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The imaginary part of the group refractive index*

R. Michael Jones

Cooperative Institute for Research in Environmental Sciences, University of Colorado/NOAA, Boulder, Colorado, USA

It is shown that the group refractive index is complex whenever the absorption coefficient depends on frequency. The imaginary part of the group refractive index is proportional to the frequency derivative of the absorption coefficient. A pulse propagating in a time-independent absorbing medium will have its dominant frequency shifted by an amount proportional to the distance traveled and the imaginary part of the group refractive index and inversely proportional to the square of the pulse length.

Keywords: group refractive index, imaginary, frequency-dependent losses

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1. Introduction

Propagation of monochromatic waves is generally well understood, even for anisotropic and absorbing media. The travel time of a wave front is proportional to the real part of the phase refractive index, and the attenuation rate is proportional to the imaginary part.

That is, for a complex phase refractive index

$$n = \mu - i\chi \ , \tag{1}$$

the real part μ is associated with the phase of the wave and the imaginary part χ is associated with the amplitude. For example, for a homogeneous medium, a wave of frequency ω that has propagated a distance z into the medium at time t is represented by

$$e^{i\overbrace{(\omega t - \frac{\omega}{c} n z)}^{\text{complex phase}}} = e^{\overbrace{i\omega(t - \frac{\mu}{c} z)}^{\text{phase}}} \underbrace{e^{-\overbrace{\frac{\omega\chi}{c}}^{\alpha} z}}_{\text{amplitude}} \ . \tag{2}$$

The quantity

$$v_p = c/\mu \tag{3}$$

gives the phase speed of the wave, and

$$\alpha = \omega\chi/c \tag{4}$$

gives the attenuation coefficient.

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Similarly, for propagation of pulses in anisotropic but nonabsorbing media it is well understood that to a first approximation, the envelope of a pulse that modulates the carrier frequency of a wave will propagate without change in shape at the group velocity $d\omega/d\mathbf{k}$, where $\mathbf{k}$ is normal to the wave front, $|k| = 2\pi/\lambda$, and λ is the wavelength. [e.g., *Stratton*, 1941; *Hines*, 1951a; *Panofsky and Phillips*, 1955; *Jeffreys and Jeffreys*, 1956; *Brillouin*, 1960; *Whitham*, 1960; *Budden*, 1961; *Jackson*, 1962; *Lighthill*, 1965].

However, even for a nonabsorbing medium, a pulse can change shape as it propagates, leading to some ambiguity in how to define the propagation velocity of a pulse. One method to circumvent this ambiguity in practice is to calculate the change in shape of the pulse as it propagates [e.g., *Felsen*, 1969; *Nicolis*, 1967; *Wait*, 1969a, b; *Vogler*, 1969; *Wait*, 1970a, b, c; *Vogler*, 1970a, b]. Such calculations usually involve an inverse Fourier transform of a calculation in the frequency domain. Often, the inverse transform can be successfully evaluated by asymptotic methods [e.g., *Jeffreys and Jeffreys*, 1956; *Price*, 1968; *Felsen*, 1969].

Pulse propagation in absorbing media, however, has some conceptual difficulties. The main difficulty arises from trying to interpret the significance of a complex group velocity. *Booker* [1939] suggested that, since a pulse is made up of a spectrum of frequencies, a complex frequency might be found in the spectrum that would make the group velocity real.

Part of the difficulty was explained by *Hines* [1951a, b]. He argued that when the arrival time of a pulse is observed, it is really the time maximum of the pulse peak that is measured. Second, he showed that (for a homogeneous medium) the travel time of the time maximum of a pulse peak is given by the product of the real part of the group refractive index and the distance traveled in the medium divided by the free-space speed of light. For a heterogeneous medium, the travel time was equal to the integral of the real part of the group refractive index divided by the free-space speed of light.

Furutsu [1952], apparently unaware of Hines's work, argued that the concept of group velocity and wave path do not exist in a dissipative medium. *Suchy* [1972a, b, c, 1974] used different arguments from Hines's to advocate using the real part of $\partial\omega/\partial\mathbf{k}$ as a more appropriate group velocity. *Suchy* [1972a] argued that the imaginary part of $\partial\omega/\partial\mathbf{k}$ has no apparent physical meaning. Later, however, *Suchy* [1974] interpreted the imaginary part of $\partial\omega/\partial\mathbf{k}$ in terms of a moving derivative of the real part of the wave number.

The difficulty with interpreting a complex group velocity is similar to that of interpreting the complex direction that occurs for ray tracing in complex space [e.g., *Poeverlein*, 1962; *Budden and Jull*, 1964; *Jones*, 1970a; *Budden and Terry*, 1971; *Keller and Streifer*, 1971; *Bertoni et al.*, 1971; *Deschamps*, 1972; *Kravtsov et al.*, 1974; *Wang and Deschamps*, 1974; *Bennett*, 1974; *Connor and Felsen*, 1974].

The seemingly peculiar behavior of wave propagation in dissipative media has led to various investigations into various aspects of propagation. For example, *Hines* [1951c, d] and *Arsaef and Kinber* [1968] consider the direction of energy flux in dissipative media. *Poeverlein* [1962] points out that absorption of waves can be represented by complex propagation vectors. *Storey and Roehner* [1970] and *Roehner* [1971] consider the direction of stationary phase for a beam of waves in an absorbing medium. *Bertoni et al.* [1971] consider the nonlocal nature of propagation in dissipative media. *Batorsky and Felsen* [1971] consider complex waveguide modes. *Denman and Buch* [1973] derive a Hamiltonian for dissipative systems. Several investigators have calculated various aspects of pulse distortion in dissipative media [e.g., *Vogler*, 1969, 1970a, b; *Wait*, 1970b].

The practical calculation of the propagation of pulses in dissipative media in terms of pulse distortion is not hindered by difficulties in interpreting a group velocity that takes on complex values. The method is the same as for lossless media (that is, an inverse Fourier transform of a frequency-domain calculation), and the results for the amplitude and phase of the resulting signal as a function of time and position are just as unambiguous as for the lossless case.

In 1970, I discovered that as a pulse propagates through a dissipative medium, the frequency of the carrier will be shifted by an amount proportional to the imaginary part of the complex group refractive index, which in turn depends on the frequency dependence of dissipation. I presented the results at a conference [*Jones*, 1970b], but failed to publish the result at that time. *Bennett* [1974], however, reported the results in the special issue of Proceedings IEEE on rays and beams, referring to my unpublished work. In the same issue, *Connor and Felsen* [1974, eq. 7], apparently independently, derive the frequency shift of a Gaussian-shaped pulse propagating in a dissipative medium, but do not mention the relation to the imaginary part of the group refractive index. *Jones* [1981] presented the results in more detail, but that report did not reach a wide audience.

2. Complex group refractive index

A complex group refractive index is defined by

$$n' \equiv \frac{d}{d\omega}(\omega n) = \frac{d}{d\omega}(\omega\mu) - i\frac{d}{d\omega}(\omega\chi), \qquad (5)$$

with real part

$$\Re(n') = \frac{d}{d\omega}(\omega\mu) \qquad (6)$$

and imaginary part

$$\Im(n') = -\frac{d}{d\omega}(\omega\chi). \qquad (7)$$

It is well known that .

$$\frac{d\omega}{dk} \qquad (8)$$

gives the group velocity. More specifically,

$$\Re\left(\frac{dk}{d\omega}\right)z = \frac{d}{d\omega}\left(\frac{\omega\mu}{c}\right)z = \frac{1}{c}\frac{d}{d\omega}(\omega\mu)\,z = \frac{\Re(n')}{c}z \qquad (9)$$

gives the travel time of the time maximum of a pulse[*Hines*, 1951a, b, c, d], giving the significance of the real part of the group refractive index.

To see the significance of the imaginary part of the group refractive index, we combine (4) and (7) to see that the imaginary part of the group refractive index

$$\Im(n') = -\frac{d}{d\omega}(\omega\chi) = -\frac{d}{d\omega}(c\alpha) = -c\frac{d\alpha}{d\omega} \quad (10)$$

is proportional to the frequency derivative of the attenuation coefficient.

3. Pulse propagation

A transmitted pulse with shape $m(t)$ and carrier frequency ω_0

$$m(t)e^{i\omega_0 t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}A(\omega)e^{B(\omega)}d\omega \quad (11)$$

that has a spectrum

$$A(\omega)e^{B(\omega)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}m(t)e^{i\omega_0 t}dt \quad (12)$$

will have its frequency shifted as it propagates through a dissipative medium.

Notice that the spectrum is divided arbitrarily into a slowly varying part A and an exponentially varying part $\exp(B)$. If the pulse $m(t)$ is symmetric, we can take $A(\omega)$ and $B(\omega)$ to be real whenever ω is real. When $m(t)$ is not symmetric, we could choose $B(\omega)$ to represent the symmetric part, in which case it would be real whenever ω is real, but then $A(\omega)$ would be complex and would have a phase that varied with frequency. Since this is counter to the assumption that $\exp(B)$ represent the quickly varying part of the spectrum, we must allow $B(\omega)$ to be complex in the general case where $m(t)$ might be asymmetric.

The medium acts like a filter. Let the pulse propagate through a homogeneous medium that multiplies a monochromatic wave of frequency ω by

$$\exp\left(-i\frac{\omega}{c}\mu z\right)\exp(-\alpha z) \ , \quad (13)$$

where z is the distance traveled (normal to a wavefront) in the medium. The spectrum of the pulse at a point z in the medium is then

$$A(\omega)\exp\left[B(\omega) - i\frac{\omega}{c}\mu(\omega)z - \alpha(\omega)z\right] \quad (14)$$

To find the dominant frequency ω_p in this spectrum, we set

$$\frac{d}{d\omega}\left|A(\omega)\exp[B(\omega) - \alpha(\omega)z]\right|_{\omega=\omega_p} = 0 \ . \quad (15)$$

When the variation of A is neglected, this gives

$$\Re\left[\frac{d}{d\omega_p}B(\omega_p)\right] - z\frac{d\alpha(\omega_p)}{d\omega_p} = 0 \ . \quad (16)$$

Expanding up to second order in a Taylor series

$$B(\omega) = B(\omega_1) + B'(\omega_1)(\omega-\omega_1) + \frac{1}{2}B''(\omega_1)(\omega-\omega_1)^2 \quad (17)$$

about the point where

$$\Re\left[\frac{dB(\omega_1)}{d\omega_1}\right] = \Re\left[B'(\omega_1)\right] = 0 \quad (18)$$

gives

$$\Re\left[B''(\omega_1)\right](\omega_p - \omega_1) - z\frac{d\alpha(\omega_p)}{d\omega_p} = 0 \ . \quad (19)$$

Thus, the frequency shift is given by

$$\omega_p - \omega_1 = \frac{z}{\Re\left[B''(\omega_1)\right]}\frac{d\alpha(\omega_p)}{d\omega_p} \ . \quad (20)$$

Using (10) gives

$$\delta\omega \equiv \omega_p - \omega_1 = -\frac{z}{c}\frac{\Im\left[n'(\omega_p)\right]}{\Re\left[B''(\omega_1)\right]} \ , \quad (21)$$

which is proportional to the imaginary part of the group refractive index.

4. Gaussian-shaped pulse

A Gaussian-shaped pulse

$$m(t) = \exp[-(t/\tau)^2] \quad (22)$$

has a pulse spectrum

$$A(\omega)\exp[B(\omega)] = \overbrace{\frac{\tau}{\sqrt{2}}}^{A}\exp\overbrace{\left[-\frac{\tau^2}{4}(\omega-\omega_0)^2\right]}^{B(\omega)} \quad (23)$$

so that

$$B(\omega) = -\frac{\tau^2}{4}(\omega-\omega_0)^2 \quad (24)$$

and

$$\omega_1 = \omega_0. \quad (25)$$

Thus, the frequency shift is

$$\delta\omega = \frac{2z\Im(n')}{c\tau^2}, \quad (26)$$

where z is the distance traveled and τ is the pulse length. A shorter pulse length (broader spectrum) gives a larger frequency shift, as expected.

5. heterogeneous medium

For pulse propagation through a heterogeneous medium in the WKB aproximation, each frequency component is multiplied by

$$a(\omega) \exp\left[-i\frac{\omega}{c}P(\omega)\right] \ , \qquad (27)$$

where $a(\omega)$ is a slowly varying function,

$$P = \int_{raypath} \mathbf{n} \cdot \mathbf{ds} \qquad (28)$$

is the complex phase path, $\mathbf{n}$ is a vector pointing in the wave-normal direction whose magnitude equals the complex phase refractive index, and $\mathbf{ds}$ is a vector pointing in the ray direction. The Taylor expansion gives

$$B''(\omega_1)(\omega_p - \omega_1) + \frac{1}{c}\Im\left[P'(\omega_p)\right] = 0 \ , \qquad (29)$$

where

$$P' \equiv \frac{d}{d\omega}[\omega P(\omega)] \qquad (30)$$

is the complex group path.

Thus, for an inhomogeneous medium, the frequency shift from (21) is

$$\delta\omega = \omega_p - \omega_1 = -\frac{\Im\left[P'(\omega_p)\right]}{cB''(\omega_1)} \ , \qquad (31)$$

or, for a Gaussian-shaped pulse,

$$\delta\omega = \frac{2\Im(P')}{c\tau^2} \ . \qquad (32)$$

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R. Michael Jones, Cooperative Institute for Research in Environmental Sciences, University of Colorado/NOAA, Boulder, CO 80309-0216, USA. (Michael.Jones@Co

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