)$ in a special case. The derivation here follows the work of *Eady* [1949], but does not make some approximations used by *Eady* [1949] and other authors [*Jones*, 1967; *Smith*, 1986]. In addition, this derivation shows explicitly the origin of the operator-ordering terms that arise from eliminating variables. The components of angular velocity of the Earth are assumed constant, so that Rossby waves are not considered here.

We begin with the entropy variable Φ [*Eady*, 1949] defined by

$$\nabla\Phi \equiv -\nabla\rho_{pot}/\rho, \tag{1}$$

where ρ is density and ρ_{pot} is the potential density. In the incompressible case where $\rho_{pot} = \rho$, the adiabatic condition $DS/Dt = 0$ (where S is entropy), which is equivalent to [*Holton*, 1992, section 7.4.1, equation (7.2.8), p. 199]

$$\frac{D\rho_{pot}}{Dt} = 0, \tag{2}$$

requires that

$$\frac{D\Phi}{Dt} = 0. \tag{3}$$

We define

$$\tilde{\mathbf{g}} \equiv \nabla p/\rho = \mathbf{g} - D\mathbf{U}/Dt - 2\mathbf{\Omega} \times \mathbf{U}, \tag{4}$$

which is the effective acceleration due to gravity, including (minus) the acceleration of the background flow. $\mathbf{U}$ is the background flow velocity and $\boldsymbol{\Omega}$ is the angular velocity of the Earth. We can write the baroclinic vector as [*Gill*, 1982, sec. 7.11, p. 238]

$$\mathbf{B} \equiv \nabla\rho_{pot}/\rho \times \tilde{\mathbf{g}} = -\nabla \times \tilde{\mathbf{g}}. \tag{5}$$

From (5) and (1), we have

$$\nabla \times \tilde{\mathbf{g}} = \nabla\Phi \times \tilde{\mathbf{g}}. \tag{6}$$

If we consider the $\tilde{\mathbf{g}}$ on the left side of (6) to be given by (4), then (6) gives one form of the vorticity equation.

Taking the curl of the vorticity equation (6) gives

$$\nabla \times (\nabla \times \tilde{\mathbf{g}}) = \nabla \times (\nabla\Phi \times \tilde{\mathbf{g}}). \tag{7}$$

The continuity equation is

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{U}) = 0. \tag{8}$$

We now consider the following four equations. First: the z component of (7), the curl of the vorticity equation, second: the z component of the vorticity equation (6), third: the adiabatic condition (3), and fourth: the continuity equation (8) for an incompressible fluid (i.e., for $D\rho/Dt = 0$).

We then write these four equations as a matrix equation. This gives

$$\begin{pmatrix} \begin{matrix} -\hat{k}_z\hat{k}_x\hat{\omega} \\ -2i\hat{k}_z\hat{k}_y\Omega_z \\ -2i\hat{k}_H^2\Omega_y \end{matrix} & \begin{matrix} -\hat{k}_z\hat{k}_y\hat{\omega} \\ +2i\hat{k}_z\hat{k}_x\Omega_z \\ +2i\hat{k}_H^2\Omega_x \end{matrix} & \begin{matrix} -2i\hat{k}_z\hat{k}_x\Omega_y \\ +2i\hat{k}_z\hat{k}_y\Omega_x \\ +\hat{k}_H^2\hat{\omega} \end{matrix} & \begin{matrix} i\tilde{g}_z\hat{k}_H^2 \\ -i\frac{\partial^2\tilde{g}_z}{\partial x^2}\hat{k}_x \\ -i\frac{\partial^2\tilde{g}_z}{\partial y^2}\hat{k}_y \\ -i\hat{k}_x\tilde{g}_x\hat{k}_z \\ -i\hat{k}_y\tilde{g}_y\hat{k}_z \end{matrix} \\ \\ \begin{matrix} \hat{k}_y\hat{\omega} \\ -2i\hat{k}_x\Omega_z \end{matrix} & \begin{matrix} -\hat{k}_x\hat{\omega} \\ -2i\hat{k}_y\Omega_z \end{matrix} & \begin{matrix} 2i\hat{k}_x\Omega_x \\ +2i\hat{k}_y\Omega_y \end{matrix} & \begin{matrix} -i\tilde{g}_y\hat{k}_x \\ +i\tilde{g}_x\hat{k}_y \end{matrix} \\ \\ 0 & 0 & 0 & \hat{\omega} \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ \Phi \end{pmatrix} = 0, \qquad (9)$$

where U, V, and W and the components of the flow velocity in the x, y, and z (vertical) directions, $\hat{\mathbf{k}} \equiv -i\nabla$, $\hat{\omega} \equiv iD/Dt \equiv i\partial/\partial t + i\mathbf{U}\cdot\nabla$, $\hat{k}_H^2 \equiv \hat{k}_x^2 + \hat{k}_y^2$.

In the 4 by 4 matrix above, notice that the fourth column in the first (top) row has five terms, and that each of the four columns in the second row has two terms. The order of the factors in most terms is significant because the derivative operators apply to all factors to their right.

The first row in (9) is equivalent to Eady's equation (I.11) except that he neglects Ω_x, Ω_y, $\tilde{g}_x$, $\tilde{g}_y$, derivatives of $\tilde{g}_z$, and the remaining term in the third column. The second row in (9) is equivalent to Eady's equation (I.9) except that he neglects Ω_x, Ω_y, $\tilde{g}_x$, and $\tilde{g}_y$. The third and fourth rows in (9) are equivalent to Eady's equations (I.3) and (I.1), respectively.

Let us neglect all derivatives on all components of $\tilde{\mathbf{g}}$ and linearize by letting $U = U_0 + u$, $V = V_0 + v$, $W = W_0 + w$, and $\Phi = \Phi_0 + \phi$. Then the linearization of (9) is

$$
\begin{pmatrix}
\begin{array}{c} -\hat{k}_z\hat{k}_x\hat{\omega}_0 \\ -i\hat{k}_z\hat{k}_xU_{0,x} \\ -i\hat{k}_z\hat{k}_yV_{0,x} \\ -2i\hat{k}_z\hat{k}_y\Omega_z \\ -2i\hat{k}_H^2\Omega_y \\ +iW_{0,x}\hat{k}_H^2 \end{array} &
\begin{array}{c} -\hat{k}_z\hat{k}_y\hat{\omega}_0 \\ -i\hat{k}_z\hat{k}_xU_{0,y} \\ -i\hat{k}_z\hat{k}_yV_{0,y} \\ +2i\hat{k}_z\hat{k}_x\Omega_z \\ +2i\hat{k}_H^2\Omega_x \\ +iW_{0,y}\hat{k}_H^2 \end{array} &
\begin{array}{c} -2i\hat{k}_z\hat{k}_x\Omega_y \\ -i\hat{k}_z\hat{k}_xU_{0,z} \\ -i\hat{k}_z\hat{k}_yV_{0,z} \\ +2i\hat{k}_z\hat{k}_y\Omega_x \\ +\hat{k}_H^2\hat{\omega}_0 \\ +iW_{0,z}\hat{k}_H^2 \end{array} &
\begin{array}{c} i\tilde{g}_z\hat{k}_H^2 \\ -i\tilde{g}_x\hat{k}_x\hat{k}_z \\ -i\tilde{g}_y\hat{k}_y\hat{k}_z \end{array} \\
\begin{array}{c} \hat{k}_y\hat{\omega}_0 \\ +i\hat{k}_yU_{0,x} \\ -i\hat{k}_xV_{0,x} \\ -2i\hat{k}_x\Omega_z \end{array} &
\begin{array}{c} -\hat{k}_x\hat{\omega}_0 \\ +i\hat{k}_yU_{0,y} \\ -i\hat{k}_xV_{0,y} \\ -2i\hat{k}_y\Omega_z \end{array} &
\begin{array}{c} 2i\hat{k}_x\Omega_x \\ +i\hat{k}_yU_{0,z} \\ -i\hat{k}_xV_{0,z} \\ +2i\hat{k}_y\Omega_y \end{array} &
\begin{array}{c} -i\tilde{g}_y\hat{k}_x \\ +i\tilde{g}_x\hat{k}_y \end{array} \\
i\Phi_{0,x} & i\Phi_{0,y} & i\Phi_{0,z} & \hat{\omega}_0 \\
\hat{k}_x & \hat{k}_y & \hat{k}_z & 0
\end{pmatrix}
\begin{pmatrix} u \\ v \\ w \\ \phi \end{pmatrix} = 0, \qquad (10)
$$

where $\hat{\omega}_0 \equiv i(\frac{\partial}{\partial t} + \mathbf{U_0} \cdot \nabla)$ and $U_{0,x} \equiv \partial U_0/\partial x$ and $\Phi_{0,x} \equiv \partial\Phi_0/\partial x$, etc. are used for compactness.

If at this point, we were to calculate the dispersion relation corresponding to the differential equation (10) by letting $\hat{\omega}_0 \to \omega$ and $\hat{\mathbf{k}} \to \mathbf{k}$ and setting the determinant of the matrix in (10) to zero, we would get the dispersion relation [*Jones*, 2005, eq. (11)] in the Boussinesq approximation [*Jones*, 2005, eq. (24)] (neglecting $\mathbf{k}_A$). To see this, notice that if we let $D\rho/Dt = 0$, $\hat{\omega}_0 \to \omega$, let $\hat{\mathbf{k}} \to \mathbf{k}$, and neglect $\mathbf{k}_A$ in [*Jones*, 2005, eq. (7)], the determinant of that matrix can be converted to that in (10)[1] (except for a simple factor). That we can convert the determinant for the matrix in [*Jones*, 2005, eq. (7)] to the determinant of the matrix in (10) shows that any differences in the dispersion relation are not from using the vorticity equation versus using the momentum equation.

Now, however, we shall calculate the differential equation for w from (10). We begin by taking the special case considered by *Eady* [1949], *Jones* [1967], and *Smith* [1986]. That is, we consider flow in only the x direction, let it be a function of z only, we neglect Ω_x, Ω_y, $\tilde{g}_x$, and $\tilde{g}_y$ and we take $\Phi_{0,x} \equiv \partial\Phi_0/\partial x$ to be zero. Then (10) becomes

$$\begin{pmatrix} \begin{array}{c}-\hat{k}_z\hat{k}_x\hat{\omega}_0 \\ -2i\hat{k}_z\hat{k}_y\Omega_z\end{array} & \begin{array}{c}-\hat{k}_z\hat{k}_y\hat{\omega}_0 \\ +2i\hat{k}_z\hat{k}_x\Omega_z\end{array} & \begin{array}{c}-i\hat{k}_z\hat{k}_x U_{0,z} \\ +\hat{k}_H^2\hat{\omega}_0\end{array} & i\tilde{g}_z\hat{k}_H^2 \\ \begin{array}{c}\hat{k}_y\hat{\omega}_0 \\ -2i\hat{k}_x\Omega_z\end{array} & \begin{array}{c}-\hat{k}_x\hat{\omega}_0 \\ -2i\hat{k}_y\Omega_z\end{array} & i\hat{k}_y U_{0,z} & 0 \\ 0 & i\Phi_{0,y} & i\Phi_{0,z} & \hat{\omega}_0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \phi \end{pmatrix} = 0, \quad (11)$$

Equation (11) is equivalent to Eady's equation (II.4) except that he has imposed geostrophic balance and he neglects the second term in row one column three.

To find a single differential equation for w, we do the following: First, add $2i\Omega_z$ times the fourth row to the second row. Then, we replace the first row by $\hat{\omega}_0^2$ times the first row plus $(2i\Omega_z\hat{\omega}_0\hat{k}_z + \tilde{g}_z\Phi_{0,y}\hat{k}_x)$ times the new second row minus $i\tilde{g}_z\hat{\omega}_0\hat{k}_H^2$ times the third row plus $(\hat{\omega}_0^2\hat{k}_z\hat{\omega}_0 - \tilde{g}_z\Phi_{0,y}\hat{\omega}_0\hat{k}_y)$ times the fourth row. This gives (after rearranging some terms)

$$\begin{pmatrix} 2i\Omega_z\hat{\omega}_0[\hat{k}_z,\hat{\omega}_0]\hat{k}_y & -2i\Omega_z\hat{\omega}_0[\hat{k}_z,\hat{\omega}_0]\hat{k}_x & \begin{array}{c}-4\Omega_z^2\hat{\omega}_0\hat{k}_z^2 \\ +\hat{\omega}_0^2\hat{k}_z\hat{\omega}_0\hat{k}_z \\ -i\hat{\omega}_0^2\hat{k}_z\hat{k}_x U_{0,z} \\ +2i\Omega_z\tilde{g}_z\Phi_{0,y}\hat{k}_x\hat{k}_z \\ -\tilde{g}_z\Phi_{0,y}\hat{\omega}_0\hat{k}_y\hat{k}_z \\ -2\Omega_z\hat{\omega}_0\hat{k}_z\hat{k}_y U_{0,z} \\ +\hat{\omega}_0^2\hat{k}_H^2\hat{\omega}_0 \\ +\tilde{g}_z\Phi_{0,z}\hat{\omega}_0\hat{k}_H^2 \\ +i\tilde{g}_z\Phi_{0,y}\hat{k}_x\hat{k}_y U_{0,z}\end{array} & 0 \\ \hat{k}_y\hat{\omega}_0 & -\hat{k}_x\hat{\omega}_0 & \begin{array}{c}i\hat{k}_y U_{0,z} \\ +2i\Omega_z\hat{k}_z\end{array} & 0 \\ 0 & i\Phi_{0,y} & i\Phi_{0,z} & \hat{\omega}_0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \phi \end{pmatrix} = 0, \quad (12)$$

where

$$[\hat{k}_z, \hat{\omega}_0] \equiv \hat{k}_z \hat{\omega}_0 - \hat{\omega}_0 \hat{k}_z = iU_{0,z} \hat{k}_x. \tag{13}$$

If, at this point, we were to let $\hat{k}_z \to k_z$ and $\hat{\omega}_0 \to \omega$, and take the determinant of (12), we would get the dispersion relation [*Jones*, 2005, eq. (28)] because $[k_z, \omega] = 0$. Instead, we add $-2i\Omega_z[\hat{k}_z, \hat{\omega}_0]$ times the second row to the first row to give

$$\begin{pmatrix} 0 & 0 & \begin{matrix} -4\Omega_z^2 \hat{\omega}_0 \hat{k}_z^2 \\ +\hat{\omega}_0^3 \hat{k}_z^2 \\ \underbrace{+\hat{\omega}_0^2 [\hat{k}_z, \hat{\omega}_0] \hat{k}_z}_{operator-ordering} \\ -i\hat{\omega}_0^2 U_{0,z} \hat{k}_z \hat{k}_x \\ \underbrace{-i\hat{\omega}_0^2 [\hat{k}_z, U_{0,z}] \hat{k}_x}_{operator-ordering} \\ +2i\Omega_z \tilde{g}_z \Phi_{0,y} \hat{k}_x \hat{k}_z \\ \underbrace{+4\Omega_z^2 [\hat{k}_z, \hat{\omega}_0] \hat{k}_z}_{operator-ordering} \\ -\tilde{g}_z \Phi_{0,y} \hat{\omega}_0 \hat{k}_y \hat{k}_z \\ -2\Omega_z \hat{\omega}_0 U_{0,z} \hat{k}_y \hat{k}_z \\ \underbrace{-2\Omega_z \hat{\omega}_0 [\hat{k}_z, U_{0,z}] \hat{k}_y}_{operator-ordering} \\ +\hat{\omega}_0^2 \hat{k}_H^2 \hat{\omega}_0 \\ +\tilde{g}_z \Phi_{0,z} \hat{\omega}_0 \hat{k}_H^2 \\ +i\tilde{g}_z \Phi_{0,y} \hat{k}_x U_{0,z} \hat{k}_y \\ \underbrace{+2\Omega_z [\hat{k}_z, \hat{\omega}_0] U_{0,z} \hat{k}_y}_{operator-ordering} \end{matrix} & 0 \\ \hat{k}_y \hat{\omega}_0 & -\hat{k}_x \hat{\omega}_0 & \begin{matrix} iU_{0,z} \hat{k}_y \\ +2i\Omega_z \hat{k}_z \end{matrix} & 0 \\ 0 & i\Phi_{0,y} & i\Phi_{0,z} & \hat{\omega}_0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ \phi \end{pmatrix} = 0, \tag{14}$$

where

$$[\hat{k}_z, U_{0,z}] \equiv \hat{k}_z U_{0,z} - U_{0,z} \hat{k}_z = -iU_{0,z,z} = -i \frac{\partial^2 U_0}{\partial z^2}, \tag{15}$$

the terms marked "operator-ordering" arise because of the dependence on the order of the differential operators. The first, second, and fourth operator-ordering terms come from the second, third, and sixth terms in the third column of the top row of the matrix in (12).

The first row in (14) is now a single differential equation for w, which can be written (after taking $\hat{\omega}_0 \to \omega_0$, $\hat{k}_x \to k_x$, $\hat{k}_y \to k_y$, $\hat{k}_H^2 \to k_H^2$, $\hat{k}_z \to -i\partial/\partial z$, $\Phi_{0,y} \to \partial\Phi_0/\partial y$, $\Phi_{0,z} \to \partial\Phi_0/\partial z$, $U_{0,z} \to \partial U_0/\partial z$ and dividing by $-\omega_0$)

$$\left(\overbrace{-4\Omega_z^2 + \omega_0^2}^{EJS}\right) \frac{\partial^2 w}{\partial z^2}$$
$$+\left(\overbrace{\underbrace{i\omega_0[\hat{k}_z, \hat{\omega}_0]}_{operator-ordering} + \omega_0 \frac{\partial U_0}{\partial z} k_x}^{cancels} \overbrace{-2\frac{\Omega_z}{\omega_0} \underbrace{\tilde{g}_z \frac{\partial \Phi_0}{\partial y}}_{=\partial B/\partial y} k_x \underbrace{+4i\frac{\Omega_z^2}{\omega_0}[\hat{k}_z, \hat{\omega}_0]}_{operator-ordering}}^{EJS} \overbrace{-i \underbrace{\tilde{g}_z \frac{\partial \Phi_0}{\partial y}}_{=\partial B/\partial y} k_y - 2i\Omega_z \frac{\partial U_0}{\partial z} k_y}^{EJ}\right) \frac{\partial w}{\partial z}$$
$$+\left(\underbrace{i\omega_0[\hat{k}_z, U_{0,z}]k_x}_{operator-ordering} \overbrace{\underbrace{+2\Omega_z[\hat{k}_z, U_{0,z}]k_y}_{operator-ordering} - \omega_0^2 k_H^2}^{J} \overbrace{\underbrace{-\tilde{g}_z \frac{\partial \Phi_0}{\partial z}}_{=N^2} k_H^2}^{EJS}\right.$$
$$\left. \overbrace{-i \underbrace{\tilde{g}_z \frac{\partial \Phi_0}{\partial y}}_{=\partial B/\partial y} \frac{\partial U_0}{\partial z} \frac{k_x k_y}{\omega_0} \underbrace{-2\frac{\Omega_z}{\omega_0}[\hat{k}_z, \hat{\omega}_0] \frac{\partial U_0}{\partial z} k_y}_{operator-ordering}}^{EJ}\right) w = 0, \tag{16}$$

where $B(y, z)$ is the horizontally varying buoyancy associated with horizontal density gradients (not to be confused with the baroclinic vector $\mathbf{B}$). To turn (16) into a dispersion relation for comparison with [*Jones*, 2005, eq. (28)], we replace $\partial/\partial z$ by ik_z. The first and second terms in the coefficient of $\partial w/\partial z$ cancel because of (13).

When the fluid is in geostrophic balance (where $\tilde{g}_z(\partial\Phi_0/\partial y) = 2\Omega_z(\partial U_0/\partial z)$ and $[\hat{k}_z, \hat{\omega}_0] \equiv i\hat{k}_x(\partial U_0/\partial z) = i\hat{k}_x\tilde{g}_z(\partial\Phi_0/\partial y)/(2\Omega_z))$, the third and fourth terms in the coefficient of $\partial w/\partial z$ will be equal, and the fifth and sixth terms will be equal. Also, the fifth and sixth terms in the coefficient of w will be equal.

For the geostrophic case, [*Eady*, 1949, equation II.7], [*Jones*, 1967, equation 10], and [*Smith*, 1986, equation 3.1] get the terms so marked by E, J, or S, respectively, except that *Smith* [1986] takes $k_y = 0$, so does not get the k_y terms. They do not get the first term in the coefficient for w, which may be small for the case they consider. The first, third, and fourth terms in the coefficient of w are in the Boussinesq approximation to the Taylor-Goldstein equation [e.g., *Gossard and Hooke*, 1975].

The operator-ordering terms arise because the order of some differential operators is important. Specifically, $(D/Dt)(\partial/\partial z)$ is not the same as $(\partial/\partial z)(D/Dt)$ and $(\partial/\partial z)(\partial U_0/\partial z)(w)$ is not the same as $(\partial U_0/\partial z)(\partial/\partial z)(w)$. The operator-ordering terms give differences between the dispersion relation derived from (16) and that in [*Jones*, 2005, eq. (28)]. This dependence of the dispersion relation on the set of dependent variables leads to an apparent ambiguity. This effect in quantum field theory is referred to as the "operator-ordering ambiguity" [*Halliwell*, 1988], a terminology that may be appropriate in the present context as well.

Appendix A:

Boussinesq, horizontal, geostrophic flow

For this section, we use the notation that the components of the background flow velocity in the x, y, and z (vertical) directions are U, V, and W, and we take $W = 0$. This gives

$$\tilde{\mathbf{\Omega}} \equiv \mathbf{\Omega} + \zeta/4 = (\Omega_x - \frac{1}{4}\frac{\partial V}{\partial z}, \Omega_y + \frac{1}{4}\frac{\partial U}{\partial z}, \Omega_z + \frac{1}{4}\frac{\partial V}{\partial x} - \frac{1}{4}\frac{\partial U}{\partial y}) \tag{A1}$$

and

$$e = \frac{1}{2}\begin{pmatrix} 2\frac{\partial U}{\partial x} & \frac{\partial V}{\partial x}+\frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x}+\frac{\partial U}{\partial y} & 2\frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial U}{\partial z} & \frac{\partial V}{\partial z} & 0 \end{pmatrix}. \quad \text{(A2)}$$

For $W = 0$, geostrophic balance requires

$$\nabla p/\rho \equiv \tilde{\mathbf{g}} = (\tilde{g}_x, \tilde{g}_y, \tilde{g}_z) \approx (2\Omega_z V, -2\Omega_z U, \tilde{g}_z). \quad \text{(A3)}$$

Using (5), (A3), and incompressibility ($\partial U/\partial x + \partial V/\partial y = 0$) gives

$$\mathbf{B} = \nabla\rho_{pot}/\rho \times \tilde{\mathbf{g}} = -\nabla \times \tilde{\mathbf{g}} = (B_x, \omega_{cy}^2, \omega_{cz}^2) = (-2\Omega_z\frac{\partial U}{\partial z}, -2\Omega_z\frac{\partial V}{\partial z}, 0) \quad \text{(A4)}$$

and

$$\nabla\rho_{pot}/\rho \approx \left(\frac{2\Omega_z}{\tilde{g}_z}\frac{\partial V}{\partial z} + \frac{2\Omega_z V}{\tilde{g}_z}\frac{1}{\rho}\frac{\partial\rho_{pot}}{\partial z}, -\frac{2\Omega_z}{\tilde{g}_z}\frac{\partial U}{\partial z} - \frac{2\Omega_z U}{\tilde{g}_z}\frac{1}{\rho}\frac{\partial\rho_{pot}}{\partial z}, \frac{1}{\rho}\frac{\partial\rho_{pot}}{\partial z}\right). \quad \text{(A5)}$$

Thus, the determinant of the matrix in [*Jones*, 2005, eq. (7)], for the incompressible case ($D\rho/Dt = 0$) in the Boussinesq approximation for a horizontal flow in geostrophic balance is

$$\begin{vmatrix} -\omega - i\frac{\partial U}{\partial x} & 2i\tilde{\Omega}_z - \frac{i}{2}(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}) & -2i\tilde{\Omega}_y - \frac{i}{2}\frac{\partial U}{\partial z} & i\tilde{g}_x & k_x \\ -2i\tilde{\Omega}_z - \frac{i}{2}(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}) & -\omega - i\frac{\partial V}{\partial y} & 2i\tilde{\Omega}_x - \frac{i}{2}\frac{\partial V}{\partial z} & i\tilde{g}_y & k_y \\ 2i\tilde{\Omega}_y - \frac{i}{2}\frac{\partial U}{\partial z} & -2i\tilde{\Omega}_x - \frac{i}{2}\frac{\partial V}{\partial z} & -\omega & i\tilde{g}_z & k_z \\ -i(\frac{2\Omega_z}{\tilde{g}_z}\frac{\partial V}{\partial z} + \frac{\tilde{g}_x}{\tilde{g}_z}\frac{1}{\rho}\frac{\partial\rho_{pot}}{\partial z}) & -i(-\frac{2\Omega_z}{\tilde{g}_z}\frac{\partial U}{\partial z} + \frac{\tilde{g}_y}{\tilde{g}_z}\frac{1}{\rho}\frac{\partial\rho_{pot}}{\partial z}) & -i(\frac{1}{\rho}\frac{\partial\rho_{pot}}{\partial z}) & -\omega & 0 \\ k_x & k_y & k_z & 0 & 0 \end{vmatrix} = 0. \quad \text{(A6)}$$

To compare (A6) with [*Kunze*, 1985, eq. (5)], we first multiply each element in (A6) by i. We divide the fourth column by $-\tilde{g}_z = |\tilde{g}_z|$, and multiply the fourth row by the same factor. We interchange the fourth and fifth columns, and set Ω_x and Ω_y to zero. Finally, we approximate $\tilde{\mathbf{g}} \equiv \nabla p/\rho$ by $\mathbf{g}$, neglect g_x and g_y, and for the incompressible case, let $\rho_{pot} = \rho$. The two matrices are then identical.

Acknowledgments. This research was supported by the NOAA Environmental Technology Laboratory.

Notes

1. Explicitly, we perform the following steps on the determinant of the matrix in [*Jones*, 2005, eq. (7)]. (1) replace row three by $k_x k_z$ times row one plus $k_y k_z$ times row two minus k_H^2 times row three; (2) replace row two by k_x times row two minus k_y times row one; (3) delete row one and column five (because column five is now zero except in the first row); (4) interchange rows one and two; (5) multiply column four by minus one.

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