A general acoustic-gravity wave dispersion relation for the atmosphere and ocean - the linear terms *

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Abstract

Including baroclinicity, vorticity, symmetric rate of strain, and all components of the Earth's angular velocity in the dispersion relation for internal acoustic-gravity waves in the atmosphere or ocean adds terms that are linear in the wavenumber. The significance of these linear terms is investigated.

1 Introduction

Dispersion relations for internal gravity waves in the atmosphere and ocean under the barotropic approximation (where surfaces of constant pressure and density are assumed to coincide) have been known since at least 1960, both with [1] and without [2] coriolis force from the Earth's rotation. The barotropic approximation for the dispersion relation [1, 3] for internal acoustic-gravity waves in a fluid includes acoustic terms as well.

Several authors include some shear components and have shown that some components of baroclinicity as well as vorticity and rate of strain can be significant in some cases [4, 5, 6, 7, 8, 9]. The general dispersion relation [10] for internal acoustic-gravity waves has terms linear in wavenumber $\mathbf{k}$, whose significance we shall investigate.

2 A General dispersion relation

A general dispersion relation [10] for internal acoustic-gravity waves including all components of the Earth's rotation, baroclinicity, vorticity, and rate of strain is

$$\begin{aligned}(\mathbf{k}^2 + \mathbf{k_A}^2)(N^2 - \omega^2) + \mathbf{k} \cdot \mathbf{S} \cdot \mathbf{k} + \mathbf{k_A} \cdot \mathbf{S} \cdot \mathbf{k_A} \\ +\mathbf{A} \cdot \mathbf{k} + (1/C^2)[\omega^4 - 4\omega^2 \tilde{\mathbf{\Omega}}^2 + \mathbf{B}^2/2 - 2i\omega\tilde{\mathbf{\Omega}} \cdot \mathbf{B} \\ +i\mathbf{B} \cdot \mathbf{e} \cdot \mathbf{B}/(2\omega) - \omega^2 tr(\tilde{\mathbf{e}}) \\ -4i\omega\tilde{\mathbf{\Omega}} \cdot \mathbf{e} \cdot \tilde{\mathbf{\Omega}} - i\omega|\mathbf{e}| + 2\tilde{\mathbf{\Omega}} \cdot \mathbf{e} \cdot \mathbf{B}] = 0, \qquad (1)\end{aligned}$$

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where S is the symmetric matrix defined by $S_{\alpha\beta} = (T_{\alpha\beta} + T_{\beta\alpha})/2$,

$$\begin{aligned}T_{\alpha\beta} \equiv -\tilde{g}_\alpha \frac{1}{\rho}\frac{\partial \rho_{pot}}{\partial x_\beta} + 4\tilde{\Omega}_\alpha\tilde{\Omega}_\beta + \frac{2i}{\omega}\tilde{\Omega}_\alpha B_\beta \\ +\tilde{e}_{\alpha\beta} + i\omega e_{\alpha\beta} - \frac{i}{\omega}(\tilde{\mathbf{g}} \times \mathbf{e} \times \nabla\rho_{pot}/\rho)_{\alpha\beta}, \qquad (2)\end{aligned}$$

$$\mathbf{A} = (4\omega\tilde{\mathbf{\Omega}} + i\mathbf{B}) \cdot (\mathbf{1} + i\mathbf{e}/\omega) \times \mathbf{\Gamma} + 2\mathbf{k}_A \cdot \tilde{\mathbf{\Omega}}\mathbf{B}/\omega, \quad (3)$$

$\omega = \sigma - \mathbf{k} \cdot \mathbf{U}$ is the intrinsic frequency, σ is the wave frequency, $\mathbf{U}$ is the background fluid velocity, $\mathbf{k}$ is the wavenumber, $\tilde{\mathbf{g}} \equiv \nabla p/\rho = \mathbf{g} - D\mathbf{U}/Dt - 2\mathbf{\Omega} \times \mathbf{U}$ is the effective vector acceleration due to gravity [including (minus) the acceleration of the background flow], p is background pressure, ρ is background density, ρ_{pot} is background potential density [11, Section 3.7.5, pp. 54-55], $\tilde{\mathbf{\Omega}} \equiv \mathbf{\Omega} + \zeta/4$, $\zeta \equiv \nabla \times \mathbf{U}$ is the vorticity, $\mathbf{\Omega}$ is the Earth's angular velocity, C is sound speed, $\mathbf{k}_A \equiv \nabla\rho/(2\rho)$,

$$N^2 = \nabla\rho_{pot} \cdot \nabla p/\rho^2 = \nabla\rho_{pot} \cdot \tilde{\mathbf{g}}/\rho \qquad (4)$$

is the square of the Brunt-Väisälä frequency [12],

$$\mathbf{B} \equiv \frac{(\nabla\rho \times \nabla p)}{\rho^2} = \frac{(\nabla\rho_{pot} \times \nabla p)}{\rho^2} = (\frac{\nabla\rho_{pot}}{\rho} \times \tilde{\mathbf{g}}) \qquad (5)$$

is the baroclinic vector [11, Section 7.11, pp. 237-238], $\mathbf{e}$ is the matrix representing the symmetric rate-of-strain tensor [13, p. 89], [14, p. 228], [15] $e_{ij} \equiv (\partial U_i/\partial x_j + \partial U_j/\partial x_i)/2$, $\tilde{e}_{ij} \equiv e_{\alpha\beta}e_{\gamma\delta}\varepsilon_{i\alpha\gamma}\varepsilon_{j\beta\delta}/2$ is the matrix of co-factors of $\mathbf{e}$, ε_{ijk} is the completely antisymmetric tensor with $\varepsilon_{123} = 1$ and $\varepsilon_{ijk} + \varepsilon_{jik} = \varepsilon_{ijk} + \varepsilon_{ikj} = 0$, and summation from 1 to 3 is implied over repeated indices. Notice that the symmetric rate-of-strain tensor is the symmetric part of the shear tensor, while the vorticity gives the antisymmetric part of the shear tensor. In (2) and (3), a cross product of a vector times a matrix is the matrix whose columns are the cross products of the vector times the corresponding columns of the original matrix. A cross product of a matrix times a vector is the matrix whose rows are the cross products of the corresponding rows of the original matrix times the vector.

A vector identity [10, equation (4)] was used to transform the dispersion relation [10] to that in (1). The particular combination of ζ and $\mathbf{\Omega}$ that combine to form $\tilde{\mathbf{\Omega}}$ agrees with other work [6], and is a result of the linearization process.

A dispersion relation depends on the differential equations from which the dispersion relation was derived. Both the dispersion relation and the set of differential equations are needed to calculate a WKB approximation. The set of differential equations from which the dispersion relation (1) was derived are available at [10, eq. (6) or Appendix A].

3 Neglecting baroclinicity and rate of strain

To see the effect of the linear terms, we must use the diagonal form for the dispersion relation. Neglecting baroclinicity and rate of strain, and assuming that the Brunt-Väisälä frequency N dominates over vorticity and the Earth's rotation, we have approximately, to order N^{-2}, [16]

$$\begin{aligned}1/C^2[\omega^4 - 4\omega^2\tilde{\mathbf{\Omega}}^2] + \mathbf{A}\cdot\mathbf{k} \\ +(k_1^2 + k_{A1}^2)\left(-\omega^2 + N^2\right) \\ +(k_2^2 + k_{A2}^2)\left[-\omega^2 + N^2 + 4\tilde{\mathbf{\Omega}}_\perp^2 + \tfrac{16}{N^2}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2\right] \\ +(k_3^2 + k_{A3}^2)\left[-\omega^2 + 4\tilde{\Omega}_z^2 - \tfrac{16}{N^2}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2\right] = 0,\end{aligned} \quad (6)$$

where the subscript z denotes a component parallel to both the $-\tilde{\mathbf{g}}$ and the $-\nabla\rho_{pot}$ directions in this barotropic example, the subscript $\perp$ denotes components normal to the z direction, and k_1, k_2, and k_3 are the components of the wavenumber in the rotated coordinate system.

Converting to the rotated coordinate system is straightforward, but tedious. A matrix T is used to make the transformation as follows: $\mathbf{k}_{\text{rotated}} = T\mathbf{k}$ and $\Lambda T = TS$ can be used to solve for T, where Λ is the diagonal matrix of eigenvalues of S. The wavenumbers in the rotated coordinate system are

$$k_1 = \mathbf{k}_\perp \times \tilde{\mathbf{\Omega}}_\perp / |\tilde{\mathbf{\Omega}}_\perp|, \quad (7)$$

$$k_2 = \frac{\mathbf{k}\cdot\tilde{\mathbf{\Omega}}_\perp + k_z\tilde{\Omega}_z\frac{\lambda_2}{N^2+\lambda_2}}{\sqrt{\tilde{\mathbf{\Omega}}_\perp^2 + \tilde{\Omega}_z^2\left(\frac{\lambda_2}{N^2+\lambda_2}\right)^2}}, \quad (8)$$

and

$$k_3 = \frac{k_z\tilde{\Omega}_z + \mathbf{k}\cdot\tilde{\mathbf{\Omega}}_\perp\frac{N^2+\lambda_3}{\lambda_3}}{\sqrt{\tilde{\Omega}_z^2 + \tilde{\mathbf{\Omega}}_\perp^2\left(\frac{N^2+\lambda_3}{\lambda_3}\right)^2}}, \quad (9)$$

where λ_2 and λ_3 are the two nonzero eigenvalues of the symmetric matrix S. When we neglect baroclinicity and rate of strain, these are

$$\lambda_2 = 4\tilde{\mathbf{\Omega}}_\perp^2 + 16\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2/N^2 \quad (10)$$

and

$$\lambda_3 = -N^2 + 4\tilde{\mathbf{\Omega}}_z^2 - 16\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2/N^2. \quad (11)$$

Neglecting baroclinicity and rate of strain, the linear terms are given by $\mathbf{A}\cdot\mathbf{k}$, where, from (3),

$$\mathbf{A} = 4\omega\tilde{\mathbf{\Omega}} \times (\mathbf{k}_A - \tilde{\mathbf{g}}/C^2). \quad (12)$$

The transformation for all vectors in the rotated coordinate system is the same as that for $\mathbf{k}$ in (7), (8), and (9). Assuming that the Brunt-Väisälä frequency N dominates over vorticity and the Earth's rotation, we have approximately, to order N^{-2}, for $\tilde{\mathbf{\Omega}}$,

$$\tilde{\Omega}_2 \approx \tilde{\mathbf{\Omega}}_\perp(1 + 4\tilde{\mathbf{\Omega}}_z^2/N^2), \quad (13)$$

$$\tilde{\Omega}_3 \approx \tilde{\mathbf{\Omega}}_z(1 - 4\tilde{\mathbf{\Omega}}_\perp^2/N^2), \quad (14)$$

and $\tilde{\Omega}_1$ is zero. We also have, to order N^{-2},

$$k_{A2} \approx 4\tilde{\mathbf{\Omega}}_\perp\tilde{\mathbf{\Omega}}_z k_{Az}/N^2, \quad (15)$$

$$k_{A3} \approx k_{Az}, \quad (16)$$

and k_{A1} is zero. Similarly, to order N^{-2},

$$\tilde{g}_2 \approx 4\tilde{\mathbf{\Omega}}_\perp\tilde{\mathbf{\Omega}}_z\tilde{g}_z/N^2, \quad (17)$$

$$\tilde{g}_3 \approx \tilde{g}_z, \quad (18)$$

and $\tilde{g}_1$ is zero. Then, the only nonzero component of $\mathbf{A}$ in the rotated coordinate system is

$$A_1 \approx 4\omega\tilde{\mathbf{\Omega}}_\perp(k_{Az} - \tilde{g}_z/C^2). \quad (19)$$

So, finally, the dispersion relation is approximately

$$\begin{aligned}1/C^2[\omega^4 - 4\omega^2\tilde{\mathbf{\Omega}}^2]\underbrace{-4\omega^2\tilde{\mathbf{\Omega}}_\perp^2(k_{Az} - \tilde{g}_z/C^2)^2/(N^2-\omega^2)}_{\text{linear term}} \\ +[k_1\underbrace{+2\omega\tilde{\mathbf{\Omega}}_\perp(k_{Az} - \tilde{g}_z/C^2)/(N^2-\omega^2)}_{\text{linear term}}]^2\left(-\omega^2 + N^2\right) \\ +(k_2^2 + k_{A2}^2)\left[-\omega^2 + N^2 + 4\tilde{\mathbf{\Omega}}_\perp^2 + \tfrac{16}{N^2}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2\right] \\ +(k_3^2 + k_{A3}^2)\left[-\omega^2 + 4\tilde{\Omega}_z^2 - \tfrac{16}{N^2}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2\right] = 0.\end{aligned} \quad (20)$$

The effect of the linear term is to add a frequency-dependent term to the dispersion and to displace the origin of the k_1^2 term. It appears that the effect is small unless the frequency is close to the Brunt-Väisälä frequency.

4 Including baroclinicity

Neglecting rate of strain, but including baroclinicity and assuming that the Brunt-Väisälä frequency N dominates over vorticity and the Earth's rotation, we have approximately, to order N^{-2}, [16]

$$\begin{aligned}
&1/C^2[\omega^4 - 4\omega^2\tilde{\mathbf{\Omega}}^2 + \mathbf{B}^2/2 - 2i\omega\tilde{\mathbf{\Omega}}\cdot\mathbf{B}] + \mathbf{A}\cdot\mathbf{k} \\
&\quad + (k_1^2 + k_{A1}^2)\left(-\omega^2 + N^2\right) \\
&\quad + (k_2^2 + k_{A2}^2)\Big[-\omega^2 + N^2 + 4\tilde{\mathbf{\Omega}}_\perp^2 + \tfrac{2i}{\omega}\tilde{\mathbf{\Omega}}\cdot\mathbf{B} + \tfrac{\mathbf{B}^2}{4N^2} \\
&\qquad - \tfrac{\tilde{\Omega}^2\mathbf{B}^2 - (\tilde{\mathbf{\Omega}}\cdot\mathbf{B})^2}{N^2\omega^2} + \tfrac{8i}{N^2\omega}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}\cdot\mathbf{B} + \tfrac{16}{N^2}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2\Big] \\
&\quad + (k_3^2 + k_{A3}^2)\Big[-\omega^2 + 4\tilde{\Omega}_z^2 - \tfrac{\mathbf{B}^2}{4N^2} + \tfrac{\tilde{\mathbf{\Omega}}^2\mathbf{B}^2 - (\tilde{\mathbf{\Omega}}\cdot\mathbf{B})^2}{N^2\omega^2} \\
&\qquad - \tfrac{8i}{N^2\omega}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}\cdot\mathbf{B} - \tfrac{16}{N^2}\tilde{\Omega}_z^2\tilde{\Omega}_\perp^2\Big] = 0,
\end{aligned} \tag{21}$$

where the subscript z denotes a component parallel to the $-\tilde{\mathbf{g}}/|\tilde{\mathbf{g}}| - \nabla\rho_{pot}/|\nabla\rho_{pot}|$ direction, the subscript $\perp$ denotes components normal to the z direction, and k_1, k_2, and k_3 are the components of the wavenumber in the rotated coordinate system.

Solving $\Lambda T_r = T_r S$ for T_r as the transformation matrix to rotate the coordinate system (where $k_{\text{new}} = T_r k_{\text{old}}$), gives approximately to order N^{-2}

$$k_1 = \mathbf{k}_\perp \times \tilde{\mathbf{\Omega}}_\perp/|\tilde{\mathbf{\Omega}}_\perp| + N^{-2} \text{ terms} , \tag{22}$$

the same as when we neglected vorticity, at least to this approximation.

The nonzero eigenvalues are given approximately, to order N^{-2} by [16]

$$\begin{aligned}
\lambda_2 = {} & 4\tilde{\mathbf{\Omega}}_\perp^2 + \tfrac{2i}{\omega}\tilde{\mathbf{\Omega}}\cdot\mathbf{B} + \tfrac{\mathbf{B}^2}{4N^2} - \tfrac{\tilde{\Omega}^2\mathbf{B}^2 - (\tilde{\mathbf{\Omega}}\cdot\mathbf{B})^2}{N^2\omega^2} \\
& + \tfrac{8i}{N^2\omega}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}\cdot\mathbf{B} + 16\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2/N^2
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\lambda_3 = {} & -N^2 + 4\tilde{\mathbf{\Omega}}_z^2 - \tfrac{\mathbf{B}^2}{4N^2} + \tfrac{\tilde{\mathbf{\Omega}}^2\mathbf{B}^2 - (\tilde{\mathbf{\Omega}}\cdot\mathbf{B})^2}{N^2\omega^2} \\
& - \tfrac{8i}{N^2\omega}\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}\cdot\mathbf{B} - 16\tilde{\Omega}_z^2\tilde{\mathbf{\Omega}}_\perp^2/N^2.
\end{aligned} \tag{24}$$

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