

# ***PRESSURE GRADIENT FORCE OF WRF***

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# Hydrostatically-balanced Perturbation Form of the Governing Equations

$$\begin{aligned} \frac{\partial U}{\partial t} + m \left[ \frac{\partial}{\partial x} (Uu) + \frac{\partial}{\partial y} (Vu) \right] + \frac{\partial}{\partial \eta} (\Omega u) + (\mu_d \alpha \frac{\partial p'}{\partial x} + \mu_d \alpha' \frac{\partial \bar{p}}{\partial x}) \\ + (\alpha / \alpha_d) (\mu_d \frac{\partial \phi'}{\partial x} + \frac{\partial p'}{\partial \eta} \frac{\partial \phi}{\partial x} - \mu_d' \frac{\partial \phi}{\partial x}) = F_U \end{aligned} \quad (2.35)$$

$$\frac{\partial \mu_d'}{\partial t} + m^2 \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + m \frac{\partial \Omega}{\partial \eta} = 0 \quad \rightarrow \quad \text{non-hydrostatic terms} \quad (2.38)$$

$$\frac{\partial \phi'}{\partial t} + \frac{1}{\mu_d} \left[ m^2 \left( U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial y} \right) + m \Omega \frac{\partial \phi}{\partial \eta} - gW \right] = 0 \quad (2.39)$$

$$\frac{\partial \Theta}{\partial t} + m^2 \left[ \frac{\partial}{\partial x} (U\theta) + \frac{\partial}{\partial y} (V\theta) \right] + m \frac{\partial}{\partial \eta} (\Omega \theta) = F_{\Theta} \quad (2.40)$$

$$p = p_0 (R_d \theta_m / p_0 \alpha_d)^\gamma \quad (2.31)$$

# Assumption

- Hydrostatic (i.e., reference state)
- No map projection factor for simplicity
- Only x direction
- No external forcing (i.e., no physics or parameterization)
- Dry atmosphere
- Reference state horizontally homogeneous

## Simplified Governing Equations

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(Uu) + \bar{\mu}\alpha \frac{\partial p'}{\partial x} + \bar{\mu} \frac{\partial \phi'}{\partial x} = 0 \quad (1)$$

$$\frac{\partial \phi'}{\partial t} + \frac{U}{\bar{\mu}} \frac{\partial \phi'}{\partial x} = 0 \quad (2)$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial x}(U\theta) = 0 \quad (3)$$

$$p = p_0 (R_d \theta / p_0 \alpha)^\gamma \quad (4)$$

## A. Solve for geopotential height

To linearize Equation (2), assume  $U = [U] + U''$  and  $[U] \gg U''$

$$\frac{\partial \phi'}{\partial t} + \frac{[U]}{\mu_d} \frac{\partial \phi'}{\partial x} = 0$$

Rewrite the equation in discrete form of 2<sup>nd</sup> order in space and Runge-Kutta 3<sup>rd</sup> in time

$$\phi_n'^* = \phi_n'^\tau + \frac{[U]\Delta t}{6\bar{\mu}\Delta x} (\phi_{n-1}'^\tau - \phi_{n+1}'^\tau)$$

$$\phi_n'^{**} = \phi_n'^\tau + \frac{[U]\Delta t}{4\bar{\mu}\Delta x} (\phi_{n-1}'^* - \phi_{n+1}'^*)$$

$$\phi_n'^{\tau+1} = \phi_n'^\tau + \frac{[U]\Delta t}{2\bar{\mu}\Delta x} (\phi_{n-1}'^{**} - \phi_{n+1}'^{**})$$

$$\begin{aligned}
\phi_n^{\prime\tau+1} &= \phi_n^{\prime\tau} + \frac{[U]\Delta t}{2\bar{\mu}\Delta x} (\phi_{n-1}^{\prime\tau} - \phi_{n+1}^{\prime\tau}) \\
&+ \frac{[U]^2(\Delta t)^2}{8\bar{\mu}^2(\Delta x)^2} (\phi_{n-2}^{\prime\tau} - 2\phi_n^{\prime\tau} + \phi_{n+2}^{\prime\tau}) \\
&+ \frac{[U]^3(\Delta t)^3}{48\bar{\mu}^3(\Delta x)^3} (\phi_{n-3}^{\prime\tau} - 3\phi_{n-1}^{\prime\tau} + 3\phi_{n+1}^{\prime\tau} - \phi_{n+3}^{\prime\tau})
\end{aligned}$$

Let  $\phi_n^{\prime\tau} = \hat{\phi} \exp[i(kn\Delta x + \omega\tau\Delta t)]$

$$\begin{aligned}
\psi^1 &= 1 + \frac{[U]\Delta t}{2\bar{\mu}\Delta x} (\psi_{-1} - \psi_1) + \frac{[U]^2(\Delta t)^2}{8\bar{\mu}^2(\Delta x)^2} (\psi_{-2} - 2 + \psi_2) \\
&+ \frac{[U]^3(\Delta t)^3}{48\bar{\mu}^3(\Delta x)^3} (\psi_{-3} - 3\psi_{-1} + 3\psi_1 - \psi_3)
\end{aligned}$$

$$\begin{aligned}\lambda \cos(\omega_r \Delta t) &= 1 - \frac{\bar{U}^2 (\Delta t)^2}{4\bar{\mu}^2 (\Delta x)^2} + \frac{\bar{U}^2 (\Delta t)^2}{4\bar{\mu}^2 (\Delta x)^2} \cos(2k\Delta x) \\ &= 1 - \frac{\gamma^2}{4} + \frac{\gamma^2}{4} \cos(2k\Delta x)\end{aligned}$$

$$\begin{aligned}\lambda \sin(\omega_r \Delta t) &= -\frac{\bar{U}\Delta t}{\bar{\mu}\Delta x} \sin(k\Delta x) - \frac{\bar{U}^3 (\Delta t)^3}{24\bar{\mu}^3 (\Delta x)^3} \sin(3k\Delta x) + \frac{\bar{U}^3 (\Delta t)^3}{8\bar{\mu}^3 (\Delta x)^3} \sin(k\Delta x) \\ &= -\gamma \sin(k\Delta x) - \frac{\gamma^3}{24} \sin(3k\Delta x) + \frac{\gamma^3}{8} \sin(k\Delta x)\end{aligned}$$

$$\begin{aligned}\lambda^2 &= 1 + \frac{\gamma^4}{16} + \frac{\gamma^4}{16} \cos^2(2k\Delta x) - \frac{\gamma^2}{2} + \frac{\gamma^4}{8} \cos(2k\Delta x) + \frac{\gamma^2}{2} \cos(2k\Delta x) \\ &\quad + \gamma^2 \sin^2(k\Delta x) + \frac{\gamma^6}{576} \sin^2(3k\Delta x) + \frac{\gamma^6}{64} \sin^2(k\Delta x) + \frac{\gamma^4}{12} \sin(k\Delta x) \sin(3k\Delta x) \\ &\quad - \frac{\gamma^6}{96} \sin(k\Delta x) \sin(3k\Delta x) - \frac{\gamma^4}{4} \sin^2(k\Delta x) \\ &= 1 + \frac{\gamma^4}{16} + \frac{\gamma^4}{16} \cos^2(2k\Delta x) - \frac{\gamma^4}{8} \cos(2k\Delta x) \\ &\quad + \frac{\gamma^6}{576} \sin^2(3k\Delta x) + \frac{\gamma^6}{64} \sin^2(k\Delta x) + \frac{\gamma^4}{12} \sin(k\Delta x) \sin(3k\Delta x) \\ &\quad - \frac{\gamma^6}{96} \sin(k\Delta x) \sin(3k\Delta x) - \frac{\gamma^4}{4} \sin^2(k\Delta x)\end{aligned}$$

A. If wavelength is  $2\Delta x$

$$\lambda^2 = 1$$

$$c/c_a = 0$$

B. If wavelength is  $4\Delta x$

$$\lambda^2 = 1 - \frac{\gamma^4}{12} + \frac{\gamma^6}{36}$$

$$c/c_a = \frac{\bar{\mu}}{\Delta tk[U]} \cos^{-1} \left[ \frac{1}{\lambda} \left( 1 - \frac{\gamma^2}{2} \right) \right] = \frac{2}{\pi\gamma} \cos^{-1} \left[ \frac{1}{\lambda} \left( 1 - \frac{\gamma^2}{2} \right) \right]$$

C. If wavelength is  $12\Delta x$

$$\lambda^2 = 1 - \frac{\gamma^4}{192} + \frac{\gamma^6}{2304}$$

$$c/c_a = \frac{\bar{\mu}}{\Delta tk[U]} \cos^{-1} \left[ \frac{1}{\lambda} \left( 1 - \frac{\gamma^2}{8} \right) \right] = \frac{6}{\pi\gamma} \cos^{-1} \left[ \frac{1}{\lambda} \left( 1 - \frac{\gamma^2}{8} \right) \right]$$

D. If wavelength is  $24\Delta x$

$$\lambda^2 = 1 - 3.74 \times 10^{-4} \gamma^4 + 8.35 \times 10^{-6} \gamma^6$$

$$c/c_a = \frac{\bar{\mu}}{\Delta tk[U]} \cos^{-1} \left[ \frac{1}{\lambda} \left( 1 - \frac{(2 - \sqrt{3})\gamma^2}{8} \right) \right] = \frac{12}{\pi\gamma} \cos^{-1} \left[ \frac{1}{\lambda} \left( 1 - \frac{(2 - \sqrt{3})\gamma^2}{8} \right) \right]$$



## B. Solve for potential temperature

Assume basic state potential temperature horizontally homogeneous

$$\frac{\partial \theta''}{\partial t} + \frac{1}{\bar{\mu}} \frac{\partial}{\partial x} (U \theta'') = 0$$

Linearize with respect to momentum, just like geopotential height

$$\frac{\partial \theta''}{\partial t} + \frac{[U]}{\bar{\mu}} \frac{\partial}{\partial x} \theta'' = 0$$

Solution is the same as for geopotential height

$$\theta'' = \hat{\theta} \exp[i(kn\Delta x + \omega\tau\Delta t)]$$

## C. Solve for momentum

Consider only external waves

$$p = p_0 (R_d \theta / p_0 \alpha)^\gamma$$

$$\Rightarrow \frac{\partial p'}{\partial x} = p_0 \left( \frac{R_d}{p_0 \alpha} \right)^\gamma \frac{\partial \theta''^\gamma}{\partial x}$$

$$\frac{\partial U}{\partial t} + \bar{\mu} \alpha \frac{\partial p'}{\partial x} + \bar{\mu} \frac{\partial \phi'}{\partial x} = 0$$

$$\Rightarrow \frac{\partial U''}{\partial t} + \bar{\mu} \alpha p_0 \left( \frac{R_d}{p_0 \alpha} \right)^\gamma \frac{\partial \theta''^\gamma}{\partial x} + \bar{\mu} \frac{\partial \phi'}{\partial x} = 0$$

$$U''_{n+1/2}{}^{\tau+T} = A(\lambda^\gamma)^T \exp\{i[k\gamma(n+1/2)\Delta x + \omega_r \gamma(\tau+T)\Delta t]\} + B\lambda^T \exp\{i[k\gamma(n+1/2)\Delta x + \omega_r \gamma(\tau+T)\Delta t]\}$$

Phase speed is the same as potential temperature and geopotential height.  
However, the amplitude change differs ...

# Summary for Simplified & Linearized Governing Equations

- 2<sup>nd</sup> order in space and Runge-Kutta 3<sup>rd</sup> in time preserves amplitude or nearly preserves amplitude in advection.
- Short waves travel more slowly.  $2\text{-}\Delta_x$  waves are stationary, while long waves travel almost as actual speed. Bad for short waves!
- The influence of thermodynamic energy on momentum is to the power of  $C_p/C_v$ , and acts on different spectrum.
- Note: it's only 2<sup>nd</sup> order in space, usually, people use 4<sup>th</sup> order or higher
- Reminder: a. Here we didn't consider non-linear problem and aliasing; b. Linearization breaks flux forms of advection.

If we don't linearize the equations, the (1)-(4) system is still solvable.

Step 1:

$$U_{n+1/2}^{''*} = U_{n+1/2}^{''\tau} + \frac{\bar{\mu}\Delta t}{3\Delta x} \left[ \alpha p_0 \left( \frac{R_d}{p_0 \alpha} \right)^\gamma (\theta^{''\gamma}|_n^\tau - \theta^{''\gamma}|_{n+1}^\tau) + (\phi_n^{\prime\tau} - \phi_{n+1}^{\prime\tau}) \right]$$

$$\phi_n^{\prime*} = \phi_n^{\prime\tau} + \frac{\Delta t}{12\bar{\mu}\Delta x} \left[ (U_{n-1/2}^{''\tau} + U_{n+1/2}^{''\tau})(\phi_{n-1}^{\prime\tau} - \phi_{n+1}^{\prime\tau}) \right]$$

$$\theta_n^{''*} = \theta_n^{''\tau} + \frac{\Delta t}{12\bar{\mu}\Delta x} \left[ (U_{n-3/2}^{''\tau} + U_{n-1/2}^{''\tau})\theta_{n-1}^{''\tau} - (U_{n+1/2}^{''\tau} + U_{n+3/2}^{''\tau})\theta_{n+1}^{''\tau} \right]$$

Step 2:

$$U''_{n+1/2}^{**} = U''_{n+1/2}{}^{\tau} + \frac{\bar{\mu}\Delta t}{2\Delta x} \left[ \alpha p_0 \left( \frac{R_d}{p_0 \alpha} \right)^{\gamma} (\theta''^{\gamma}|_n^* - \theta''^{\gamma}|_{n+1}^*) + (\phi'_n{}^* - \phi'_{n+1}{}^*) \right]$$

$$\phi'_n{}^{**} = \phi'_n{}^{\tau} + \frac{\Delta t}{8\bar{\mu}\Delta x} \left[ (U''_{n-1/2}{}^* + U''_{n+1/2}{}^*) (\phi'_{n-1}{}^* - \phi'_{n+1}{}^*) \right]$$

$$\theta''_n{}^{**} = \theta''_n{}^{\tau} + \frac{\Delta t}{8\bar{\mu}\Delta x} \left[ (U''_{n-3/2}{}^* + U''_{n-1/2}{}^*) \theta''_{n-1}{}^* - (U''_{n+1/2}{}^* + U''_{n+3/2}{}^*) \theta''_{n+1}{}^* \right]$$

Step 3:

$$U_{n+1/2}^{\prime\prime\tau+1} = U_{n+1/2}^{\prime\prime\tau} + \frac{\bar{\mu}\Delta t}{\Delta x} \left[ \alpha p_0 \left( \frac{R_d}{p_0 \alpha} \right)^\gamma (\theta^{\prime\prime\gamma} \Big|_n^{**} - \theta^{\prime\prime\gamma} \Big|_{n+1}^{**}) + (\phi_n^{\prime**} - \phi_{n+1}^{\prime**}) \right]$$

$$\phi_n^{\prime\tau+1} = \phi_n^{\prime\tau} + \frac{\Delta t}{4\bar{\mu}\Delta x} \left[ (U_{n-1/2}^{\prime\prime**} + U_{n+1/2}^{\prime\prime**}) (\phi_{n-1}^{\prime**} - \phi_{n+1}^{\prime**}) \right]$$

$$\theta_n^{\prime\tau+1} = \theta_n^{\prime\tau} + \frac{\Delta t}{4\bar{\mu}\Delta x} \left[ (U_{n-3/2}^{\prime\prime**} + U_{n-1/2}^{\prime\prime**}) \theta_{n-1}^{\prime\prime**} - (U_{n+1/2}^{\prime\prime**} + U_{n+3/2}^{\prime\prime**}) \theta_{n+1}^{\prime\prime**} \right]$$

Let

$$U_{n+1/2}''^\tau = \hat{U}_1 \exp\{i[k\gamma(n+1/2)\Delta x + \omega\gamma\tau\Delta t]\} + \hat{U}_2 \exp\{i[k(n+1/2)\Delta x + \omega\tau\Delta t]\}$$

$$\theta_n''^\tau = \hat{\theta} \exp[i(kn\Delta x + \omega\tau\Delta t)]$$

$$\phi_n'{}^\tau = \hat{\phi} \exp[i(kn\Delta x + \omega\tau\Delta t)]$$

You can try it ...

		$\gamma = \frac{[U]\Delta t}{\bar{\mu}\Delta x}$		
		0.1	0.5	0.9
wavelength	$2\Delta x$	$\lambda = 1$ $c/c_a = 0$	$\lambda = 1$ $c/c_a = 0$	$\lambda = 1$ $c/c_a = 0$
	$4\Delta x$	$\lambda \cong 1$ $c/c_a = 0.637$	$\lambda = 0.995$ $c/c_a = 0.632$	$\lambda = 0.960$ $c/c_a = 0.638$
	$12\Delta x$	$\lambda \cong 1$ $c/c_a = 0.955$	$\lambda \cong 1$ $c/c_a = 0.953$	$\lambda \cong 0.997$ $c/c_a = 0.949$
	$24\Delta x$	$\lambda \cong 1$ $c/c_a = 0.989$	$\lambda \cong 1$ $c/c_a = 0.988$	$\lambda \cong 1$ $c/c_a = 0.987$