

# Reply

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**Abstract:** *Canuto and Dubovikov raise a number of objections to the Fox-Kemper, Ferrari, and Hallberg (FFH) parameterization of restratification by submesoscale eddies, and they claim that their own new parameterization is superior in a number of regards and better explains some simulation data used by FFH. A more careful analysis of the behavior of the FFH parameterization in certain limits and a dedicated reanalysis of the simulation data used to test the FFH parameterization reveals that these claims are unfounded.*

Canuto and Dubovikov (2009b) (CD) suggest that the Fox-Kemper et al. (2008b) (FFH) model for submesoscale restratification of the mixed layer (ML) is limited to a Richardson number (Ri) regime  $5 < \text{Ri} < 10^3$ . To do so, they select simulation data from the figures in FFH and replot it with their own submesoscale model (CD model) from another manuscript (Canuto and Dubovikov 2009a).

CD make a number of claims in their paper:

1. The FFH model cannot be used for large Richardson number due to either geometric covariance arguments or expansion in powers of  $f^{-1}$ .
2. The CD model is superior to the FFH model based on the data in Figure 14e of FFH in the regime  $\text{Ri} < 5$ .
3. The CD model is capable of predicting the ratio of Eddy Kinetic Energy  $K$  to mean flow kinetic energy  $K_M$  at the mixed layer base as a function of Richardson number alone.
4. The CD model correctly includes the effects of wind on stratification and destratifica-

tion at the submesoscale.

Here we show that these claims are unfounded.

In Fox-Kemper and Ferrari (2008), the FFH model is shown to represent the submesoscale quantities (denoted by primes) from those quantities in a coarser resolution model (coarse variables are denoted by double overbars). The buoyancy equation for an evolving buoyancy front in mixed layer of the coarse model becomes,

$$\bar{\bar{b}}_t + \nabla \cdot \left[ \bar{\bar{\mathbf{u}b}} + \overline{\mathbf{u}'b'} \right] = -\nabla \cdot \mathcal{F}. \quad (1)$$

Where  $b = -g\rho/\rho_0$  is the buoyancy, and  $\mathbf{u}$  is the three dimensional velocity. FFH argue that the bulk of the MLE eddy restratification  $\overline{\mathbf{u}'b'}$  may be represented using an eddy-induced streamfunction,

$$\overline{\mathbf{u}'b'} \approx \Psi \times \nabla \bar{\bar{b}}, \quad (2)$$

$$\Psi = C_e \frac{H^2 \nabla \bar{\bar{b}}^z \times \hat{\mathbf{z}}}{|f|} \mu(z), \quad (3)$$

$$\mu(z) = \left[ 1 - \left( \frac{2z}{H} + 1 \right)^2 \right] \left[ 1 + \frac{5}{21} \left( \frac{2z}{H} + 1 \right)^2 \right].$$

Where  $f$  is Coriolis parameter, and  $H$  and  $\nabla \bar{\bar{b}}^z$  are the ML depth and buoyancy gradient averaged vertically over the ML in the coarse GCM. The efficiency factor used here is  $C_e = 0.06$ , which is the best fit for simulations without diurnal convection.

Fox-Kemper et al. (2008a) regularize the FFH parameterization to converge near the equator to the same limit as the Young (1994) and Ferrari and Young (1997) frictional

scaling:

$$\Psi = C_e \frac{H^2 \nabla \bar{b} \times \hat{z}}{\sqrt{f^2 + \tau^{-2}}} \mu(z). \quad (4)$$

The finite frictional timescale,  $\tau$ , is proportional to the timescale for momentum to mix (by viscosity or convective or shear turbulence) across the mixed layer depth. The appearance of friction is consistent with the result noted in both Boccaletti et al. (2007) and Fox-Kemper et al. (2008b) that subgrid-scale momentum mixing has an minor but detectable impact on restratification rate over the range of  $f$  and Ekman number simulated.

## 1. Geometric Covariance and Powers of $f$

CD argue for the abstract principles of geometric covariance and analyticity in any submesoscale parameterization. In practice, neglecting these principles would lead to unacceptable physical consequences in a global climate model. The FFH parameterization has been used in a growing number of global climate models for a number of years (Fox-Kemper et al. 2008a), so clearly these principles are obeyed. For example, geometric covariance ensures that the eddy parameterization restratifies in both the Northern and Southern hemisphere. The FFH model, (3) and (4), is geometrically covariant. The original form of the FFH parameterization, (3), was formulated studying simulations where  $f \neq 0$ , and is singular and thus not analytic at the equator ( $f = 0$ ). The global model form, (4), is well-behaved at the equator and analytic over all real  $f$ . Appendix A presents a mathematical treatment of

these issues.

The form advocated by Fox-Kemper et al. (2008a), (4), extends the physical principle underlying the FFH parameterization: overturning mixed layer fronts by ageostrophic or eddy-induced effects. The streamfunction in (2) parameterizes the eddy fluxes that overturn the front. As one approaches the equator, frictional and large Rossby number slumping of fronts overtake the geostrophic constraints of Rossby adjustment and indeed the eddy-induced slumping. The rate of slumping of a frictionally-constrained and rotationally-constrained front in the absence of eddies is treated by Young (1994) and Ferrari and Young (1997). The frictional slumping rate in these works is valid near the equator, but invalid in the extratropics as submesoscale eddy effects are neglected (see Boccaletti et al. 2007, for a discussion).

CD imply in section IV that the CD parameterization is special in its ability to possess both geometric covariance and analyticity. Apparently pure deduction is not sufficient to select a unique form, as a wide class of functions satisfies these constraints (examples in Appendix A). However, numerical experimentation can select a unique form, and the next section demonstrates FFH outperforms CD in this regard.

## 2. Comparison to Simulation Data

FFH, Boccaletti et al. (2007), and Haine and Marshall (1998) all find that the baroclinic instability branch dominates the instabilities only for Richardson number greater than one. For  $Ri < 1$ , symmetric instabilities dominate and for  $Ri < 1/4$ , shear instabilities dominate.

These unbalanced instabilities are small and marginally resolved at best in many simulations. Furthermore, Tandon and Garrett (1994) demonstrate that an initially unbalanced mixed layer (e.g., one that results from an impulsive mixing event) geostrophically adjusts and then oscillates about  $Ri = 1$ . For these reasons, Fox-Kemper et al. (2008b) did not focus extensively on the behavior of the baroclinic branch for  $Ri < 1$ , nor is this regime of interest here.

A surprising result of FFH is that the eddy scaling of linear theory (e.g., Stone 1972a,b) differs from the scaling at finite amplitude. During early instability growth, FFH and Boccaletti et al. (2007) found excellent agreement with the linear instability theory of Stone (1972a). When eddy kinetic energy  $K$  rivals the mean kinetic energy  $K_M$ , however, Stone's scaling fails as nonlinear saturation and an inverse cascade take over. Only the finite amplitude scenario is addressed by the FFH theory, because no significant restratification occurs in the linear stage of instability.

CD state that their model appears superior to the FFH parameterization based on the fact that a few data points appear low in Figure 14e of FFH and claim that their model explains better the small  $Ri$ , small  $K/K_M$  (i.e., linear regime) behavior. Once again, the linear regime was not the focus of FFH because restratification is negligible during this phase—eddy effects are small during the linear stages and do not modify the stratification at leading order. However in response to the criticism raised by Canuto and Dubovikov, the analysis of the FFH simulation data was extended to test quantitatively the scaling over a wider range of  $Ri$  including the linear regimes. Notice that this required analysis of

unpublished data, because contrary to the statements of Canuto and Dubovikov (2009b), the data points presented in FFH were not sufficient to discuss the low Ri behavior. So, a reanalysis of the FFH simulation data was performed to test quantitatively whether these claims are true generally.

Recall that Figure 14e of FFH plots the time-mean Richardson number over the complete simulation (which is steadily increasing with time) versus the scaled magnitude of the eddy-induced streamfunction (which is noisy, but not increasing with time). In this figure, the time-mean over each simulation was used to reduce the noise, and since little or no dependence on Ri was noted it was inconsequential which Ri was used. FFH chose the time-mean of Ri for the figure and independence from the initial value of Ri is also discussed.

To test the Richardson number close to  $Ri = 1$  as Canuto and Dubovikov (2009b) insist, the time averaging must be made over shorter windows—here two times the fastest growing mode timescale is used as an averaging window,  $2\tau_s$  (see equation (3) of Fox-Kemper et al. 2008b). Thus, during the linear stage  $K$  will grow by an order of magnitude over each time window. Furthermore, only simulations that are geostrophically balanced are useful, as Rossby adjustment yields  $Ri = 1$  immediately.<sup>1</sup> FFH chose to neglect times when  $K/K_M < 0.1$  to eliminate the linear stage of evolving instabilities. To fully refute the Canuto and Dubovikov (2009b) scaling, all times where  $K/K_M > 0.01$  are now used, ensuring that any bias is in favor of the CD parameterization over FFH. In sum, a subset of the 241 runs Fox-Kemper et al. (2008b) are averaged over windows where mixed layer depth and frontal strength were unambiguous, resulting in 603 time windows without overlapping from 68

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<sup>1</sup>The runs with diurnal mixing are too noisy for this purpose so will not be mentioned further.

simulations used.

The results from these time-windows are binned by Richardson number from 1 to 9000 and shown in Fig. 1. The data points are plotted at the mean Richardson number for each bin, and the errorbars shown are  $\pm\sigma/\sqrt{N}$ , where  $\sigma$  is the standard deviation over all time-mean results in the bin and  $N$  is the number of distinct simulations contributing to each bin.<sup>2</sup> It is clear from Fig. 1 that the FFH parameterization is close to the simulation results throughout the whole range of Richardson number, while the CD parameterization agrees only in a range from about  $1 < Ri < 10$ ; not coincidentally the CD and FFH parameterizations differ by less than 65% over this range. Consistent with Figure 14e of FFH, the Green (1970) and Stone (1972b) scalings do not agree with data.

[Figure 1 about here.]

From the dataset of windowed time-mean values, a student t-test provides a statistical assessment for rejection of a theory over a particular range of Ri. The null hypothesis  $\overline{\overline{w'b'}} = \overline{\overline{w'b'}}_{\text{theory}}$  may be rejected if the deviations between data and theory are statistically significant given the scatter of the data. The results of this analysis are given in Table 1. Apparently, CD are correct that their theory cannot be rejected by the simulation data in the range  $Ri < 5$ , but CD are incorrect when they claim, ‘However, since  $\overline{\overline{w'b'}}_{\text{FFH}}$  does not depend on Ri, we conclude that (1a) [FFH] cannot be valid in that regime and is therefore incomplete’. In fact, the more careful test here of the simulation data reveals too much noise

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<sup>2</sup>Each averaging window is correlated with the preceding and following windows in the same simulation, thus they are not statistically independent. Different simulations are independent, however.

to reject the FFH parameterization with standard statistical confidence limits. Only the FFH parameterization cannot be rejected over all Ri ranges considered.<sup>3</sup>

[Table 1 about here.]

### 3. Ratio of Eddy to Mean Kinetic Energy

According to CD, their parameterization predicts the ratio  $K/K_M$  of eddy to mean kinetic energy as a function of Ri.

$$\frac{K}{K_M} \approx X(\text{Ri}) = \frac{5\text{Ri} - \sqrt{\text{Ri}^2 + 15\text{Ri}}}{4(\text{Ri} + \sqrt{\text{Ri}^2 + 15\text{Ri}})} \quad (5)$$

Fig. 2 of CD shows agreement with a single simulation discussed in FFH covering the range  $0 < \text{Ri} < 30$ . Since this scaling is crucial to the other claims made, it is compared against snapshots from the 68 simulations described above (Fig. 2). In the range  $0 < \text{Ri} < 30$ , some of the simulations are near  $K/K_M/X(\text{Ri}) = 1$ , consistent with Fig. 2 of CD. However, their function  $X(\text{Ri})$  is clearly incompatible over the full range of Ri and differing initial conditions of Ri. Just among the runs starting with no initial stratification ( $\text{Ri} = 0$ ),  $X(\text{Ri})$  is about a factor of 5 too small in the  $0.5 < \text{Ri} < 10$  range and a factor of ten too large for  $\text{Ri} > 30$ .  $X(\text{Ri})$  misses the behavior in simulations with initial stratification by more than three orders of magnitude.

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<sup>3</sup>Interestingly, the Green (1970) theory cannot be rejected over all Ri. However, the Green coefficient was tuned to fit a similar dataset in FFH, so the overestimate at large Ri and the underestimate at small Ri scatter evenly about the data (undetectable with a t-test).

[Figure 2 about here.]

## 4. Wind

There is insufficient detail given in CD to assess the handling of wind effects at the submesoscale in Canuto and Dubovikov (2009a). It would be interesting to know how the nonlinear Ekman effects (Thomas and Rhines 2002), restratification and destratification by up- and down-front winds (Thomas and Ferrari 2008), and intrathermocline anomalous PV (Thomas 2008) are handled by the CD parameterization.

## 5. Conclusions

The authors readily agree with Canuto and Dubovikov (2009b) that the FFH parameterization is limited and potentially only relates to the simulations studied in Fox-Kemper et al. (2008b)—a tiny subset of the possible scenarios relevant to submesoscale eddy-frontal interactions. The neglect of wind is problematic as noted in FFH, but recent analyses and simulations have shown that the FFH scaling proves robust in more general settings including wind (e.g., Capet et al. 2008; Mahadevan et al. 2008). However, Canuto and Dubovikov (2009b) go much farther than noting the lack of wind effects in their critique of the FFH parameterization. They claim that their Canuto and Dubovikov (2009a) parameterization is superior in a number of respects. This note demonstrates that none of the arguments or analysis given in Canuto and Dubovikov (2009b) are substantive. While the FFH parame-

terization is not perfect it is superior to the CD parameterization in explaining the results of simulation data from Fox-Kemper et al. (2008b).

There is insufficient detail in Canuto and Dubovikov (2009b) to judge the parameterization of Canuto and Dubovikov (2009a) overall. However, one hopes that the submesoscale parameterization is quite different from the mesoscale parameterization of Canuto and Dubovikov (2005), because the results here in Figs. 1, 2, and the t-test indicate that the CD scalings—which converge for large Ri to the Stone (1972b) scaling as explained in FFH—may be rejected statistically and are an order of magnitude too small in the large Ri limit relevant for the mesoscale. In contrast, the FFH scaling applies over all relevant submesoscale Ri, and indeed over all  $Ri > 1$  studied.

## A Mathematics of Geometric Covariance and Analyticity

CD suggest that any submesoscale parameterization should obey geometric covariance and be analytic so that it is expressible as a series of powers of  $f^{-1}$  as  $f^{-1}$  approaches 0, as geostrophic relationships can be solved by series expansion in powers of  $f^{-1}$ .

The principle of geometric covariance is satisfied by the FFH parameterization, as it provides a pseudovector eddy-induced streamfunction which may be used to derive vector eddy-induced velocities  $\mathbf{v}^\dagger = \nabla \times \Psi$  and eddy fluxes of buoyancy, (2). It is clear then that the streamfunction must be a pseudovector, and indeed the FFH definition provides

a pseudovector (3), as all quantities on the right hand side are either a scalar or a vector and the one cross-product results in a pseudovector. Thus, the FFH parameterization is geometrically covariant.

CD argue that one might have to remove the absolute value from  $|f|$  to eliminate a singularity and thereby violate geometric invariance, and we agree. One obvious consequence: the parameterization for submesoscale eddy *restratification* would only *restratify* in the Northern Hemisphere—in the Southern hemisphere it would *destratify*. The FFH parameterization is used for global modeling, so this change is unacceptable.

CD suggest removing the absolute value to eliminate the singularity inherent in the square root operator,  $|f| = \sqrt{f^2}$ . The singularity becomes important as  $f$  approaches zero. However, difficulties as  $f$  approaches zero are readily apparent in both the FFH and CD models and were mentioned specifically in Fox-Kemper and Ferrari (2008). Both the FFH parameterization in (3) and the CD parameterization predict infinite vertical fluxes at the equator (see equation (4b) of CD). Since all the simulations in FFH have  $f \neq 0$ , the  $f = 0$  limit is not discussed there.

However, the FFH parameterization has been used in a growing number of global climate models for a number of years (Fox-Kemper et al. 2008a). To do so, the equator is usually treated as in (4). The regularized FFH parameterization 1) preserves the geometric covariance of (3), 2) regularizes the square root singularity so that the FFH parameterization is analytic and complex differentiable over all real  $f$ , 3) converges to a physical limit at the equator, and 4) allows for a series expansion of the FFH parameterization in powers of  $f$  near

zero (equatorial limit) and  $f^{-1}$  near zero (geostrophic limit). Respectively, the equatorial and geostrophic expansions are

$$\begin{aligned} \Psi &= C_e H^2 \tau \nabla \bar{b}^z \times \hat{\mathbf{z}} \mu(z) \cdot \\ &\quad \left[ 1 - \frac{1}{2} f^2 \tau^2 + \frac{3}{8} f^4 \tau^4 - O(f^6 \tau^6) \right], \\ &= \frac{C_e H^2 \nabla \bar{b}^z \times \hat{\mathbf{z}} \mu(z)}{|f|} \cdot \\ &\quad \left[ 1 - \frac{1}{2 f^2 \tau^2} + \frac{3}{8 f^4 \tau^4} - O\left(\frac{1}{f^6 \tau^6}\right) \right]. \end{aligned}$$

CD imply in section IV that the CD parameterization is special in its ability to possess both geometric covariance and expandability in powers of  $f^{-1}$ . However, regularization makes a large class of suitable functions available. For example,

$$\begin{aligned} (f^2 + \tau^{-2})^{n/2} &= |f|^n \cdot \\ &\quad \left[ 1 - \frac{n}{2 f^2 \tau^2} + \frac{n^2 - 2n}{8 f^4 \tau^4} - O\left(\frac{1}{f^6 \tau^6}\right) \right] \end{aligned}$$

for any integer  $n$  and for arbitrarily small but finite  $\tau$ , which potentially includes both the CD and FFH cases.

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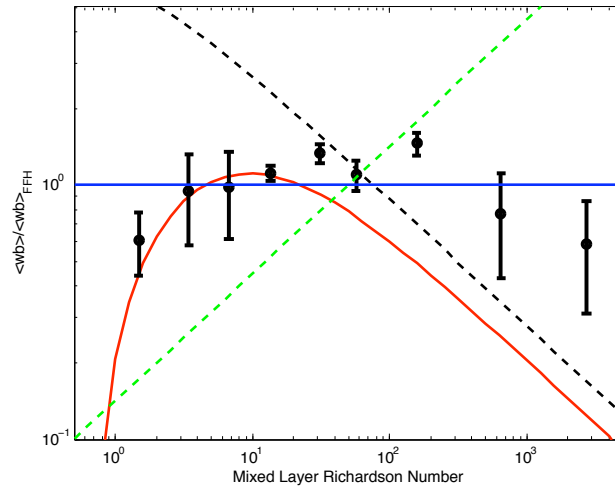


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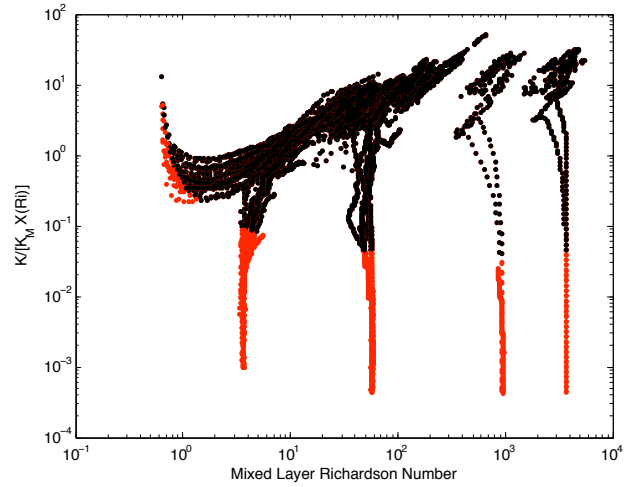


Figure 2: Mean of  $K/K_M/X(Ri)$  at every 4hours where mixed layer diagnosis was unambiguous and  $K/K_M > 0.05$ . Red color indicates the linear instability phase of the simulations where  $K/K_M < 0.1$ .

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Table 1: Results of student t-test for various theories with  $\overline{w'b'} = \overline{w'b'}_{\text{theory}}$  as the null hypothesis. The results are the same for 0.05 and 0.01 significance levels and with 2 or 4  $\tau_s$  time-averaging windows.

Theory	All Ri	Ri < 5	Ri < 100	Ri > 100
CD	Reject	Can't Reject	Reject	Reject
FFH	Can't Reject	Can't Reject	Can't Reject	Can't Reject
Stone	Reject	Reject	Reject	Reject
Green	Can't Reject	Reject	Reject	Reject