Diagonal form for internal gravity wave dispersion relations for the ocean and atmosphere*

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Abstract. To fully investigate properties of a dispersion relation for internal waves in the ocean or atmosphere that includes baroclinicity or rate of strain, it is necessary to rotate the coordinate axes to diagonalize the dispersion relation by eliminating cross terms in the wavenumber. Failure to eliminate from the dispersion relation cross terms in the wavenumber can lead to significant errors in estimating maximum and minimum frequencies of propagation. We give the details here to perform the diagonalization.

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1. Introduction

Jones [2005] gives a general dispersion relation for internal acoustic-gravity waves including all components of the Earth's rotation, baroclinicity, vorticity, and rate of strain. Because that dispersion relation contains cross terms in the wavenumber $\mathbf{k}$, it is necessary to rotate the coordinate axes to eliminate those cross terms when analyzing some of the properties of the dispersion relation. The dispersion relation is expressed in terms of a matrix S that must be diagonalized to eliminate all of the cross terms in $\mathbf{k}$.

Jones [2006] (using some of the details derived here) gives an approximate dispersion relation in diagonal form by rotating the axes to eliminate cross terms in $\mathbf{k}$ and estimates the effects of vorticity, baroclinicity and rate of strain on the minimum and maximum propagation frequencies for internal gravity waves.

Section 2 describes the process of diagonalizing the dispersion relation. Appendix A gives the general dispersion relation [*Jones*, 2005] for internal acoustic-gravity waves including all components of the Earth's rotation, baroclinicity, vorticity, and rate of strain.

Appendix B derives approximate formulas to calculate the eigenvalues of the matrix S needed to diagonalize the dispersion relation. The eigenvalues of S are calculated by *Jones* [2006] neglecting rate of strain, and here, in appendix C, including rate of strain. Appendix D gives some formulas needed for some of the derivations.

2. Diagonalizing the dispersion relation

The eigenvalues λ of S are determined by the eigenvalue equation (B1), which leads to the cubic equation (B2), whose coefficients are in terms of the trace (sum of the diagonal terms) of S, the trace of $\tilde{S}$, the matrix of the co-factors of S, and the determinant of S. Equations (B3), (B4), and (B5) give approximations for the three eigenvalues of S when the determinant $|S|$ can be neglected to first order. The formulas for $tr(S)$ and $tr(\tilde{S})$ have many terms when we include rate of strain, and are given in Appendix C.

Appendix A: General dispersion relation

We use a general dispersion relation [*Jones*, 2005] for internal acoustic-gravity waves including all components of the Earth's rotation, baroclinicity, vorticity, and rate of strain.

$$\begin{gathered}(\mathbf{k}^2 + \mathbf{k_A}^2)(N^2 - \omega^2) + \mathbf{k}\cdot\mathbf{S}\cdot\mathbf{k} + \mathbf{k_A}\cdot\mathbf{S}\cdot\mathbf{k_A} \\ +\mathbf{A}\cdot\mathbf{k} + (1/C^2)[\omega^4 - 4\omega^2\tilde{\mathbf{\Omega}}^2 + \mathbf{B}^2/2 - 2i\omega\tilde{\mathbf{\Omega}}\cdot\mathbf{B} \\ +i\mathbf{B}\cdot\mathbf{e}\cdot\mathbf{B}/(2\omega) - \omega^2 tr(\tilde{\mathbf{e}}) \\ -4i\omega\tilde{\mathbf{\Omega}}\cdot\mathbf{e}\cdot\tilde{\mathbf{\Omega}} - i\omega|\mathbf{e}| + 2\tilde{\mathbf{\Omega}}\cdot\mathbf{e}\cdot\mathbf{B}] = 0, \qquad \text{(A1)}\end{gathered}$$

where S is the symmetric matrix defined by

$$\begin{gathered}S_{\alpha\beta} \equiv \underbrace{-\frac{1}{2\rho}\left(\frac{\partial\rho_{pot}}{\partial x_\alpha}\tilde{g}_\beta + \frac{\partial\rho_{pot}}{\partial x_\beta}\tilde{g}_\alpha\right)}_{1}\underbrace{+4\tilde{\Omega}_\alpha\tilde{\Omega}_\beta}_{2} \\ \underbrace{+\frac{i}{\omega}\left(\tilde{\Omega}_\alpha B_\beta + \tilde{\Omega}_\beta B_\alpha\right)}_{3}\underbrace{+\tilde{e}_{\alpha\beta}}_{4}\underbrace{+i\omega e_{\alpha\beta}}_{5} \\ \underbrace{-\frac{i}{2\omega}[(\tilde{\mathbf{g}}\times\mathbf{e}\times\nabla\rho_{pot}/\rho)_{\alpha\beta} + (\tilde{\mathbf{g}}\times\mathbf{e}\times\nabla\rho_{pot}/\rho)_{\beta\alpha}]}_{6}, \qquad \text{(A2)}\end{gathered}$$

(the term numbers will be referred to later),

$$\mathbf{A} = (4\omega\tilde{\mathbf{\Omega}} + i\mathbf{B})\cdot(\mathbf{1} + i\mathbf{e}/\omega)\times\mathbf{\Gamma} + 2\mathbf{k}_A\cdot\tilde{\mathbf{\Omega}}\mathbf{B}/\omega, \qquad \text{(A3)}$$

$\omega = \sigma - \mathbf{k}\cdot\mathbf{U}$ is the intrinsic frequency, σ is the wave frequency, $\mathbf{U}$ is the background fluid velocity, $\mathbf{k}$ is the wavenumber, $\tilde{\mathbf{g}} \equiv \nabla p/\rho = \mathbf{g} - D\mathbf{U}/Dt - 2\mathbf{\Omega}\times\mathbf{U}$ is the effective vector acceleration due to gravity [including (minus) the acceleration of the background flow], p is background pressure, ρ is background density, ρ_{pot} is background potential density [e.g., *Gill*, 1982, Section 3.7.5, pp. 54-55], $\tilde{\mathbf{\Omega}} \equiv \mathbf{\Omega} + \zeta/4$, $\zeta \equiv \nabla\times\mathbf{U}$ is the vorticity, $\mathbf{\Omega}$ is the Earth's angular velocity, C is sound speed, $\mathbf{k}_A \equiv \nabla\rho/(2\rho)$,

$$N^2 = \nabla\rho_{pot}\cdot\nabla p/\rho^2 = \nabla\rho_{pot}\cdot\tilde{\mathbf{g}}/\rho \qquad \text{(A4)}$$

is the square of the Brunt-Väisälä frequency [e.g., *Jones*, 2001],

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$$\mathbf{B} \equiv \frac{(\nabla\rho \times \nabla p)}{\rho^2} = \frac{(\nabla\rho_{pot} \times \nabla p)}{\rho^2} = (\frac{\nabla\rho_{pot}}{\rho} \times \tilde{\mathbf{g}}) \quad \text{(A5)}$$

is the baroclinic vector [e.g., *Gill*, 1982, Section 7.11, pp. 237-238], $\mathbf{e}$ is the matrix representing the symmetric rate-of-strain tensor [*Aris*, 1962, p. 89], [*Cole*, 1962, p. 228], [*Monin and Yaglom*, 1987, section 6.3, p. 389]

$$e_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad \text{(A6)}$$

$$\tilde{e}_{ij} \equiv e_{\alpha\beta} e_{\gamma\delta} \varepsilon_{i\alpha\gamma} \varepsilon_{j\beta\delta}/2 \quad \text{(A7)}$$

is the matrix of co-factors of $\mathbf{e}$, ε_{ijk} is the completely antisymmetric tensor with $\varepsilon_{123} = 1$ and $\varepsilon_{ijk}+\varepsilon_{jik} = \varepsilon_{ijk}+\varepsilon_{ikj} = 0$, and summation from 1 to 3 is implied over repeated indices. Notice that the symmetric rate-of-strain tensor is the symmetric part of the shear tensor, while the vorticity gives the antisymmetric part of the shear tensor. In (A2) and (A3), a cross product of a vector times a matrix is the matrix whose columns are the cross products of the vector times the corresponding columns of the original matrix. A cross product of a matrix times a vector is the matrix whose rows are the cross products of the corresponding rows of the original matrix times the vector.

A vector identity [*Jones*, 2005, equation (4)] was used to transform the dispersion relation in the work of *Jones* [2005] to that in (A1). The particular combination of ζ and $\mathbf{\Omega}$ that combine to form $\tilde{\mathbf{\Omega}}$ agrees with the analysis of *Kunze* [1985], and is a result of the linearization process.

A dispersion relation depends on the differential equations from which the dispersion relation was derived. Both the dispersion relation and the set of differential equations are needed to calculate a WKB approximation. The set of differential equations from which the dispersion relation (A1) was derived are available [*Jones*, 2005, eq. (6) or Appendix A].

Appendix B: Rotating the axes to diagonalize the matrix *S*

Equation (A1) will have no cross terms in components of $\mathbf{k}$ if the matrix S in (A2) is diagonal. Because S is symmetric, it can be diagonalized by rotating the axes. The elements of that diagonal matrix will be given by the eigenvalues of S, which are the three solutions for λ of the eigenvalue equation

$$|S - \lambda I| = 0, \quad \text{(B1)}$$

where I is the unit 3×3 matrix.
Equation (B1) gives a cubic equation for λ,

$$\lambda^3 - tr(S)\lambda^2 + tr(\tilde{S})\lambda - |S| = 0, \quad \text{(B2)}$$

where $\tilde{S}$ is the matrix of co-factors of S, tr(S) is the trace (sum of the diagonal elements) of S, and $|S|$ is the determinant of S. Although (B2) can be solved in closed form, we shall use approximate solutions because they are simpler.

Equations (A4) and (A5) show that $B << N^2$ when the baroclinicity is smaller than a few degrees. Except very near neutral stability, N dominates over vorticity and rate of strain [*Kunze*, 1985]. Under those conditions, we can neglect $|S|$ to first order. This gives the approximate solutions to (B2) as

$$\lambda_1 \approx |S|/tr(\tilde{S}) + tr(S)|S|^2/tr(\tilde{S})^3, \quad \text{(B3)}$$

$$\lambda_2 \approx \tfrac{1}{2}tr(S) + \sqrt{tr(S)^2/4 - tr(\tilde{S})} - \frac{|S|}{2tr(\tilde{S})}\left[1 - \frac{tr(S)}{2\sqrt{tr(S)^2/4 - tr(\tilde{S})}}\right], \quad \text{(B4)}$$

and

$$\lambda_3 \approx \tfrac{1}{2}tr(S) - \sqrt{tr(S)^2/4 - tr(\tilde{S})} - \frac{|S|}{2tr(\tilde{S})}\left[1 + \frac{tr(S)}{2\sqrt{tr(S)^2/4 - tr(\tilde{S})}}\right]. \quad \text{(B5)}$$

Equations (B3), (B4), and (B5) give approximations for the three eigenvalues of S in terms of $tr(S)$, $tr(\tilde{S})$, and $|S|$, when we can neglect $|S|$ to first order. We keep the terms only to lowest order in $1/N^2$. This allows us to neglect $|S|$ in (B3), (B4), and (B5).

Appendix C: Rotating the Axes including Rate of strain

For S given by(A2), we have

$$|S| = \underbrace{-2(\tilde{\mathbf{\Omega}}\cdot\mathbf{B})^2}_{(1,1,2)} \underbrace{-\frac{i\mathbf{B}^2}{2\omega}\mathbf{B}\cdot\tilde{\mathbf{\Omega}}}_{(1,1,3)}$$
$$\underbrace{-(\mathbf{B}\times\nabla\rho_{pot}/\rho\cdot\tilde{\mathbf{\Omega}})(\mathbf{B}\times\tilde{\mathbf{g}}\cdot\tilde{\mathbf{\Omega}})/\omega^2}_{(1,3,3)}$$
$$\underbrace{-\mathbf{B}\cdot\tilde{\mathbf{e}}\cdot\mathbf{B}}_{(1,1,4)} \underbrace{-i\omega\mathbf{B}\cdot\mathbf{e}\cdot\mathbf{B}}_{(1,1,5)} \underbrace{+\frac{i}{2\omega}\mathbf{B}\cdot(\tilde{\mathbf{g}}\times\mathbf{e}\times\nabla\rho_{pot}/\rho)\cdot\mathbf{B}}_{(1,1,6)}$$
$$\underbrace{+4\tilde{\mathbf{g}}\times\tilde{\mathbf{\Omega}}\cdot\tilde{\mathbf{e}}\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,2,4)} \underbrace{+4i\omega\tilde{\mathbf{g}}\times\tilde{\mathbf{\Omega}}\cdot\mathbf{e}\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,2,5)}$$
$$\underbrace{-\frac{2i}{\omega}\tilde{\mathbf{g}}\times\tilde{\mathbf{\Omega}}\cdot(\frac{\nabla\rho_{pot}}{\rho}\times\mathbf{e}\times\tilde{\mathbf{g}}+\tilde{\mathbf{g}}\times\mathbf{e}\times\frac{\nabla\rho_{pot}}{\rho})\cdot\tilde{\mathbf{\Omega}}\times\frac{\nabla\rho_{pot}}{\rho}}_{(1,2,6)}$$
$$\underbrace{+\frac{2i}{\omega}\tilde{\mathbf{g}}\times\mathbf{B}\cdot\tilde{\mathbf{e}}\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,3,4)} \underbrace{-2\tilde{\mathbf{g}}\times\mathbf{B}\cdot\mathbf{e}\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,3,5)}$$
$$\underbrace{+\frac{2}{\omega^2}\tilde{\mathbf{g}}\times\mathbf{B}\cdot(\nabla\rho_{pot}/\rho\times\mathbf{e}\times\tilde{\mathbf{g}})\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,3,6)}$$
$$\underbrace{+\frac{2}{\omega^2}\tilde{\mathbf{g}}\times\mathbf{B}\cdot(\tilde{\mathbf{g}}\times\mathbf{e}\times\nabla\rho_{pot}/\rho)\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,3,6)}$$
$$\underbrace{-\frac{4}{\omega^2}\tilde{\mathbf{\Omega}}\times\mathbf{B}\cdot\tilde{\mathbf{e}}\cdot\mathbf{B}\times\tilde{\mathbf{\Omega}}}_{(3,3,4)} \underbrace{-\frac{4i}{\omega}\tilde{\mathbf{\Omega}}\times\mathbf{B}\cdot\mathbf{e}\cdot\mathbf{B}\times\tilde{\mathbf{\Omega}}}_{(3,3,5)}$$
$$\underbrace{+\frac{4i}{\omega^3}\tilde{\mathbf{\Omega}}\times\mathbf{B}\cdot(\tilde{\mathbf{g}}\times\mathbf{e}\times\nabla\rho_{pot}/\rho)\cdot\mathbf{B}\times\tilde{\mathbf{\Omega}}}_{(3,3,6)} \underbrace{+|\tilde{\mathbf{e}}|}_{(4,4,4)} \underbrace{-i\omega^3|\mathbf{e}|}_{(5,5,5)}$$
$$\underbrace{+\frac{i}{8\omega^3}|\nabla\rho_{pot}/\rho\times\mathbf{e}\times\tilde{\mathbf{g}}+\tilde{\mathbf{g}}\times\mathbf{e}\times\nabla\rho_{pot}/\rho|}_{(6,6,6)} \quad \text{(C1)}$$

plus additional terms, which we shall neglect along with the rest of $|S|$. The numbers below the terms in (C1) and in the equations below give the term numbers in (A2) from which the terms came.

$$
\begin{aligned}
tr(S) &\equiv S_{xx} + S_{yy} + S_{zz} \\
&= \underbrace{-N^2}_{1} \underbrace{+4\tilde{\Omega}^2}_{2} \underbrace{+\frac{2i}{\omega}\tilde{\Omega}\cdot\mathbf{B}}_{3} \underbrace{+tr(\tilde{e})}_{4} \underbrace{+i\omega\,\overbrace{tr(e)}^{=0}}_{5} \\
&\quad \underbrace{+\frac{i}{\omega}(N^2\,\overbrace{tr(e)}^{=0} - \tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\nabla\rho_{pot}/\rho)}_{6} \\
&= \underbrace{-N^2}_{1} \underbrace{+4\tilde{\Omega}^2}_{2} \underbrace{+\frac{2i}{\omega}\tilde{\Omega}\cdot\mathbf{B}}_{3} \underbrace{+tr(\tilde{e})}_{4} \\
&\quad \underbrace{-\frac{i}{\omega}(N^2\mathbf{e}_{zz} + \frac{\mathbf{B}^2\mathbf{e}_{zz} - (\mathbf{B}\times\hat{\imath}_z)\cdot\mathbf{e}\cdot(\mathbf{B}\times\hat{\imath}_z)}{2N^2 + 2\sqrt{N^4+\mathbf{B}^2}})}_{6},
\end{aligned}
\tag{C2}
$$

where use is made of formulas in appendix D, $\hat{\imath}_z$ is a unit vector in the z direction, and from here on, the subscript z denotes a component parallel to the $-\tilde{\mathbf{g}}/|\tilde{\mathbf{g}}| - \nabla\rho_{pot}/|\nabla\rho_{pot}|$ direction. We also have

$$
\begin{aligned}
& tr(\tilde{S}) = \\
& S_{yy}S_{zz} - S_{yz}^2 + S_{xx}S_{zz} - S_{xz}^2 + S_{xx}S_{yy} - S_{xy}^2 \\
&= \tfrac{1}{2}\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}S_{\alpha\beta}S_{\gamma\delta} = \tfrac{1}{2}\left[tr(S)^2 - tr(S^2)\right] \\
&= \textstyle\sum_{i=1}^{6}\sum_{j=1}^{6}\frac{1}{2}\left[tr(S_i)tr(S_j) - tr(S_iS_j)\right] \\
&= \underbrace{-\frac{1}{4}\mathbf{B}^2}_{(1,1)} \underbrace{-4\tilde{\mathbf{\Omega}}\times\tilde{\mathbf{g}}\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,2)} \\
& \underbrace{-\frac{i}{\omega}\tilde{\mathbf{\Omega}}\times\mathbf{g}\cdot\mathbf{B}\times\nabla\rho_{pot}/\rho}_{(1,3)} \underbrace{-\frac{i}{\omega}\mathbf{B}\times\mathbf{g}\cdot\tilde{\mathbf{\Omega}}\times\nabla\rho_{pot}/\rho}_{(1,3)} \\
& \underbrace{+(\tilde{\mathbf{\Omega}}\times B)^2/\omega^2}_{(3,3)} \underbrace{+\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\tilde{e}_{\alpha\beta}\Phi_{,\gamma}\tilde{g}_\delta}_{(1,4)} \\
& \underbrace{+4\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\tilde{e}_{\alpha\beta}\tilde{\Omega}_\gamma\tilde{\Omega}_\delta}_{(2,4)} \underbrace{+\frac{2i}{\omega}\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\tilde{e}_{\alpha\beta}\tilde{\Omega}_\gamma B_\delta}_{(3,4)} \\
& \underbrace{+i\omega\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}e_{\alpha\beta}\Phi_{,\gamma}\tilde{g}_\delta}_{(1,5)} \underbrace{+4i\omega\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}e_{\alpha\beta}\tilde{\Omega}_\gamma\tilde{\Omega}_\delta}_{(2,5)} \\
& \underbrace{-2\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}e_{\alpha\beta}\tilde{\Omega}_\gamma B_\delta}_{(3,5)} \\
& \underbrace{-\frac{i}{2\omega}(\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta} + \varepsilon_{i\alpha\delta}\varepsilon_{i\beta\gamma})\varepsilon_{\alpha ac}\varepsilon_{\beta bd}\Phi_{,a}\tilde{g}_b e_{cd}\Phi_{,\gamma}\tilde{g}_\delta}_{(1,6)} \\
& \underbrace{-\frac{4i}{\omega}\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\varepsilon_{\alpha ac}\varepsilon_{\beta bd}\Phi_{,a}\tilde{g}_b e_{cd}\tilde{\Omega}_\gamma\tilde{\Omega}_\delta}_{(2,6)} \\
& \underbrace{+\frac{1}{2\omega^2}(\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta} + \varepsilon_{i\alpha\delta}\varepsilon_{i\beta\gamma})\times}_{(3,6)} \\
& \underbrace{(\varepsilon_{\alpha ac}\varepsilon_{\beta bd} + \varepsilon_{\alpha bc}\varepsilon_{\beta ad})\Phi_{,a}\tilde{g}_b e_{cd}\tilde{\Omega}_\gamma B_\delta}_{(3,6)} \\
& \underbrace{+\frac{1}{2}\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\tilde{e}_{\alpha\beta}\tilde{e}_{\gamma\delta}}_{(4,4)} \underbrace{+i\omega\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\tilde{e}_{\alpha\beta}\tilde{e}_{\gamma\delta}}_{(4,5)} \\
& \underbrace{-\frac{\omega^2}{2}\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}e_{\alpha\beta}e_{\gamma\delta}}_{(5,5)} \\
& \underbrace{-\frac{i}{\omega}\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\varepsilon_{\gamma eg}\varepsilon_{\delta fh}\Phi_{,e}\tilde{g}_f\tilde{e}_{\alpha\beta}e_{gh}}_{(4,6)} \\
& \underbrace{+\varepsilon_{i\alpha\gamma}\varepsilon_{i\beta\delta}\varepsilon_{\gamma eg}\varepsilon_{\delta fh}\Phi_{,e}\tilde{g}_f e_{\alpha\beta}e_{gh}}_{(5,6)} \\
& \underbrace{-\frac{2}{\omega^2}[N^2 tr(\mathbf{e}) - \tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\frac{\nabla\rho_{pot}}{\rho}]^2}_{(6,6)} \\
& \underbrace{+\frac{1}{2\omega^2}tr[(\tilde{\mathbf{g}}\times\mathbf{e}\times\frac{\nabla\rho_{pot}}{\rho} + \frac{\nabla\rho_{pot}}{\rho}\times\mathbf{e}\times\tilde{\mathbf{g}})^2]}_{(6,6)}.
\end{aligned}
\tag{C3}
$$

After some algebra and some identities, we get

$$
\begin{aligned}
& tr(\tilde{S}) = \\
& \underbrace{-\frac{1}{4}\mathbf{B}^2}_{(1,1)} \underbrace{-4N^2\tilde{\Omega}^2 + 4\tilde{\mathbf{g}}\cdot\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}\cdot\frac{\nabla\rho_{pot}}{\rho}}_{(1,2)} \underbrace{+(\tilde{\mathbf{\Omega}}\times B)^2/\omega^2}_{(3,3)} \\
& \underbrace{-\frac{2i}{\omega}N^2\tilde{\mathbf{\Omega}}\cdot\mathbf{B} + \frac{i}{\omega}\tilde{\mathbf{g}}\cdot\tilde{\mathbf{\Omega}}\ \mathbf{B}\cdot\frac{\nabla\rho_{pot}}{\rho} + \frac{i}{\omega}\tilde{\mathbf{g}}\cdot\mathbf{B}\ \tilde{\mathbf{\Omega}}\cdot\frac{\nabla\rho_{pot}}{\rho}}_{(1,3)} \\
& \underbrace{-[N^2 tr(\tilde{e}) - \tilde{\mathbf{g}}\cdot\tilde{e}\cdot\frac{\nabla\rho_{pot}}{\rho}]}_{(1,4)} \underbrace{-i\omega[N^2 tr(e) - \tilde{\mathbf{g}}\cdot e\cdot\frac{\nabla\rho_{pot}}{\rho}]}_{(1,5)} \\
& -\underbrace{\frac{i}{\omega}[N^4 tr(e) - N^2\tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\nabla\rho_{pot}/\rho + \mathbf{B}\cdot\mathbf{e}\cdot\mathbf{B}]}_{(1,6)} \\
& +\underbrace{4[\tilde{\mathbf{\Omega}}^2 tr(\tilde{e}) - \tilde{\mathbf{\Omega}}\cdot\tilde{e}\cdot\tilde{\mathbf{\Omega}}]}_{(2,4)} + \underbrace{4i\omega[\tilde{\mathbf{\Omega}}^2 tr(e) - \tilde{\mathbf{\Omega}}\cdot e\cdot\tilde{\mathbf{\Omega}}]}_{(2,5)} \\
& \underbrace{+\frac{4i}{\omega}[N^2\tilde{\mathbf{\Omega}}^2 tr(e)]}_{(2,6)} \underbrace{+\frac{4i}{\omega}[\tilde{\mathbf{\Omega}}^2\tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\nabla\rho_{pot}/\rho]}_{(2,6)} \\
& \underbrace{-\frac{4i}{\omega}[\tilde{\mathbf{g}}\cdot(\tilde{\mathbf{\Omega}}\times\mathbf{e}\times\tilde{\mathbf{\Omega}})\cdot\nabla\rho_{pot}/\rho]}_{(2,6)} \\
& \underbrace{+\frac{2i}{\omega}[\tilde{\mathbf{\Omega}}\cdot\mathbf{B}tr(\tilde{e}) - \tilde{\mathbf{\Omega}}\cdot\tilde{e}\cdot\mathbf{B}]}_{(3,4)} \underbrace{-2[\tilde{\mathbf{\Omega}}\cdot\mathbf{B}tr(e) - \tilde{\mathbf{\Omega}}\cdot e\cdot\mathbf{B}]}_{(3,5)} \\
& \underbrace{+\frac{1}{2\omega^2}[-4N^2 tr(\mathbf{e}) + 4\tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\nabla\rho_{pot}/\rho]\tilde{\mathbf{\Omega}}\cdot\mathbf{B}}_{(3,6)} \\
& \underbrace{-\frac{1}{2\omega^2}[2\tilde{\mathbf{g}}\cdot(\tilde{\mathbf{\Omega}}\times\mathbf{e}\times\mathbf{B})\cdot\nabla\rho_{pot}/\rho]}_{(3,6)} \\
& \underbrace{-\frac{1}{2\omega^2}[2\tilde{\mathbf{g}}\cdot(\mathbf{B}\times\mathbf{e}\times\tilde{\mathbf{\Omega}})\cdot\nabla\rho_{pot}/\rho]}_{(3,6)} \\
& \underbrace{+i\omega[tr(e)tr(\tilde{e}) - 3|e|]}_{(4,5)} \underbrace{-\omega^2 tr(\tilde{e})}_{(5,5)}
\end{aligned}
$$

$$\underbrace{-N^2tr(e)^2+tr(e)\tilde{\mathbf{g}}\cdot e\cdot\nabla\rho_{pot}/\rho+2\tilde{\mathbf{g}}\cdot\tilde{e}\cdot\nabla\rho_{pot}/\rho}_{(5,6)}$$

$$\underbrace{+tr(\tilde{e})}_{(4,4)}\underbrace{-\frac{i}{\omega}[-N^2tr(e)tr(\tilde{e})]}_{(4,6)}\underbrace{-\frac{i}{\omega}[tr(\tilde{e})\tilde{\mathbf{g}}\cdot e\cdot\nabla\rho_{pot}/\rho]}_{(4,6)}$$

$$\underbrace{-\frac{i}{\omega}[tr(e^2)\tilde{\mathbf{g}}\cdot e\cdot\nabla\rho_{pot}/\rho]}_{(4,6)}\underbrace{+\frac{i}{\omega}(\tilde{\mathbf{g}}\cdot e^3\cdot\nabla\rho_{pot}/\rho)}_{(4,6)}$$

$$\underbrace{-\frac{2}{\omega^2}[N^2tr(\mathbf{e})-\tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\nabla\rho_{pot}/\rho]^2}_{(6,6)}$$

$$\underbrace{+\frac{1}{\omega^2}[(\tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\tilde{\mathbf{g}})(\nabla\rho_{pot}/\rho\cdot\mathbf{e}\cdot\nabla\rho_{pot}/\rho)+(\tilde{\mathbf{g}}\cdot\mathbf{e}\cdot\nabla\rho_{pot}/\rho)^2]}_{(6,6)}$$

$$\underbrace{-\frac{1}{\omega^2}[2N^2\tilde{\mathbf{g}}\cdot\hat{\mathbf{e}}\cdot\nabla\rho_{pot}/\rho]}_{(6,6)}\underbrace{-\frac{1}{\omega^2}[\tilde{\mathbf{g}}^2\nabla\rho_{pot}/\rho\cdot\hat{\mathbf{e}}\cdot\nabla\rho_{pot}/\rho]}_{(6,6)}$$

$$\underbrace{-\frac{1}{\omega^2}[(\nabla\rho_{pot}/\rho)^2\tilde{\mathbf{g}}\cdot\hat{\mathbf{e}}\cdot\tilde{\mathbf{g}}]}_{(6,6)}, \qquad \text{(C4)}$$

where $\hat{\mathbf{e}}\equiv\tilde{\mathbf{e}}+\mathbf{e}^2/2$.
Using results from appendix D, we get finally

$$tr(\tilde{S})=-N^2\left[\underbrace{4\tilde{\mathbf{\Omega}}_\perp^2}_{(1,2)}\underbrace{+\frac{2i}{\omega}\tilde{\Omega}\cdot\mathbf{B}}_{(1,3)}\underbrace{+\tilde{e}_{xx}+\tilde{e}_{yy}}_{(1,4)}\right.$$

$$\underbrace{+i\omega(e_{xx}+e_{yy})}_{(1,5)}\underbrace{-\frac{iN^2}{\omega}\mathbf{e}_{zz}}_{(1,6)}\underbrace{-\frac{4i}{\omega}\tilde{\mathbf{\Omega}}^2\mathbf{e}_{zz}}_{(2,6)}\underbrace{+\frac{4i}{\omega}(\tilde{\mathbf{\Omega}}\times\mathbf{e}\times\tilde{\mathbf{\Omega}})_{zz}}_{(2,6)}$$

$$\underbrace{-\frac{2\tilde{\mathbf{\Omega}}\cdot\mathbf{B}}{\omega^2}\mathbf{e}_{zz}}_{(3,6)}\underbrace{+\frac{1}{\omega^2}(\tilde{\mathbf{\Omega}}\times\mathbf{e}\times\mathbf{B})_{zz}}_{(3,6)}\underbrace{+\frac{1}{\omega^2}(\mathbf{B}\times\mathbf{e}\times\tilde{\mathbf{\Omega}})_{zz}}_{(3,6)}$$

$$\left.\underbrace{+\frac{i}{\omega}tr(\tilde{\mathbf{e}})\mathbf{e}_{zz}}_{(4,6)}\underbrace{+\frac{i}{\omega}tr(\mathbf{e}^2)\mathbf{e}_{zz}}_{(4,6)}\underbrace{-\frac{i}{\omega}(\mathbf{e}^3)_{zz}}_{(4,6)}\underbrace{-2\tilde{\mathbf{e}}_{zz}}_{(5,6)}\right]$$

$$\underbrace{+\frac{\mathbf{B}^2\mathbf{E}_{zz}-(\mathbf{B}\times\hat{\imath}_z)\cdot\mathbf{E}\cdot(\mathbf{B}\times\hat{\imath}_z)}{2N^2+2\sqrt{N^4+\mathbf{B}^2}}}$$

only the (1,6) term is significant.

$$\underbrace{-\frac{1}{4}\mathbf{B}^2}_{(1,1)}\underbrace{+(\tilde{\mathbf{\Omega}}\times B)^2/\omega^2}_{(3,3)}\underbrace{-\frac{i}{\omega}\mathbf{B}\cdot\mathbf{e}\cdot\mathbf{B}}_{(1,6)}\underbrace{-4i\omega\tilde{\mathbf{\Omega}}\cdot\mathbf{e}\cdot\tilde{\mathbf{\Omega}}}_{(2,5)}$$

$$\underbrace{+4\tilde{\mathbf{\Omega}}^2tr(\tilde{\mathbf{e}})}_{(2,4)}\underbrace{-4\tilde{\mathbf{\Omega}}\cdot\tilde{\mathbf{e}}\cdot\tilde{\mathbf{\Omega}}}_{(2,4)}\underbrace{+2\tilde{\mathbf{\Omega}}\cdot\mathbf{e}\cdot\mathbf{B}}_{(3,5)}$$

$$\underbrace{+\frac{2i}{\omega}\tilde{\mathbf{\Omega}}\cdot\mathbf{B}tr(\tilde{\mathbf{e}})}_{(3,4)}\underbrace{-\frac{2i}{\omega}\tilde{\mathbf{\Omega}}\cdot\tilde{\mathbf{e}}\cdot\mathbf{B}}_{(3,4)}\underbrace{-3i\omega|e|}_{(4,5)}\underbrace{+tr(\tilde{e})}_{(4,4)}\underbrace{-\omega^2tr(\tilde{e})}_{(5,5)}$$

$$\underbrace{-\frac{4}{\omega^2}\left[N^4\hat{e}_{zz}+\frac{N^2\mathbf{B}^2\hat{e}_{zz}}{2N^2+2\sqrt{N^4+\mathbf{B}^2}}\right]}_{(6,6)}$$

$$\underbrace{-\frac{4}{\omega^2}\left[-\frac{1}{4}(\mathbf{B}\times\hat{\imath}_z)\cdot\hat{\mathbf{e}}\cdot(\mathbf{B}\times\hat{\imath}_z)+(\mathbf{e}\times\mathbf{B})^2_{zz}\right]}_{(6,6)}, \qquad \text{(C5)}$$

where

$$\mathbf{E}\equiv\underbrace{4\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}}_{(1,2)}\underbrace{+\frac{i}{\omega}\tilde{\mathbf{\Omega}}\mathbf{B}+\frac{i}{\omega}\mathbf{B}\tilde{\mathbf{\Omega}}}_{(1,3)}\underbrace{+\tilde{\mathbf{e}}}_{(1,4)}\underbrace{+i\omega\mathbf{e}}_{(1,5)}$$

$$\underbrace{+\frac{iN^2}{\omega}\mathbf{e}}_{(1,6)}\underbrace{+\frac{4i}{\omega}\tilde{\mathbf{\Omega}}^2\mathbf{e}}_{(2,6)}\underbrace{-\frac{4i}{\omega}\tilde{\mathbf{\Omega}}\times\mathbf{e}\times\tilde{\mathbf{\Omega}}}_{(2,6)}$$

$$\underbrace{+\frac{2\tilde{\mathbf{\Omega}}\cdot\mathbf{B}}{\omega^2}\mathbf{e}}_{(3,6)}\underbrace{-\frac{1}{\omega^2}\tilde{\mathbf{\Omega}}\times\mathbf{e}\times\mathbf{B}}_{(3,6)}\underbrace{-\frac{1}{\omega^2}\mathbf{B}\times\mathbf{e}\times\tilde{\mathbf{\Omega}}}_{(3,6)}$$

$$\underbrace{-\frac{i}{\omega}tr(\tilde{\mathbf{e}})\mathbf{e}}_{(4,6)}\underbrace{-\frac{i}{\omega}tr(\mathbf{e}^2)\mathbf{e}}_{(4,6)}\underbrace{+\frac{i}{\omega}\mathbf{e}^3}_{(4,6)}\underbrace{+2\tilde{\mathbf{e}}}_{(5,6)}\ . \qquad \text{(C6)}$$

Equations (B3), (B4), and (B5) give approximations for the three eigenvalues of S with $tr(S)$ given by (C2), $tr(\tilde{S})$ given by (C5), and we neglect $|S|$.

Appendix D: Supplemental formulas

To simplify the calculations, we choose the z axis to be in a particular direction. Since both $\tilde{\mathbf{g}}$ and $\nabla\rho_{pot}$ point approximately down, we might imagine that choosing the z direction parallel to either $\tilde{\mathbf{g}}$ or $\nabla\rho_{pot}$ would be useful. Choosing the z direction to be in the $-\tilde{\mathbf{g}}$ direction would give

$$\tilde{\mathbf{g}}\cdot\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}\cdot\nabla\rho_{pot}/\rho=N^2\tilde{\Omega}_z^2+(\mathbf{B}\times\tilde{\mathbf{\Omega}})\cdot\tilde{\mathbf{\Omega}}. \qquad \text{(D1)}$$

On the other hand, choosing the z-axis to be in the $-\nabla\rho_{pot}$ direction would give

$$\tilde{\mathbf{g}}\cdot\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}\cdot\nabla\rho_{pot}/\rho=N^2\tilde{\Omega}_z^2-(\mathbf{B}\times\tilde{\mathbf{\Omega}})\cdot\tilde{\mathbf{\Omega}}. \qquad \text{(D2)}$$

That the baroclinic last term has the opposite sign in the two cases suggests that choosing the z axis half way between the two, (that is, in the $-\tilde{\mathbf{g}}/|\tilde{\mathbf{g}}|-\nabla\rho_{pot}/|\nabla\rho_{pot}|$ direction) would make that baroclinic term vanish to first order in $\mathbf{B}$. We can write

$$\frac{\tilde{\mathbf{g}}}{|\tilde{\mathbf{g}}|}=\frac{\tilde{g}_z}{|\tilde{\mathbf{g}}|}\hat{\imath}_z-\frac{(\mathbf{B}\times\hat{\imath}_z)}{(N^4+\mathbf{B}^2)^{1/4}\sqrt{2N^2+2\sqrt{N^4+\mathbf{B}^2}}}, \qquad \text{(D3)}$$

and

$$\frac{\nabla\rho_{pot}}{|\nabla\rho_{pot}|}=\frac{(\nabla\rho_{pot})_z}{|\nabla\rho_{pot}|}\hat{\imath}_z+\frac{(\mathbf{B}\times\hat{\imath}_z)}{(N^4+\mathbf{B}^2)^{1/4}\sqrt{2N^2+2\sqrt{N^4+\mathbf{B}^2}}}, \qquad \text{(D4)}$$

where $\hat{\imath}_z$ is a unit vector in the z direction, and from here on, the subscript z denotes a component parallel to the $-\tilde{\mathbf{g}}/|\tilde{\mathbf{g}}|-\nabla\rho_{pot}/|\nabla\rho_{pot}|$ direction. This gives

$$\tilde{\mathbf{g}}\cdot\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}\cdot\nabla\rho_{pot}/\rho=N^2\tilde{\Omega}_z^2+\frac{\mathbf{B}^2\tilde{\Omega}_z^2-(\mathbf{B}\times\hat{\imath}_z)\cdot\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}\cdot(\mathbf{B}\times\hat{\imath}_z)}{2N^2+2\sqrt{N^4+\mathbf{B}^2}}, \qquad \text{(D5)}$$

$$\tilde{\mathbf{g}} \cdot \mathbf{e} \cdot \nabla \rho_{pot}/\rho = N^2 \mathbf{e}_{zz} + \frac{\mathbf{B}^2 \mathbf{e}_{zz} - (\mathbf{B} \times \hat{\imath}_z) \cdot \mathbf{e} \cdot (\mathbf{B} \times \hat{\imath}_z)}{2N^2 + 2\sqrt{N^4 + \mathbf{B}^2}}, \tag{D6}$$

where (D6) is also valid if any symmetric matrix, such as $\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}$ or $(\tilde{\mathbf{\Omega}}\mathbf{B} + \mathbf{B}\tilde{\mathbf{\Omega}})$ is substituted for $\mathbf{e}$,

$$\underbrace{\tilde{\mathbf{g}} \cdot \mathbf{e} \cdot \tilde{\mathbf{g}}}_{(6,6)} = \frac{|\tilde{\mathbf{g}}|}{|\nabla \rho_{pot}/\rho|} \times \left[N^2 \mathbf{e}_{zz} + \frac{\mathbf{B}^2 \mathbf{e}_{zz} + (\mathbf{B} \times \hat{\imath}_z) \cdot \mathbf{e} \cdot (\mathbf{B} \times \hat{\imath}_z)}{2N^2 + 2\sqrt{N^4 + \mathbf{B}^2}} - (\mathbf{e} \times \mathbf{B})_{zz} \right], \tag{D7}$$

and

$$\underbrace{\nabla \rho_{pot}/\rho \cdot \mathbf{e} \cdot \nabla \rho_{pot}/\rho}_{(6,6)} = \frac{|\nabla \rho_{pot}/\rho|}{|\tilde{\mathbf{g}}|} \times \left[N^2 \mathbf{e}_{zz} + \frac{\mathbf{B}^2 \mathbf{e}_{zz} + (\mathbf{B} \times \hat{\imath}_z) \cdot \mathbf{e} \cdot (\mathbf{B} \times \hat{\imath}_z)}{2N^2 + 2\sqrt{N^4 + \mathbf{B}^2}} + (\mathbf{e} \times \mathbf{B})_{zz} \right]. \tag{D8}$$

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