Quantum Gravity and Mach's Principle: resolving the inconsistency between General Relativity and quantum theory by considering the interaction of local physics and distant matter

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Preface to the 2008 edition

This is a collection of my essays that I wrote over about 40 years on various aspects of the problem of trying to reconcile gravitation and quantum theory through the interaction of distant matter on local inertial frames (the latter sometimes referred to as Mach's principle).

I am putting these ideas together now for two reasons. First, to put them in one place so that I can refer to them as I try to put together a coherent theory for reconciling gravitation and quantum theory. Second, in case I am not able to get any of these ideas published, at least they may not be lost.

Each chapter is a separate essay. The various essays were written at different times, and are presented in chronological order as well as I could correctly identify the correct time of writing.

There is some duplication of ideas among the essays, and I have not attempted to edit the essays to make a coherent presentation. There is also some disagreement among the essays, reflecting either a change in my views over the years, or showing that I may have forgotten what I wrote earlier. Some of the essays may have been drafts for later essays.

There is also some variation in style among the essays. I have not tried to impose a uniform style.

Preface

I have corrected some errors, and have added an introduction (chapter 1).

I have also added some afterthoughts to some of the chapters to reflect how my thinking has changed since I wrote those chapters. In chapter 27, “the gravitational vector potential,” I have discussed the significance and meaning of a gravitational vector potential, and suggested some other possibilities. In chapter 29, the 1979 version of “the quantum basis for Mach’s principle,” in chapter 31, the 1980 version of “the quantum basis for Mach’s principle,” in chapter 49, “Quantum Selection of a Classical Cosmology,” in chapter 75, “The criteria for a solution of the field equations to be a classical limit of a quantum cosmology,” and in chapter 85, the June 2006 version of “The criteria for a solution of the field equations to be a classical limit of a quantum cosmology,” I discuss mistakes I made in those manuscripts, and consider how I should present that material in the future. In chapter 71, “physics in a sparse universe,” I add some new material about physics in a universe in which the Planck length is comparable with the size of the universe.

In chapter 75, “The criteria for a solution of the field equations to be a classical limit of a quantum cosmology,” and in chapter 85, the June 2006 version of “The criteria for a solution of the field equations to be a classical limit of a quantum cosmology,” I point out that I need to change my notation for propagators.

I added several new chapters, including chapter 9, “Gravitation and wave mechanics,” chapter 10, “Motivation,” chapter 11, “Thoughts on the origin of inertia,” chapter 12, “A gravitational interpretation of Planck’s constant,” chapter 13, “Motivation - 2,” chapter 14, “The use of natural units,” chapter 15, “Gravitational mass,” chapter 16, “Pair production,” chapter 17, “Einstein’s theory does not contain Mach’s principle,” chapter 18, “My motivation for this research - connection between quantum theory and gravitation through Mach’s principle,” chapter 19, “Why I feel that the concept of a photon is a useful mathematical device rather than a reality,” chapter 20, “Ideas on Cards,” chapter 86, “An integral form of Einstein’s gravity,” chapter 87, “An electron is waves of what?,” chapter 88, “A wave function applies to an ensemble,” chapter 89, “Gravitational Field representation of General Relativity,” chapter 90, “A realistic interpretation of quantum field theory,” chapter 91, “Dirac equation in the frame of the electron,” chapter 92, “Inertia of light,” chapter 93, “Quantum theory - philosophies versus interpretations versus models,” chapter 94, “A wave function allows forecasts, but is not a real field,” chapter 95, “Gravitational vector potential revisited,” chapter 96, “Inertial frames,” chapter 97, “Fine-grained and coarse-grained histories,” chapter 98, “Nonlinear acoustic-gravity waves in a fluid,” chapter 99, “Kings of the jungle,” chapter 100, “Myths in physics,” chapter 101, “Renormalization,” chapter 102, “Phase interference can solve the rotation problem,” chapter 103, “Possible limitations for the uncertainty relations,” chapter 104, “Box normalization,” chapter 105, “Discrete General Relativity,” chapter 106, “The rotation problem,” chapter 107, “The zero-point energy problem,” chapter 108, “Are the wave functions associated with the four fundamental interactions of a different kind from each other?,” chapter 109, “Representing Geometry as Gravity,” chapter 110, “Gravity Without Geometry - III,” chapter 111, “Refractive index for a particle wave in a gravitational field,” chapter 112, “The quantum N-body problem,” chapter 113, “When did our universe become classical?,” chapter 114, "A realistic quantum model," chapter 115, "The acceleration problem," chapter 116, "Relative acceleration of matter and inertial frames," chapter 117, "The fluctuating field interpretation of quantum mechanics," chapter 118, "Relative acceleration of matter and inertial frames," chapter 119, "Quantum gravity and the relative acceleration of matter and inertial frames," chapter 120, "Comment on arXiv preprint 'From maximum force to physics in 9 lines – and implications for quantum gravity' by Christoph Schiller," chapter 122, "Quantum gravity: reconciling gravitation and quantum theory," and have added more comments to chapter 123, "Epilog."

I may include more of my essays that were not in the previous edition.

Chapters 46 (Effective rest mass of a photon in a plasma), 84 (Resolving the zero-point energy problem by dropping an assumption in quantum theory), 86 (An integral form of Einstein's gravity) need revision. Chapter 46 needs some additional references. In Chapter 84, I need to add some comments relating to the e-mail exchange I had with Jeff Grove in January 2013. In Chapter 86, I need to add some references and develop the integral form of gravitation further.

Chapters 91 (Dirac equation in the frame of the electron), 92 (Inertia of light), 95 (Gravitational vector potential revisited), 96 (Inertial frames), 97 (Fine-grained and coarse-grained histories), 98 (Nonlinear acoustic-gravity waves in a fluid), 99 (King on the Mountain), 100 (Myths in physics), 101 (Renormalization), and 122 (Quantum gravity: reconciling gravitation and quantum theory) are unfinished.

I still have some essays I wrote a long time ago that I have not yet scanned and added to the book.

There are so many pages now that the printed version of the book is split into two volumes.

Boulder
August 2022

Acknowledgments

I would like to thank everyone with whom I discussed some of these concepts presented here. I would especially like to thank David Peterson, David Bartlett, Gary Bornzin, Jay Palmer, and Douglas Gough. In addition, I would like to thank Derek Raine, Stephen Hawking, Julian Barbour, Bruno Bertotti, Don Page, Ronald Adler, Richard Arnowitt, Roger Penrose, and Hubert Goenner for useful discussions.

I would especially like to thank W. H. Hooke for his encouragement in 1977 to continue this research.

I also thank Gary Bornzin for the following Groucho Marx quote:

Time flies like an arrow.

Fruit flies like a banana.

Contents

[1] ca 1962

[2] ca 1967
[3] ca 1967
[4] ca 1969
[5] ca 1969. See section 20.12 in chapter 20.
[6] ca 1970. This is the original handwritten version that I found after 2008
[7] Essay number 6, ca April 1971
[8] Essay number 7, August 1971
[9] Essay number 8, ca 1971-1972

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[10] ca 1974

[11] 1974. Electromagnetic Lense-Thirring effect

[12] 9 October 1974

[13] 23 October 1974

[14] 20 November 1974

[15] 18 December 1974

[16] Essay number 25 in the printed list of essays. October 1979. Early draft of the paper submitted to General Relativity and Gravitation in October 1979

[17] Submitted to General Relativity and Gravitation 15 October 1979
[18] July 1980. Presented at the Ninth International Conference on General Relativity and Gravitation, 14-19th July 1980, Friedrich Schiller University, Jena, DDR, on pages 483-484 in Volume 2 of the Abstracts of Contributed Papers
[19] Submitted to Foundations of Physics 7 August 1980
[20] Draft 15 June 1981. The summary seems a little short.
[21] Draft 17 June 1981
[22] draft 20 June 1981
[23] July 1981

[24] ca July 1981
[25] Written sometime after 9 October 1981
[26] File qcos.tex, DRAFT - 4 December 1983
[27] Joensu, Finland, 1985

[28] Joensu, Finland, 1985, or slightly later
[29] ca 1985
[30] draft - 16 Oct 1986.
[31] 8 December 1989, Corrected 1 May 2015
[32] 18 April 1994. Presented at (and submitted to the Proceedings of) the conference on Mach's Principle: "From Newton's Bucket to Quantum Gravity", held at Tübingen, Germany, July 26-30, 1993, File: q-cos2.tex

[33] essay, written before 1 April 1997
[34] essay2, written sometime before 1 April 1997
[35] essay3, April 1997
[36] 29 January 1998
[37] 30 January, 2 February 1998, file what2.tex
[38] 3 February 1998, file what3.tex

[39] file what4.tex, written between 3 and 10 February 1998
[40] file what5.tex, 10 February 1998
[41] file essay5.tex, written sometime before 1 April 1998, possibly 19 February 1998, Copied from essay4 on 23 January 1998 and starting to alter on that date.
[42] 8 March 1998
[43] 28 February 1998 with some comments by Jay Palmer and David Peterson

[44] Samos 30 August - 4 Sept. 1998
[45] Draft: 17 September 1998

[46] written before April 1999
[47] 4 November 1999
[48] 30 December 1999
[49] 25 February 2002
[50] September 2010
[51] 23 June 2002, File: q-cos6.tex, adding the cosmological constant Λ

[52] August 2002, Sarlat, France
[53] draft: 7 October 2002. Note added, March 2014: I realize this is not a correct derivation.
[54] 13 March 2003
[55] 28 June 2004
[56] ca August 2004
[57] 29 March 2005
[58] 29 December 2006

[59]draft: 16 January 2007
[60]Added 26 January 2015
[61]file qcos09.tex, Tuesday 14 April 2009

[62] David Peterson pointed out that what I am proposing in this chapter has long been known as the "ensemble interpretation" of quantum mechanics. A description of it can be found in Wikipedia, including both the pros and cons of the interpretation. As far as I can tell, all of the cons listed there can be easily refuted. Albert Einstein was one of the greatest supporters of the ensemble interpretation. I am starting to suspect, however, that what I am suggesting here differs somewhat from the ensemble interpretation.

[63] February 2010, additional comment 18 March 2011
[64] 16 February 2010
[65] 14 October 2010

[66] mostly written in 2006, finished in 2012

[67] Added 21 September 2017

[68] big.elephant.tex
[69] big.gvp.tex
[70] big.p.tex

Chapter 1

Introduction

Reconciling gravitation and quantum theory is probably the most important problem of theoretical physics today.

Although this endeavor is often referred to as "quantum gravity" or "quantizing the gravitational field," that terminology is misleading in that it suggests first, that gravitation can be quantized in the same way as any other interaction, and second, that quantum theory will not have to be altered in the process of reconciliation.

There are two reasons why gravitation is a special case. First, at least as represented in General Relativity as geometry, it forms the background on which the rest of physics, including quantum theory, operates. Second, quantum theory has a strong implicit link to gravitation through inertia (which is a gravitational phenomenon) because the wave properties of particles seem to be associated with momentum (and therefore inertia).

The two main lines of attack on this reconciliation problem in the quantum gravity community are string theory and loop quantum gravity. String theory deals with difficulties caused by treating particles as points. Because that problem needs to be dealt with in any case, string theory (or something similar) will continue to play an important role in physics. Loop quantum gravity tries to deal with the reconciliation problem by choosing different variables, and that may be one of the keys to solving the reconciliation problem.

Because neither of those lines of attack deal with the problem of quantum theory being based on the existence of a background inertial frame, they will not be able to solve the reconciliation problem by themselves, although they may contribute to the solution.

Both quantum theory and General Relativity will probably have to be altered. There are several reasons why the problem is difficult.

The order of the chapters is chronological, but their discussion in this introduction is organized by topic. (I still have to revise this introduction considerably.)

1.1 Questions to be answered

1. Does gravitation intrinsically form a background geometry for quantum processes and other fields or does that just happen for our universe because there is a lot of matter in the universe?

2. If the latter, what is the correct way to treat interactions in such a way that all interactions are treated in the same way?

1.2 Altering General Relativity

Mashhoon [7] has proposed a nonlocal special relativity based on an acceleration-induced nonlocality.

1.3 Geometry versus inertia

The hardest part seems to be separating the geometry from the gravitation. Geometry has inertia, but it is difficult to separate the inertial aspect of geometry from other aspects of geometry.

Chapter 92 calculates the inertia of light. More specifically, it tries to write Maxwell's equations in a curved background as a set of linear first-order differential equations in such a way that the inertial and gravitational effects are explicit rather than implicit. It is not finished.

Chapter 46 talks about the effective rest mass of a photon in a plasma. The effective mass of a photon in a plasma is proportional to the plasma frequency. Or, equivalently, the effective rest frequency of a photon in a plasma equals the plasma frequency. Equivalently, the square of the effective rest mass of a photon in a plasma is proportional to the product of the electron density and the ratio of the square of charge on the electron and the mass of the electron.

If we want to compare this with the Higgs field creating effective rest mass of particles, then we should consider the electron density as analogous to the strength of the Higgs field, and the ratio of the square of the charge on the electron and the mass of the electron as analogous to the strength with which the Higgs field interacts with the particle in question. Considering e^2/m_e as the strength with which a photon interacts with a plasma is very reasonable. This concept seems very easy to generalize, as is now considered in section 46.5 in that chapter. In a related calculation, chapter 48 considers the inertia of a charged particle in a charged universe.

Chapter 54 considers whether the universe can be open. Chapter 68 gives a derivation of special relativity. Chapter 78 considers the conjectured spacetime structure of the universe. Chapter 79 points out situations in which particles can exceed the speed of light. Chapter 83 considers the origin of geometry. Chapters 5, 22, 24, and 25 also consider geometry and inertia.

Part of the reason for the difficulty with the gravitational field, is that if we treat it as geometry, then we have difficulty separating those parts of the geometry that have nothing to do with gravitation from those parts that are intrinsically gravitational. Chapters 51, 52, 53, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 72, 73, 109, and 110 deal with this aspect of the problem.

1.4 Mass and charge are not quite parallel.

Although both mass and charge are sources of fields, mass is also part of the dispersion relation for matter waves. That is, mass plays at least two different roles, and those two roles need to be better dealt with.

Maybe it is the dual role played by mass that makes reconciling gravitation and quantum theory so difficult.

1.5 Where is the gravitational field in General Relativity?

In Newtonian gravity, it was simple. There was a gravitational field that was the gradient of a scalar field. To find the gravitational force on a body of mass m, we simply multiplied the gravitational field by m. When the theory of electromagnetism was developed in the 19th century, we found that force fields could be a little more complicated. We have an electromagnetic field $F_{\mu\nu}$ which we multiply by a current J^ν to give the Lorentz force.

Now, we come to General Relativity. The total force on a body is zero, and is given by $T^{\alpha\beta}_{;\beta} = 0$. This includes the gravitational force as expressed by the geodesic equation plus the Lorentz force for electromagnetism. The geodesic equation includes inertia, of course, showing that inertia really is a gravitational force.

We now know that inertia is a gravitational force. However, (except in the frame of the particle) there is an $\ddot{x}$ term in the geodesic equation. This is a strange term for a force term from a field, such as the gravitational field. It does not look like the gradient of a potential, for example. More specifically, it does not look like anything in the electromagnetic force in the Lorentz force.

The only solution I have been able to find for this difficulty is to restrict use of a force equation to the frame of the particle, where that term does not appear. However, there is no rest frame for massless particles, such as a photon, but photons still have inertia.

One of the problems with representing gravitation as geometry is that the geodesic equation does not seem to look like other force laws, such as the Lorentz force. Chapters 47, 51, 52, and 72 deal with this problem.

Another aspect of the geodesic equation is that in the frame of the body or particle, inertial forces take a form that is similar to electromagnetic induction forces. In fact, when we transform back to an arbitrary frame, the 4-velocity of the body or particle (when extended to be a field) behaves similar to an electromagnetic vector potential. In fact, it is actually possible to define locally (ignoring sources) a gravitational vector potential. Chapters 27, 72, and 123 discuss the gravitational vector potential.

Chapter 95 looks again at the gravitational vector potential that was first considered in chapter 27. It is not finished.

Although the gravitational vector potential is not a completely satisfactory solution, there is another possibility. How about $G^{\mu\nu}{}_{;\nu} - T^{\mu\nu}{}_{;\nu} = 0$? Maybe separating out the $U^{\mu\nu} \equiv U^\mu U^\nu$ part of $T^{\mu\nu}$.

But where is the gravitational field? A gravitational field would be a field that depends only on location in spacetime in some arbitrarily-chosen coordinate system. We would then multiply that field by something that depends on how a body of mass m is moving in that coordinate system. What are some possibilities?

The electromagnetic force is a field times a current, that is $F_{\mu\nu}J^\nu$. The corresponding thing for gravitation might be $F_{\alpha\beta\gamma}T^{\beta\gamma}$. But instead we have something like $T^{\mu\nu}{}_{;\nu} = 0$. So, where is the gravitational field?

Maybe we can write the geodesic equation in the frame of the particle, assume it has the form $F_{\alpha\beta\gamma}T^{\beta\gamma}$ in all frames and transform back accordingly. Chapter 89 does something similar to this, but it is not finished. If that does not work, treating $T^{\mu\nu}$ as a source for gravitation may not be the correct way to do it. Or, another possibility is that maybe the electromagnetic form for force has that form only because of inertial frames. Maybe there is a more general form. In addition, the above procedure would not work for massless particles, such as photons, which also feel gravitation and inertia. If that procedure actually works for massive particles, we would have to find a method for massless particles.

The first thing that needs to be done is to make a list of the properties that a gravitational field should have. I will start that list here.

1. The field components that act on a body or particle should be independent of the properties of that body or particle, including the motion of that body or particle.

2. If all components of the gravitational field are zero, there should be no gravitational force on the body or particle.

3. The gravitational field components should transform as some kind of tensor when changing frames.

There is a possibility that it is not possible to find a field that satisfies all of the above criteria. I think the clue is to realize that inertial frames can be considered part of the gravitational field. With regard to the strength of the inertial field, we need to recognize that inertial mass is only proportional to gravitational mass, not equal to it, and the gravitational constant G is really an inertial constant. (In fact, the significance of the equivalence principle is that inertia is a gravitational force, and nothing more.)

I am fairly certain we can satisfy all three conditions. Part of the solution is to realize that the coordinates of a local inertial frame give part of the gravitational field. Using the transformation for a connection on p. 262 of Misner, Thorne, and Wheeler should help. Knowing the local inertial frame does not give the strength of the inertial field. To get that, we realize that inertial mass and gravitational mass are not equal, only proportional. The proportionality constant gives the strength.

Yes, the gravitational field is given by the tensor ${F^{\alpha}}_{\beta\gamma}$. See section 89.2.

As it turns out, it is not possible to find a tensor gravitational field that satisfies all of the properties that a gravitational field tensor should satisfy. See section 89.12.1. Maybe it is still possible to find a tensor gravitational field by requiring only that such a field equals (the negative of) the connection along the trajectory of a body or particle.

1.6 Replace General Relativity by a theory in which the universe is spatially flat on the average?

Cosmological observations indicate that the universe is spatially flat on the whole. However, replacing General Relativity with a theory in which the universe is exactly spatially flat on the average would be a bad idea because it would give inertia with no source.

1.7 Altering QM

We need to understand ordinary quantum theory before we can reconcile quantum theory with gravitation. Chapters 32, 33, 34, 35, 36, 38, 39, 40, 41, 43, 76, 80, 84, 87, and 90 try to do this.

One aspect of understanding quantum theory is to realize that fields, not particles, are fundamental. Among other things, that means that we probably need to consider wave functions to represent actual physical fields, such as electric or magnetic fields. Then, $\psi^*\psi$ would not represent the probability density for a particle to be at some location, but we would calculate the amplitude for a process, such as detection in some apparatus, with the corresponding probability for that process.

One aspect of quantum theory that needs to be examined is choice of modes that comes from box normalization, along with the corresponding creation and annihilation operators. Although this mathematical procedure works in many situations, it clearly does not represent the real situation, and by trying to use it in difficult situations may not work well, for example in non-inertial frames [8]. See chapter 104.

Another aspect of quantum theory that needs to be looked at is what happens when an atom emits a photon. The standard quantum theory solution is that the atom emits a spherical wave which is interpreted as giving the probability of transmission of the photon in any direction. Or, more accurately, the square of the wave function gives the probability of detecting the photon at that place. Once the photon is actually detected at some place, then that means the atom must have had a back reaction in the opposite direction at the time of emission to conserve momentum. Since the direction of emission is not determined until the photon is actually detected, how do we reconcile that with the necessity of a back reaction at the time of emission?

I do not know what the standard answer is to that question. However, here is a possibility. There are quantum fluctuations everywhere all of the time. The magnitude of those fluctuations is small enough not to be detected as a violation of the uncertainty principle. I have not done the calculation yet, but I suspect the back reaction from the emission of one photon is also not detectable without violating the uncertainty principle.

1.7.1 Added 7 March 2015

It is generally agreed that reconciling General Relativity (gravitation) with quantum theory is the most important problem of theoretical physics today. However, to do that, it is first probably necessary to actually have a completely general quantum theory. As far as I know, a completely general quantum theory does not yet actually exist. (If anyone knows differently, please let me know.) I summarize below what I believe does exist in terms of quantum theories.

First, some definitions:

Quantum mechanics or wave mechanics covers the case of a single particle in a prescribed background field. There are some extensions, which I will mention below.

Quantum field theory is a somewhat general theory of the interactions of particles and fields. There are two versions, which I will describe below.

Quantum theory is a completely general theory of the interactions of fields and particles. As far as I am aware, it does not actually exist, but if someone knows differently, I would like to hear about it. There are, however, some general principles that most people would think should be in any reasonable quantum theory. Philosophical problems with quantum theory are discussed, but all of the examples, such as wave function collapse or Schroedinger's cat, are from ordinary quantum mechanics.

Quantum mechanics or wave mechanics: There are three theories:

Schrödinger's equation applies to non-relativistic particles of arbitrary spin. Heisenberg's formulation in terms of matrices has been shown to be equivalent to Schrödinger's equation. Sometimes the term "wave mechanics" is applied to Schrödinger's equation while "quantum mechanics" is used for Heisenberg's formulation, but they are equivalent.

Dirac's equation applies to Fermions (particles whose spin is an odd multiple of 1/2) and is fully relativistic.

The Klein-Gordon equation applies to Bosons (particles with integral spin) and is also fully relativistic.

Extensions:

Quantum mechanics has been extended to apply to an arbitrary number of identical non-interacting particles.

There is also an extension to a curved space-time background, but applications that can actually be solved are limited.

Schrödinger's equation can be applied to two particles that interact by a central force. In that case, the wave equation can be separated into a wave equation for center-of-mass motion and another equation for the relative motion. Application to an electron and a proton leads to the reduced mass for the electron for solutions to the hydrogen atom.

Quantum field theory:

A professor mentioned many years ago that there were two versions of quantum field theory. A very rigorous version that was not able to actually make a calculation, and a non-rigorous version that could make calculations, but that there was some question about its validity. If the situation has changed, I would like to hear about it. Quantum electrodynamics (QED) is in the latter category, as is QCD along with the standard model, electroweak theory, Higgs theory, etc. Renormalization is part of the latter version. Although renormalization works and agrees very well with measurements, there is some doubt about its validity because it requires infinite bare masses and infinite bare charges, even though bare masses and bare charges can never be observed. Also, the theories that can make calculations are perturbation theories, which leads to some questions about validity.

Quantum theory: As I mentioned, as far as I know, there is no general quantum theory, but if anyone knows differently, I would like to hear about it. The quantum N-body problem is discussed in chapter 112.

1.8 The Measurement Problem

Chapters 32, 33, 34, 35, 36, 38, 39, 40, 41, 43, and 76 consider the measurement process, including the problem of measurement in a quantum cosmology.

1.9 Reality or knowledge?

Any investigation into the fundamental nature of quantum theory must consider the question of whether the wave function represents reality or knowledge. Chapter 80 considers this question.

Related to this is the meaning of the Born rule. See, for example, the preprint by Jarlskog [9].

1.10 Zero-point energy

The zero-point energy problem cannot be ignored in any research that deals with fundamental questions in quantum theory and cosmology. Chapters 84 and 107 deal with this problem.

1.11 Nature of the wave function: waves of what?

When we apply the Schrödinger equation to a particle (say, an electron) in an external field, we usually think of the resulting wave function as a wave that represents the electron. We usually interpret the solution in that way, whether the external field is constructed from large apparatus, or is from a central field representing the nucleus of an atom.

However, when we apply the Schrödinger equation to two particles (such as an electron and a proton), we can separate the resulting equation into a wave equation for the center of mass and an equation for the relative position between the electron and the proton, the latter giving the reduced mass as an effective mass for the electron. Because the latter equation treats the electron and proton on an equal basis, and because both the equation and solution are symmetric with regard to the electron and proton, it does not seem appropriate to consider the resulting wave function as a wave function for the electron, but rather a wave function for the internal state of the atom, with the relative position as the independent variable.

It may be because the electron is much lighter than the proton that we usually consider the wave function to belong to the electron. However, the interpretation that assigns the wave function to the system rather than to the electron should apply no matter how heavy the proton. Therefore, even in the case of an electron in an external field is supplied by large apparatus, we should still assign the wave function to the system rather than to the electron.

The next question concerns a free particle, such as an electron. Surely then, we should assign the wave function to the electron, not to the system. However, there are two points to consider. First, there are no truly free particles, in the sense that the particle is interacting with an inertial frame, which is really a stand in for an interaction with all of the masses (energy) of the universe.

Second, there are problems with normalization of the wave function for a free particle, leading to the treatment of a free particle in a different way from a bound particle.

That our usual treatment of quantum mechanics does not recognize a free particle as interacting with anything may be at the crux of the problem between quantum theory and General Relativity (gravitation) suggests that this may be the place where quantum theory needs to be revised. Notice, however, that $\psi^*\psi$ still gives the probability of detecting the corresponding particle. We need to take that into account in our revision of quantum theory.

Whether a wave function represents the particle (an electron) or represents the system, there is still the question of what the waves are made of. Chapter 87 suggests that an electron wave function is an electroweak wave whose components actually include electric and magnetic fields as an electromagnetic wave.

Because there are four fundamental interactions, there may be four fundamental kinds of wave functions that do not interfere with each other. Chapter 108 discusses this problem.

Chapter 111 calculates the refractive index for a particle wave in a gravitational field.

1.12 Actual field fluctuations?

Chapters 90, 114, and 117 propose that the fluctuations calculated in quantum field theory are actual field fluctuations, and that the absolute square of wave functions, when integrated over some volume, give the probability that one quantum of energy for that field is actually in that volume.

Although this idea may apply to Bosons, it is not clear if it should also apply to Fermions.

If we consider a spherical wave (either for a single photon or for a single particle, such as an electron), we can take the components of the momentum normal to the propagation direction to be zero (or very nearly zero). In that case, the position of the photon or particle normal to the direction of propagation is completely (or nearly completely) undetermined. Therefore, when the photon or particle is detected somewhere, there is less mystery about the "collapse of the wavefunction."

1.13 Implicit versus explicit inertia

When we write the Dirac equation in an inertial frame, we are only implicitly using inertia. It would probably be better to include inertia explicitly, if possible. Maybe if we we could write the Dirac equation in the frame of the particle, we have the possibility of using inertia explicitly. Chapter 91 tries to write the Dirac equation in the frame of the electron. In particular, section 91.4 points out that there does not seem to be a closed-form formalism for the Dirac many-body problem. This chapter is not finished.

1.14 Philosophies, interpretations, and models

Chapter 93 contrasts philosophies, interpretations, and models. It argues that "models" is a better description for what is usually called "interpretations." Chapter 88 points out that a wave function applies to an ensemble only. Chapter 94 argues that the wave function in ordinary (first-quantized) quantum mechanics is more like a weather forecast in that it is associated with the ensemble average of fields rather than actual fields. Chapter 37 talks about exclusive alternatives in quantum theory. Chapter 44 asks whether quantum theory applies to the whole universe. Chapter 81 considers fluctuations due to gravitational waves. Chapters 2, 6, 7, 8, 24, 25, and 26 consider the proper way to understand quantum physics.

1.15 Renornalization

It is clear why we need renormalization. That it works so well gives confidence that something about is correct. However, that the correction is infinite and the bare mass and bare charge are both infinite suggests that something about it is not correct.

I suspect that in addition to renormalizing mass and charge, we need to renormalize the whole formula. That is, the bare formula is different from the apparent formula. More specifically, the form of the bare interaction may be different from the apparent interaction.

See chapter 101.

1.16 Spin

The Dirac equation applies to spin-half particles. However, it is not clear where spin enters. There is no term that one can point to that is the spin. For example, I do not know of a generalization to the Dirac equation for arbitrary half-integral spin. Somehow, we would have more confidence in the correctness of the equation if we could apply it to arbitrary spin. The two-component second-order form of the Dirac equation may be closer to having terms for spin, since it contains the Pauli spin matrices.

1.17 Choice of variables

When formulating a theory of gravitation, the choice of variables may be important. For example, using fields instead of potentials avoids taking derivatives, and therefore avoids explicitly using geometry. similarly, an integral form of the field equations may allow expressing the field equations without geometry.

Chapter 86 gives a complete integral form for Maxwell's equations, and tries to give a complete integral form for Einstein's field equations.

Chapter 89 tries to represent General Relativity in terms of gravitational fields rather than potentials (the metric). Chapter 4 also considers using other variables for gravitation.

1.18 Mach's principle

Mach's principle is often misunderstood. In fact, in any group of scientists who profess to believe in some form of Mach's principle, there will be as many opinions about what Mach's principle means as there are scientists in the group. Chapter 30 talks about Machian cosmologies as wave packets. I presented it in Jena in 1980. Chapter 42 talks about the principles in Mach's principle. One of Mach's main points is that only relative coordinates can be determined, and therefore, only relative coordinates should enter physical law. However, it is difficult to formulate General Relativity in relative coordinates. Chapter 69 considers symmetries in relative coordinates.

Chapter 70 gives the essence of Mach's ideas. Chapters 21 and 23 also consider Mach's ideas, and how they might apply in quantum theory.

1.19 Why we do not observe a relative rotation of our local inertial frame and the distant stars.

It is often recognized that some solutions of Einstein's field equations do not satisfy almost anybody's idea of Mach's principle. One obvious example is flat, empty space. That is, Minkowski spacetime. In 1979, I realized that the reason is that without matter, there would be many

Minkowski spacetimes, in relative rotation or relative acceleration with respect to each other. Since there would be no matter to choose one frame over another, they would all have the same probability. We would have all of them at once, since the phase (calculate from the action) would be zero for each of them.

Another example, is with matter, but a relative rotation of matter and inertial frames. I realized at the same time that the frames with relative rotation would cancel each other out by phase interference.

Although we do not have a quantum theory of gravity, we have some ideas of what some of the properties of such a theory might be like, including quantum cosmology. Using some of these ideas, we can try to answer the question of why we do not observe a relative rotation of our local inertial frame and the distant stars. Although this topic is slightly off the main theme, it gives insight into the connection between quantum theory and inertia. Chapters 28, 29, 31, 49, 50, 65, 66, 67, 74, 75, 85, 102, 106, and 113 deal with this aspect of the problem. Chapter 77 derives the no-boundary proposal.

1.20 Why we do not observe relative acceleration of our local inertial frame and the distant stars.

Chapters 115, 116, 118, and 119, deal with this problem. It is because with quantum cosmology, cosmologies with significant relative acceleration of matter and inertial frames would cancel each other out by phase interference in a path-integral calculation.

1.21 Putting QM and GR together

Chapter 122 tries to put quantum theory and gravitation together.

When we think of a wave function for a particle on a background inertial frame, what we really have is a wave function that describes the situation of the relation between that particle and that inertial frame. An inertial frame is not an arbitrary frame. Inertial frames have specific properties. Therefore, the parameters that describe an inertial frame are physical parameters of the system.

We can just as easily consider the coordinates of an inertial frame relative to a particle as coordinates of the particle relative to an inertial frame. In fact, the correct way to think about the situation is in terms of the relative coordinates between the inertial frame and the particle. This may be a different kind of relativity. A relativity between a local inertial frame and a field. The wave function may represent not the field, but the relativity of the field and inertial frame.

It seems that the most promising program to combine quantum theory and gravitation is to alter both quantum theory and General Relativity in very specific ways. These are:

1. Revise General Relativity so that it can be expressed as a field theory as expressed in chapter 89.

2. Revise quantum theory so that instead of being based on an inertial frame as a background, it instead treats the inertial frame as a gravitational field.

1.22 Wave properties of particles seem to be associated with inertia.

Chapter 23 points out that the wave properties of particles seem to be associated with inertia.

Somehow, the wave properties of particles seem to be associated with inertia, but we do not know why. To account for that, we have the Schrödinger equation and the Dirac equation. However, these are just ad hoc equations fixed up to give the correct relationship between energy and momentum. We do not know the meaning of the resulting wave functions, except to calculate amplitudes for processes.

For Maxwell's equations, we do know the meaning of the solutions. They are electric and magnetic fields. But because of Compton scattering of electrons and photons, we have a hint of the meaning of electron wave functions. Compton scattering conserves energy and momentum. However, energy and momentum conservation can be interpreted in terms of wave interference, expressed in terms of frequency and wavenumber. To get wave interference, there must be some correspondence between the corresponding field variables. Thus, at least part of the electron wave function must include electric or magnetic fields.

One thing that would be useful would be to write the Schrödinger or Dirac equation in a form in which the gravitational interaction (including inertia) would explicitly enter.

1.23 A sparse universe

I refer to a "sparse universe" as one in which the connection between the matter distribution and inertial frames could be discovered by experiment. We do not have such a universe, but thinking about the possibility may shed some light on the connection between quantum theory, inertial frames, and gravitation. Chapters 71 and 82 consider sparse universes.

1.24 Other additions still to add

In the Stern-Gerlach experiment, spin projection is analogous to a magnetoionic component of radiowaves in the ionosphere. This realization should help to clarify the meaning of spin.

One thing I need to do is try to think up ways to disprove the main ideas I express in this book. Or, to think up experiments that would contradict those ideas. In that way, either I can show that some of these ideas are wrong, or, if I cannot contradict them, their credence will be enhanced.

1.25 Miscellaneous

Chapter 12 gives a gravitational interpretation of Planck's constant. Chapter 13 gives the motivation for this research. Chapter 14 discusses the use of natural units. Chapter 15 discusses gravitational mass versus inertial mass. Chapter 16 discusses pair production. Chapter 17 points out that Einstein's theory does not contain Mach's principle. Chapter 18 discusses my motivation for this research - connection between quantum theory and gravitation through Mach's principle. Chapter 19 discusses why I feel that the concept of a photon is a useful mathematical device rather than a reality.

Chapter 96 considers the role of inertial frames. Chapter 97 looks at fine-grained versus course-grained histories. Chapter 98 looks at nonlinear acoustic-gravity waves in a fluid. Chapter 99 looks at the situation where both gravitation and quantum theory act like they are both kings on the mountain. Chapter 100 considers various myths in physics. Chapter 103 looks at uncertainty relations. Chapter 105 tries to develop a discrete representation of General Relativity.

1.26 The whole picture

Chapter 122 considers the whole picture.

1.27 Where are we?

Chapter 123 considers where we are.

Chapter 2

Particles in a time well[1]

One possible explanation for why we must go forward in time is that there may be a potential well that carries us forward in time. We, and all so-called "stable" particles, are carried along in that potential well.

The sides of the potential well are not infinitely high. The particles we call "unstable" have energies above the potential well, and are able to move freely back and forth in time. When they cross the potential well, we see them for a short time, until (in our usual interpretation) they "decay."

The above two paragraphs are from memory. I just found the 18-page handwritten essay[2]. It begins by postulating a quantum theory that is similar to that in Leighton's textbook [10], but generalized to 4 spacetime dimensions. That part doesn't seem very interesting.

Then we get to the interesting part on page 6:

Not only does each particle in the system have three spatial coordinates to describe its position, but also a time coordinate. The probability of making a measurement of the time coordinate of a particle and finding it to be within t and $t + \mathrm{d}t$ is the probability of the particle existing in the same time coordinate in which we exist and must make our measurements from. The objects and particles of our ordinary experience are trapped (bound) in a "time well," similar to a square-well potential in ordinary spatial coordinates. This time well is very narrow, however, and allows very little movements. Thus, most of the ordinary objects around us, show a very high preference for a certain time, (the one they are trapped in).

This means that the wave function is sharply peaked at a particular time. The probability of a particle being outside this time is essentially zero, and is one at this time. There are some particles, however, which are not trapped as we are. They can come and go seemingly at will. They are the unstable particles, which decay at some rate. Like hyperons, for instance.

And, from page 10:

If the time coordinate for one of the particles is sharply defined (The particle has a preference for existence at one time over others.), then the corresponding total energy will not be sharply defined. The depth of the well must be deep enough to allow for enough uncertainty in the total energy. If the well is not deep enough, the particle will no longer be confined to that time coordinate, and will escape. We observe particles escaping from our time state frequently. They are the unstable particles, which decay with time. Their wave functions aren't sharply enough peaked to remain trapped or bound in one particular time state.

[1] ca 1962

[2] Now discarded. A short summary is given in section 20.12 in chapter 20

Chapter 3

Proposal for a problem in gravitation (computer program)[1]

This is a single typewritten sheet[2] that proposes a computer program to numerically integrate Einstein's field equations simultaneously with integrating the geodesic equation for each particle in the universe. I thought it might lead to some insight. I didn't know about chaos for nonlinear equations at that time. See section 20.7 in chapter 20.

[1] ca 1967
[2] Now discarded

Chapter 4

Thoughts on gravitation[1]

(before I forget them)

If we consider the inertia of a body to be some sort of gravitational force due to the stars, then what will be the interpretation of energy and momentum? In analogy with E&M, they will be sources of the gravitational field, and also the things which the gravitational field acts on. Energy will be like charge, and will be the source of a scalar field. Momentum will be like currents, and will be the source of a vector field. Current opinion says that a scalar-vector field won't work for gravitation, that a tensor or scalar-tensor field is needed. Even so, energy and momentum will still be the source.[2] Maybe E&M also needs a tensor theory. See section 20.10 in chapter 20.

We have measured the group velocity of light, but have we measured the phase velocity? We usually assume them to be the same in free space; however, it may be that a gravitational field is dispersive. We could calculate the phase velocity, if we could simultaneously measure the frequency and the wave length. See section 20.10 in chapter 20.

4.1 Quantum mechanics

One problem with unifying quantum mechanics and relativity is that the uncertainty principle is built in to present QM theory, but is inconsistent with relativity, in that the speed of light is assumed certain. See section 20.9 in chapter 20.

The basis of QM is that a wavelength is associated with any momentum, and a frequency is associated with any energy. That is, waves are associated with the sources of gravitation. Notice that we have not observed waves to be associated with the sources of any of the other fields including E&M. This indicates that QM is a gravitational phenomenon. See section 20.3 in chapter 20.

Electrons have an intrinsic magnetic moment. This acts as a source in E&M, a little magnet. Electrons also have an intrinsic angular momentum. If momentum is a source of a gravitational field, then an angular momentum will be a source of gravitational field, a gravitational magnet. Then the orbital angular momentum of an electron in an atom will also be the source of a gravitational field, and there should be a gravitational interaction between these two. A gravitational spin-orbit coupling. This should be easy to calculate, and I plan to do this.[3] See section 20.9 in chapter 20.

[1] ca 1967

[2] note added in October 1979: $T^{\mu\nu}$ is the source.

[3] Note added in 2008: I have not made that calculation.

Chapter 5

On the application of Maxwell's equations to a positively-charged universe[1]

In a paper "On the origin of inertia" in 1953[11], D. W. Sciama showed that non-relativistically at least, the inertia of a body could be due to a vector gravitational potential of the stars, in analogy to the vector electromagnetic potential. Although there seem to be no observations to choose this theory over Newton's formulation, this theory does have some advantages.[2]

1. There would be no difference between the stars being fixed and the Earth rotating, and the Earth fixed with the stars revolving about the Earth.

2. All coordinate systems are acceptable reference frames, including accelerating and rotating frames.

3. You don't need to add in ficticious inertial forces when using non-inertial frames. These forces would be real forces due to the movement of the stars.

4. This theory explains why a coordinate system at rest relative to the fixed stars is an inertial frame.

5. The equivalence of inertial mass and passive gravitational mass is a consequence of the theory.

6. The attractiveness of the gravitational force is a consequence of the theory.

7. The gravitational constant is shown to be actually an inertial constant which depends on the average density of the universe.

8. What appears to be the intrinsic inertia of a body is shown to depend more on distant masses than on close masses.

If this theory is true, then it will have profound implications in all of physics. In particular, it will have an effect on quantum mechanics, since all three quantum mechanics equations, Schrödinger's equation, the Dirac equation, and the Klein-Gordon equation, are based on the concepts of kinetic

[1] ca 1967. See section 20.5 in chapter 20.

[2] Note added in 2008: Comparing with Newtonian theory may have been unfair. General Relativity also solves many of the above problems.

energy and momentum, both of which are concepts based on the idea of intrinsic inertia.[3] To find out what these implications are, I want to first investigate the concepts of energy and momentum in light of the origin of inertia.[4] Energy and momentum can be looked at in two ways (maybe even more).

1. Energy and momentum are good to have. We can run machines and build things and run our cars with energy. Momentum is also good to have, because for a given amount of energy, the more momentum we have, the more ordered and correlated our energy is and therefore the more useful it is according to the second law of thermodynamics. (A laser is often more useful than a light bulb of the same power.)

2. Energy is something which is conserved in force fields which depend only on position. Therefore, it is a useful concept to use in solving problems to find say, speed as a function of position. Momentum is also conserved in some cases, so it is also a useful concept in solving problems.

I am going to use the second way of looking at energy and momentum primarily. To do this, I am going to use an example of a positively charged universe. When Sciama made his calculations, he just used E&M equations and applied them to masses. His result, that the force is attractive should also apply to protons in a positively charged universe or to electrons in a negatively charged universe. This really sounds strange, that protons could attract one another electrically. I'm not considering nuclear forces or weak interactions here.

Consider the following example. Take all of the electrons out of the universe but a few for experimenting. Now we have a universe that has a high positive charge density. If we now write the equations of motion for a few electrons and protons, we can still formally write $F = ma$, but when we do this, not only will all of the gravitational forces be negligible, but so will the inertia of these bodies compared to the E&M inertial induction. Therefore, we can really neglect gravitation and write our equation of motion as $F = 0$. When we do this, we will find the following qualitative results.

1. Two protons will attract one another with an acceleration proportional to $1/r^2$.

2. Two electrons will repel in the same way.

3. An electron will be attracted by a proton, but a proton will be repelled by an electron, with the result that the two will accelerate in the same direction with the proton in the lead and the electron following.

This last result really sounds strange. What has happened to conservation of energy and momentum? Let's make some analogies with gravitation. If we had only protons, then we could talk of a kinetic energy $\frac{1}{2}ev^2$ and a pontential energy e^2/r . When we consider electrons too, its like something with a negative mass. It's kinetic energy would be $-\frac{1}{2}|e|v^2$, and the potential energy would be the same as we now use. It would also have a momentum $-|e|\mathbf{v}$ compared with the proton's momentum $+|e|\mathbf{v}$. Thus, the case of the proton and electron moving off in the same direction does conserve momentum and energy. It's a strange situation. The protons like each other, but hate the electrons. The electrons hate each other, but like the protons. Problem solving will work fine using these extended concepts of momentum and energy. You should remember, however, that at least in this case, we know that the origin of inertia and momentum and kinetic energy is a vector potential of the charged stars. If the electron and proton that are moving away together get going

[3] The results of these equations in the regions where they apply won't change, but the form of the equations may change. Some sort of correspondence principle should apply.

[4] They are sources of gravitational fields.

fast enough, their gravitational inertia might get big enough to be noticible. In order to conserve momentum and energy, the electron and proton would move slightly together. Could this be a Lorentz contraction?

Chapter 6

Quantum explanation of polaroid[1]

- How many photons per second are emitted by a one milliwatt 500 kHz transmitter? About 10^{25} .
- What is the average distance between photons? About 10^{-14} cm.
- Compare this to a wavelength. 6×10^4 cm.
- What power should we use to have one photon per wavelength? About 10^{-22} Watts.
- What is the polarization of a photon? (linear, circular, or what?) Answer: anything.
- What is the effect of polaroid on a single photon?

The answer to the last question is more difficult. It seems that it is the following:

A free photon is a single plane wave with a certain amplitude that is independent of position.[2] With the polaroid, the wave function is a constant on each side of the polaroid, but changes abruptly at the polaroid. Thus, we can take the amplitude of the photon wave function to change from A_1 to A_2 at the polaroid. We take the polarization of the initial wave and break it into two components, one of which is parallel with the polaroid, and one perpendicular. Call these $C_\parallel$ and $C_\perp$. Then $A_2 = A_1 C_\parallel$. The polarization of the photon after passing through the polaroid is now the same as the polaroid.

Similarly, what happens when a single linearly polarized photon is incident on a bi-refringent medium? The photon splits into two waves, each with amplitude proportional to the component of polarization of the original photon. Thus, a single photon then exists as not a single plane wave, but a superposition of two plane waves traveling at different phase velocities, each with its own characteristic polarization.[3] Of course, we can also look at it as the photon must choose one or the other and does so on the basis of probability.[4]

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[1] ca 1969

[2] Note added in 2008: Actually, it could also be a wave packet of any shape, although it partly depends on the definition of a photon.

[3] Note added in 2008: A single photon can be split and be recombined later or not depending on what kind of measurement is made.

[4] The action of polaroid is a macroscopic phenomenon. An EM wave interacts with a single atom. See section 20.11 in chapter 20.

Chapter 7

Interaction can cause wave packets[1]

Free electrons and photons are both monochromatic plane waves and they fill space completely, having a wave amplitude which is independent of position. In both cases, it is possible to make wave packets in both time and space by superimposing individual electrons or photons with different frequencies or different directions, thus having a frequency spectrun or angular spectrum of plane waves. However, it is also possible to make a wave packet with just one electron. An example is the track of an electron in a bubble chamber. The track is caused by collisions of a charged particle (the electron) with the hydrogen molecules, which ionizes the hydrogen atom and produces the bubble. After each collision, the electron loses a little energy, and thus its frequency is lowered. Thus, the electron in this case, doesn't have a single frequency, but a spectrum of frequencies, and this is a wave packet in time.

In addition, each collision will not be exactly head on, but some may push the electron to one side, and some to the other, thus giving also an angular spectrum of plane waves and thus, also a wave packet in space. Thus, the electron in such a case behaves like a classical particle, being localized in time and space. In such cases, the interference of a particle with itself essentially makes it a classical particle. We know from experience that in calculations, we treat the particle as a classical particle, and don't take wave properties or quantum properties into account in calculating trajectories. Presumably, we could also make a wave packet out of a single photon.

[1] ca 1969. See section 20.12 in chapter 20.

Chapter 8

The Tao-Te-Ching and path-integrals[1]

The Tao-Te-Ching must have been written by someone who knows about the path-integral formulation of quantum mechanics. Tao means "the way." A generalization of the path integral formulation of quantum mechanics is to consider not just "all paths," but to consider "all ways." Just reading the Tao-Te-Ching convinces me that the writer knew about the path-integral formulation of quantum mechanics.

1. . . . Therefore let there always be non-being, so we may see their sublety,
   and let there always be being, so we may see their outcome.
   The two are the same,
   But after they are produced, they have different names.

2. . . . being and non-being produce each other;
   . . .
   Therefore the sage manages affairs without action
   and spreads doctrine without words.

3. By acting without action, all things will be in order.

4. Tao is empty (like a bowl).
   It may be used but its capacity is never exhausted.
   It is bottomless, perhaps the ancestor of all things.
   . . .
   It seems to have existed before the Lord.

5. Heaven and Earth are not humane.
   . . .
   The sage is not humane.
   . . .
   How Heaven and Earth are like a bellows!
   While vacuous, it is never exhausted.
   When active, it produces ever more.

6. The spitit of the valley never dies
   . . .
   [It] is the root of Heaven and Earth.
   It is continuous, and seems to be always existing.
   Use it and you will never wear it out.

[1]ca 1970. This is the original handwritten version that I found after 2008

7. . . .

8. . . .

9. . . .

10. . . .

11. Utility of non-being, advantage of being

12. . . .

13. . . .

14. . . . Hold on to the Tao of old in order to master the things of the present.
From this, one may know the primeval beginning (of the universe)
This is called the bond of Tao.

15. evolution

16. . . .

17. rulers

18. . . .

19. . . .

20. folk wisdom

21. form, beginning

22. . . .

23. Nature says few words.

24. . . .

25. There was something undifferentiated and yet complete,
which existed before Heaven and Earth.
. . .
nature

26. . . .

27. . . . A good reckoner uses no counters.
. . .
A well-tied knot needs no rope and yet none can untie it.

28. . . .

29. . . .

30. . . . (For) after things reach their prime, they begin to grow old,
which means being contrary to Tao.

31. . . .

32. - even application of physical law
    - beware of analysis

33. . . .

34. . . .

35. . . .

36. . . .

37. Tao invariably takes no action, and yet there is nothing left undone.

38. . . .

39. . . . evolution - survival of fitest

40. . . . And being came from non-being.

41. . . . When the lowest type of men hear of Tao,
    They laugh heartily at it.
    If they did not laugh at it, it would not be Tao.

42. Tao produced the one.
    The one produced the two.
    The two produced the three.
    And the three produced the ten thousand things.

43. . . . Non-being penetrates that in which there is no space.
    . . .
    And the advantage of taking no action.

44. . . .

45. . . .

46. . . .

47. One may know the world without going out of doors.
    One may see the Way of Heaven without looking through the windows.

48. . . .

49. . . .

50. . . .

51. . . .

52. . . . beginning = mother, sons = things

53. . . .

54. . . .

55. . . . whatever is contrary to Tao will soon perish.

Chapter 9

Gravitation and wave mechanics[1]

- This problem is similar to the generalization of classical mechanics, to special relativity, wave mechanics, or quantum mechanics in that in each case a new parameter is added to describe the system (mean density of the universe, speed relative to that of light, wave-length, or size of smallest particle).[2] However, this problem is unlike the previous 3 in 2 ways:

  1. In the classical mechanics limit, we have $v/c = 0$, $\lambda = 0$, or $h = 0$. Here we have $\rho_U \neq 0$.
  2. We cannot change ρ_U or h. We can change v/c and λ.
     Show that

$$-\nabla\phi - \frac{\partial A}{\partial t} + v \times \nabla \times A = 0 \tag{9.1}$$

in the frame of the particle leads to the Schrödinger equation in the center-of-mass frame.

$$\phi = c^2 \int \frac{\rho}{r}\,\mathrm{d}V = \hbar \int \frac{\rho_\omega}{r}\,\mathrm{d}V \tag{9.2}$$

$$A = c^2 \int \frac{\rho v}{r}\,\mathrm{d}V = \hbar \int \frac{\rho_k}{r}\,\mathrm{d}V \tag{9.3}$$

Let E be the energy of a particle. Then

$$E = \phi + \frac{A^2}{2\phi}\ . \tag{9.4}$$

$$E^2 = \phi^2 + A^2 = \text{ constant.} \tag{9.5}$$

$$2E\frac{\mathrm{d}E}{\mathrm{d}t} = 2\phi\frac{\mathrm{d}\phi}{\mathrm{d}t} + 2A\frac{\mathrm{d}A}{\mathrm{d}t} = 0\ . \tag{9.6}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} \approx \frac{A}{\phi} \cdot \nabla\phi\ ? \tag{9.7}$$

$$0 = A \cdot \nabla\phi + A \cdot \frac{\mathrm{d}A}{\mathrm{d}t}\ ? \tag{9.8}$$

Let E_m be the total energy of particle m. Let p_m be the momentum of particle m. Then

$$E_m^2 = \phi_m^2 + A_m^2\ , \tag{9.9}$$

[1]Essay number 6, ca April 1971

[2]See section 20.1 in chapter 20.

where

$$\phi_m = \sum_{i \neq m} \frac{E_i}{r_{im}} \tag{9.10}$$

$$A_m = c \sum_{i \neq m} \frac{p_i}{r_{im}} = c p_m \tag{9.11}$$

- There is something wrong. We want the calculations for a particular particle to be made in a coordinate system fixed on it.

- Einstein - space has no intrinsic physical properties. Material bodies in the space give it properties.[3]

- It is clear that the speed of light is determined by the density of stars in the universe, since these stars determine an inertial frame and the speed of light is with respect to an inertial frame.[4]

- D and E are directly electromagnetic quantities since they are defined in terms of charges and currents. E and B, however, are defined in terms of forces and acceleration of charges. E and B are therefore dependent on the inertia of an electron and therefore on the density of stars in the universe. Therefore, ϵ_0 and μ_0 are constants that depend on the density of stars in the universe. ϵ_0 is proportional to ρ_0 and μ_0 is proportional to $1/\rho_0$. Thus, the speed of light c is independent of ρ_0. However, I don't really believe this because of my previous paragraph.[5]

- For c to change, we need a change in the product of μ_0 and ϵ_0 or the ratio of B to E, or the ratio in inertia, parallel and perpendicular to the path.[6]

  Maxwell's equations are for an inertial frame. We need to first generalize them to a non-inertial frame and then recognize the contribution of the stars.

- In calculating the gravitational interaction of a star, we need to include both the rest mass and kinetic energy. But, if the kinetic energy is really itself just due to the inertia it has because of its gravitational interaction with the other stars, it sounds very confusing.[7]

[3]See section 20.1 in chapter 20.
[4]See section 20.4 in chapter 20.
[5]See section 20.4 in chapter 20.
[6]See section 20.4 in chapter 20.
[7]See section 20.4 in chapter 20.

Chapter 10

Motivation[1]

The ideas I have are the following:

1. It seems clear to me that, regardless of the mathematical details, the main hypothesis of Sciama's paper, [12] that the inertia of a body is not intrinsic but due to a gravitational interaction with the stars, is correct.

2. It then follows that the momentum and kinetic energy of a body (being inertial quantities) are really manifestations of a gravitational interaction of that body with the stars.

3. Wave mechanics is based completely on the wave properties observed to be associated with momentum (de Broglie wavelength) and kinetic energy ($E = h\nu$). All 3 basic equations of wave mechanics (Schrödinger, Dirac, Klein-Gordon) are set up to give the correct (de Broglie) wavelength and frequency when applied to a free particle.

4. But if momentum and kinetic energy are gravitational quantities and if wave mechanics is based on the wave properties associated with momentum and kinetic energy, then wave mechanics should have a strong theoretical connection with gravitation.

5. No such theoretical connection is now known. I want to try to discover what that connection is, that is, to discover a theory which explicitly contains such a connection and which (obviously) agrees with the observed world.

6. Such a theory should be able to answer questions such as the following, and studying such questions may lead to the correct theory:

   If the mean density of the universe were half what it is now, then the inertial mass of the electron would be half what it now is. What would be the effect on various physical constants? Considering the radius of the first Bohr orbit in the hydrogen atom, there are several possibilities:[2]

   (a) Planck's constant h, the fine-structure constant α, and the speed of light c remain the same and the radius of the first Bohr orbit a_0 doubles.
   (b) a_0, h, and c remain the same and α doubles.
   (c) a_0, h, and α remain the same and c doubles.
   (d) a_0, α, and c remain the same and h is halved.
   (e) something more complicated.

[1] Essay number 7, August 1971

[2] See section 20.6 in chapter 20.

I doubt that (a) is true. Probably (e) is true, in which case it will be necessary to consider more explicit possibilities for (e). I intuitively suspect that both c and α will change. I think that c will increase because I suspect that the speed of light is not an intrinsic upper limit on particle speeds but rather a limit somehow imposed by the gravitational interaction of the particle with the stars. I doubt that α is a fundamental constant of nature because it is such a strange number and no one (including Eddington) has discovered a good mathematical reason for such a number.

I am hoping that looking at all of the relevant equations in a systematic manner will reveal a possible theoretical connection between gravitation and wave mechanics.

Chapter 11

Thoughts on the origin of inertia[1]

I think I've found the key to difference between electromagnetic forces and gravitational forces. It's based on this. We can't tell the difference experimentally between something that is true in general for all nature, and something that is true only for us because of the distribution of mass in the universe for instance. In such a case, the choice should be made on the basis of which gives the simpler and more constant theory.

Not only rest mass, but also kinetic energy is a source of a gravitational field.[2] But what is kinetic energy? It is a measure of the amount of work required to get it up to a certain velocity. But this work is done against inertia. Now, if inertia is a result of the action of the star field, then kinetic energy is a result of the action of the star field. Then also, the gravitation which seems to have kinetic energy as its source may be due to the action of the star field. If this is true, then the difference between E&M and gravitation may not be theoretical but only due to the fact that there are many stars in the universe, which have mass but are not charged.[3]

The gravitational effect of light is completely due to its kinetic energy. Is this also due to the star field? It seems to be an inertial property that bodies with finite rest mass cannot reach the speed of light and that light can travel only at the speed of light. Is it possible that the speed of light is not a theoretical constant, but only a parameter determined by the mass distribution of the stars?[4]

[1] Essay number 8, ca 1971-1972

[2] See the discussion by [13, Weinberg, pp. 124-125 & p. 171].

[3] See section 20.8 in chapter 20.

[4] See section 20.4 in chapter 20.

Chapter 12

A gravitational interpretation of Planck's constant, or The physical significance of Planck's constant, or Eliminating Planck's constant from physical law.[1]

12.1 Introduction

12.2 Inertia is a gravitational effect

Sciama[12, Sciama, 1953] proposed that the inertia of a body (that is, its resistance to change in motion) is due to a gravitational interaction of the stars. H made a non-relativistic calculation, in which he assumed that gravitation contained both an electric-like and a magnetic-like interaction. He showed that the gravitational constant G is not a measure of the strength of the gravitational interaction, but is instead a measure of the distribution of mass and energy in the universe. He derived the following approximate formula

$$G = \frac{1}{2\pi\rho\tau^2} , \tag{12.1}$$

where ρ is the mean density of mass-energy in the universe, τ is the inverse Hubble constant, and the Hubble constant gives the rate of expansion of the universe. The exact formula depends on th particular cosmological model used for the mass-energy distribution in the universe.

He made a calculation of the mean density of the universe using the known values of G and τ. He got 10^{-27} grams/cm^2, somewhat higher than the usual observational estimates of his day of 10^{-30} grams/cm^2 but is in fairly good agreement with present day estimates.

Davidson [14, 1957] showed that Einstein's General Relativity is consistent with this view. He made a similar calculation, and got the following value for the gravitational constant.

$$G = \frac{3}{4\pi\rho\tau^2} , \tag{12.2}$$

[1]Essay number 9, ca 1972; compare with the summary in section 20.12 in chapter 20. I will add the rest of this essay if I get time.

Davidson's result agrees with Sciama's except for a factor of 2/3. The difference between Davidson's and Sciama's formulas could be due to the difference in cosmological models. Sciama's formula was only rough estimate, in any case.

The main point is that G is really a measure of the mass and velocity distribution of the universe. If this point of view is taken, then energy, momentum, angular momentum, kinetic energy, rest mass energy, and gravitational potential energy are all gravitational effects. Energy associated with other types of interactions (like electromagnetic) then becomes associated with the combined interactions of gravitation and this other type of interaction. To make full use of these concepts, one must completely reorient one's thinking.

12.3 Wave properties of gravitational quantities

A quantum of light with an angular frequency ω has energy

$$E = \hbar\omega \ , \tag{12.3}$$

where

$$\hbar = \frac{h}{2\pi} = 1.05443 \times 10^{-34} \text{ Joule seconds} \tag{12.4}$$

is Planck's constant divided by 2π.

Chapter 13

Motivation - 2[1]

One statement of Mach's principle is that the inertial properties of matter are not intrinsic, but are due to a gravitational interaction with the rest of matter in the universe. Briefly, Mach's principle is based on the assumption that only relative motion of matter is observable and acceleration of a body relative to "absolute space" has no meaning. If only relative motion is relevant, then all coordinate systems should be equally valid including non-inertial frames. In non-inertial frames, however, it is necessary to introduce so-called inertial forces having no apparent origin. The observation that the stars seem to accelerate relative to non-inertial frames suggests that the inertial forces are related to the motion of the stars relative to those frames. Moreover, an explicit interaction must exist to relate the motion of the stars to the inertial forces, and the observed equivalence of inertial and gravitational mass suggests that this interaction is gravitational. D. W. Sciama explains this argument more fully in his paper, "On the origin of Inertia" in the Monthly Notices of the Royal Astronomical Society, 1953 [12].

I feel that a theoretical relation should exist between gravitation and quantum theory because inertial mass plays a fundamental though obscure role inthe equations governing quantum processes (e.g. the Dirac equation, the Klein-Gordon equation, and the Schrödinger equation all contain the inertial mass of the particle to which the equation is applied.). If, as argued in the preceding paragraph, the inertial of a particle is not intrinsic, but determined by a gravitational interaction with the rest of matter in the universe, then quantum theory should explicitly contain that interaction.

This problem has so aroused my curiosity that I very much want to find such a theoretical connection between gravitation theory and quantum theory.

[1] Essay number 10, ca January 1973

Chapter 14

The use of natural units[1]

In making calculations when the form of the laws of physics is either known or assumed, then it is useful to use "natural" units in which certain universal constants have the value of one. In addition to the usefulness in calculations, this practice is further supported in theoretical work by the premise that only non-dimensional quantities such as the fine structure constant or the rations of various fundamental masses, frequencies, or lengths are really measureable. That the speed of light, for instance, can not be measured by itself, but only non-dimensional expressions containing the speed of light can be measured.

If we consider the present laws of physics to be only an approximation to nature, and if in trying to construct better approximations, we want to consider possibilities in which the speed of light, for instance, is not a universal constant, then the use of natural units would be a hindrance for two reasons. First, if the speed of light were not a universal constant, but depended, for instance, on the mean density of matter in the universe, then the speed of light would be an expression rather than a number. Using a system of units not explicitly containing c as a parameter would make it impossible to substitute trial expressions for c. The second reason is more psychological. The use of the laws of physics expressed in a form not explicitly containing the speed of light discourages thinking about the speed of light as a concept, and excludes even considering the possibility that the speed of light is not a universal constant.

In extending the laws of physics to better approximate nature, one of the processes is that of generalizing the old laws by introducing a new parameter such that the generalization reduces to the old laws for suitable limiting values of the new parameters. This proxess may very likely be impeded by using a form of the laws of physics having few parameters. It may be useful to use MKS units for instance because they contain parameters for the permittivity and permeability of free space, which may not be universal constants.

[1] Essay number 11, ca 1973; compare with the shorter summary in section 20.1 in chapter 20.

Chapter 15

Gravitational mass[1]

It is observed that gravitational mass and inertial mass are equivalent. Actually, it only observed that they are proportional. In fact, if Sciama's (1953) paper, "On the origin of inertia"[12] is correct in ascribing gravitational mass as being fundamental and inertial mass as being due to a gravitational interaction with the rest of matter in the universe according to Mach's principle, then the proportionality constant is not really a constant, but depends on the mean density of mass in the universe.

If then gravitational mass and inertial mass are not interchangeable, then it becomes useful to define a quantity called gravitational mass separately from the quantity inertial mass. It is useful to do this for two reasons: First, so that in talking about the two kinds of mass we can distinguish between the two in both concept and in equations. Second, to change our "mind set" so that it is natural to distinguish between the two.

We shall define ordinary mass as being inertial mass. Just as inertial mass is defined in terms force and acceleration, we can define gravitational mass in terms of Newton's law of gravitation. I shall use the symbols M_G or m_G for gravitational mass and the symbol ρ_G for gravitational mass density. I define a "grav" as the unit of gravitational mass in the cgs units in the following way: The force of attraction between two bodies in dynes is given by

$$F = \frac{M_G m_G}{r^2} \ , \tag{15.1}$$

where M_G and m_G are gravitational masses of the two bodies in gravs and r is the distance between the two bodies in centimeters. Comparing with Newton's second law,

$$F = ma \ , \tag{15.2}$$

we find that one grav is one cm dyne$^{1/2}$ or one cm$^{3/2}$ gm$^{-1/2}$ sec^{-1}. Comparing with the usual form of Newton's law of gravitation involving usual (inertial) masses

$$F = \frac{Mm}{r^2} \ , \tag{15.3}$$

we must have

$$m_G = m\sqrt{G} \ . \tag{15.4}$$

Thus, a body having a gravitational mass of one grav or one cm dyne$^{1/2}$ has an inertial mass of about 3.88 kilograms. Of course, we expect that this ratio is not fundamental, but should depend on cosmological parameters such as the mean density of gravitational mass in the universe.

[1] Essay number 12, ca 1973; compare with shorter summary in section 20.7 in chapter 20.

Chapter 16

Pair production[1]

Let us consider pair production or annihilation for an electron-positron pair. Let us consider the simplest case, in which the orbital angular momentum of the system is zero. For calculation, we shall consider the case of production, but the case of annihilation is just the time reverse of the system. Then, the total energy of the electron-positron system, including rest energy, kinetic energy, and potential energy is

$$2\hbar\omega - \frac{e^2}{2x} = \alpha \ , \tag{16.1}$$

where $\hbar\omega$ is the rest-mass energy plus kinetic energy of each particle, electron or positron, $2x$ is the separation of the electron and positron, and α is the total energy of the system, which is a constant. Solving for ω gives

$$\omega = \frac{e^2}{4\hbar x} + \frac{\alpha}{2\hbar} \ . \tag{16.2}$$

The group velocity of the electron or the positron is

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = v_g = \frac{c^2 k}{\omega} = c\sqrt{1 - \frac{\omega_m^2}{\omega^2}} = c\frac{\sqrt{\omega^2 - \omega_m^2}}{\omega} = c\frac{\sqrt{\left(\frac{e^2}{4\hbar x} + \frac{\alpha}{2\hbar}\right)^2 - \omega_m^2}}{\frac{e^2}{4\hbar x} + \frac{\alpha}{2\hbar}} = c\frac{\sqrt{\left(\frac{e^2}{4\hbar} + \frac{\alpha x}{2\hbar}\right)^2 - \omega_m^2 x^2}}{\frac{e^2}{4\hbar} + \frac{\alpha x}{2\hbar}} \ , \tag{16.3}$$

where ω_m is the rest frequency of the electron or positron. Consider a coordinate system in which x is the position of the electron, $-x$ is the position of the positron, and pair production occurs at $t = 0$ and $x = 0$. Then, we can find the trajectories of the electron and the positron by integrating (16.3) to give

$$ct = \int \frac{\frac{e^2}{4\hbar} + \frac{\alpha x}{2\hbar}}{\sqrt{\left(\frac{e^2}{4\hbar} + \frac{\alpha x}{2\hbar}\right)^2 - \omega_m^2 x^2}}\,\mathrm{d}\,x = \int \frac{e^2 + 2\alpha x}{\sqrt{\left(e^2 + 2\alpha x\right)^2 - 16\hbar^2\omega_m^2 x^2}}\,\mathrm{d}\,x$$

$$= \int \frac{e^2 + 2\alpha x}{\sqrt{\left(4\alpha^2 - 16\hbar^2\omega_m^2\right)x^2 + 4e^2\alpha x + e^4}}\,\mathrm{d}\,x = \int \frac{\gamma + x}{\sqrt{\delta^2 x^2 + 2\gamma x + \gamma^2}}\,\mathrm{d}\,x \ , \tag{16.4}$$

where

$$\gamma \equiv \frac{e^2}{2\alpha} \ , \tag{16.5}$$

$$\delta \equiv \sqrt{1 - \beta^2} \ , \tag{16.6}$$

[1] Essay number 13, ca 1973-1974, updated in August 2022; compare with shorter summary in section 20.11 in chapter 20.

and

$$\beta \equiv \frac{2\hbar\omega_m}{\alpha} \ . \tag{16.7}$$

We can integrate (16.4) in closed form to give

$$ct = \frac{\sqrt{\delta^2 x^2 + 2\gamma x + \gamma^2}}{\delta^2} - \frac{\gamma}{\delta^2} - \frac{\gamma\beta^2}{\delta^3} \ln \frac{\delta^2 x + \gamma + \delta\sqrt{\delta^2 x^2 + 2\gamma x + \gamma^2}}{\gamma + \delta\gamma} \ , \tag{16.8}$$

where the constant of integration has been chosen so that $t = 0$ when $x = 0$.

For small x, (16.4) or (16.8) gives

$$ct \approx x \ . \tag{16.9}$$

That is, the electron and positron are each moving at about light speed at the time of pair production. Notice also from (16.2) that $\omega \to \infty$ as $x \to 0$. This unphysical effect results from considering the electron and positron as point particles and calculating classical trajectories.

A more realistic result would come by using the Schrödinger equation to treat the two-particle case. In addition, it would be necessary to allow the orbital angular momentum to be nonzero. An accurate result could come from using S-matrix theory for scattering of a photon by an electron and then converting that result analytically by exchanging the x and t axes to get the result for pair production. This is one of the standard methods of doing the calculation.

Chapter 17

Einstein's theory does not contain Mach's principle[1]

Einstein seems to be satisfied that his theory satisfies Mach's principle because the equations have the same form in all reference frames and because matter does induce inertial effects in his theory. However, his equations predict that as we continue to reduce the amount of matter in the universe, that we will continue to have inertial frames as we know them up until the limit when there are only a few or no bodies left. I would say such behavior is non-Machian, and I think Mach would have agreed. Mach never accepted General Relativity, perhaps for that reason.

I would expect that as the number of bodies in the universe decreases until there are only a few left, that we would experience something completely different from ordinary inertial frames. Thus, although General Relativity may be an excellent approximation when there are many bodies in the universe (when the proper boundary conditions are applied), it probably does not apply otherwise.

I suspect that a more general theory that applies also to a small number of bodies will properly include quantum effects and may indicate the proper boundary conditions to apply to General Relativity when there are a large number of bodies present.

Hoyle and Narlikar have such a theory, but it tries to satisfy Mach's principle by letting rest mass be determined by distant matter. Thus it neglects inertial effects of distant matter on light.

[1] Essay number 14, ca 1974; maybe this is summarized in the first point in section 20.4 in chapter 20.

Chapter 18

Motivation for this research - connection between quantum theory and gravitation through Mach's principle[1]

Mach's principle states roughly that the inertial properties of a body are determined (or affected) by the distribution of matter in the universe. The interaction causing the effect is probably gravitational, and it is now known that Einstein's equations are at least consistent with some form of Mach's principle. Since kinetic energy and momentum can be considered to be the main manifestations of inertia, it appears very likely that the momentum and kinetic energy (and perhaps rest-mass energy) of a body depend not only on its motion but also on the distribution of matter in the universe.

In addition, the wave properties of a particle depend onits momentum and energy. If the latter in turn depend on the distribution of matter in the universe, then either the wave properties of particles depend on the distribution of matter in the universe or Planck's constant depends on the distrubution of matter in the universe, or some combination of both.

Following from the above rather vague conjectures, a whole series of questions arise. Do the size of Bohr orbits and the energy levels in the hydrogen atom depend on the distribution of matter and mean density in the universe? Is it possible that as the universe evolves that the spectra of atmos are changing so that "old" radiation (coming from distant parts of the universe) emitted when the universe was probably denser shows a red or blue shift? Clearly, the observed red shift would be a combination of that effect plus that due to expansion. Is there an observation that would separate these two effects?

An endless series of such questions come to mind. To shed any light on the matter (and to find out whether such investigation is even worthwhile) requires calculating the observable effect (if any) of some specific conjecture like, "Is Planck's constant proportional to the mean density of the universe?". In many cases the consequences of such conjectures will clearly disagree with observation. In others, the consequences may be no different from those predicted by present theories, thus allowing one to choose a theory onthe basis of philosophy. For some conjectures, it is likely that the consequences will differ from those of present theories in only subtle ways, thus pointing to a small series of careful observations to select which theory agrees.

Because I am so curious, and to shed some light on these questions, I would like to make a series of such calculations.

[1]Essay number 15, ca 1974; summarized in section 20.3 in chapter 20.

Chapter 19

Why I feel that the concept of a photon is a useful mathematical device rather than a reality[1]

If I understand the definition of a photon, it is a monochromatic electromagnetic wave quantized in the usual way. If it is really monochromatic, then by the uncertainty principle, it must have always existed in the past and will always exist in the future. Thus, it cannot interact, and will never be observable. Clearly, any radiation existing in such a form will have no effect on us. In other words, virtual photons can be observed, but real photons, if they exist, can have no effect on the observable world.

When time-dependent perturbation theory is used to give approximate solutions to the time-dependent Schrödinger equation, the first-order perturbation is interpreted as a single photon interaction and the second-order perturbation is interpreted as a two-photon interaction. It is clear (at least to me) that in this case, the use of photons and their accompanying Feynman diagrams represents the mathematical approximations to the differential equation, and not reality.

I propose that the same is true also for the relativistic case and to all situations where quantum electrodynamics is applied; that the photon picture of the interaction with the Feynman diagrams represent terms in an approximate solution to some differential equation. The difference is, that for these cases, we don't know what that differential equation is yet.

[1]Essay number 16, ca April 1974; summarized in section 20.11 in chapter 20.

Chapter 20

Ideas on Cards[1]

abstract

In the 60s and 70s, I would type ideas onto 3 by 5 cards. I now have many of these cards, and if I get time, I shall try to scan and ocr them into this chapter. I organized the cards into categories, such as Mach, gravitation, E&M, QM, Mechanics, or Main topic. Sometimes the ideas combined two of those categories, so I also had cards listed by two categories.

Some of the ideas are not original. Some I no longer believe, but they helped me focus my ideas.

20.1 Mach

- Mach - Time can be measured only by position.

  We can also measure time by the decay rate of unstable particles.

- Russell misunderstands Mach's comment that the universe is only given once. See page 284 in the Science of Mechanics 6th English edition (1960)[15].

- Mach - Effects of cosmology on local physics

- Relevance of generalizations of Mach's principle in theoretical physics: Once we have accepted the possibility of a direct relationship between distant matter and the inertial properties of local matter suggested by Mach's principle, we have allowed our minds the freedom of considering other relationships between local properties and distant matter. This series of papers considers the possibility that the local wave properties of matter are strongly connected with distant matter as a direct extension of Mach's principle using philosophical arguments relating to the interpretation of the de Broglie relations. The relationship between local properties and cosmological parameters may extend farther than we might first imagine. The advances that can be made in physics resulting from new ways of looking at things justify new viewpoints even without any obvious reason why such a viewpoint is relevant. Thus I am suggesting that looking for relationships between local properties and cosmological parameters might give insignt into any field of theoretical physics, particularly in the fields of elementary particles and quantum field theory.[2] [3]

[1] ca 1967-1974
[2] ca April 1974
[3] Find the right Lagrangian. - October 1979

Perhaps the whole form of the Hamiltonian and Lagrangian formulations of mechanics depends on an interaction with the rest of matter in the universe.[3]

- We can't tell the difference experimentally between something which is true in general for all nature, and something which is only true for us because of the distribution of matter in the universe.
- Trying to incorporate the effect of the stars in a theory is analogous to generalizing classical mechanics to special relativity, wave mechanics, or quantum mechanics in that in each case a new parameter is added to describe the system (mean density of the universe, speed relative to that of light, wavelength, or size of smallest particle).[4] It is different in 2 ways:
  1. In the classical mechanics limit, we have: $v/c = 0$, $\lambda = 0$, $\hbar = 0$.
     in our case, we have $\rho_U = \text{constant} \neq 0$.
  2. We cannot change ρ_U or $\hbar = 0$. We can change v/c and λ.
- The use of natural units

  Systems of units in which c and/or $\hbar$ are equal to one should not be used when considering the possibility that c and/or $\hbar$ are not universal constants but are instead related to (say) the mean density of matter in the universe.[5]
- It is necessary to be careful in interpreting the relationship between macroscopic and microscopic properties of matter.
- Einstein - Space has no intrinsic physical properties. Material bodies in the space give it properties.[6]
- In considering the various possibilities for Mach's principle, we need only calculate the results for each system, to see which agrees with observation.
- Media modify propagation. We must consider the ultimate medium: the universe, which, rather than modifying propagation, determines it.

20.2 Mechanics

Maybe Hamilton's equations should take a different form in general.

20.3 Main Topic: Finding a connection between gravitation and quantum mechanics through thinking about Mach's principle

1. Search for a theoretical relationship between gravitation and quantum theory through Mach's principle.

   October 1979 - It turned out to be a search for a theoretical relationship between gravitation and quantum theory through thinking about Mach's principle. Also, the relationship turned out to be that interaction, all kinds, not just gravitation, reduces the quantum effect.

[4] ca April 1971. See chapter 9.
[5] ca 1973
[6] ca April 1971. See chapter 9.

2. Some form of Mach's principle is probably correct. Einstein's field equations are at least consistent with some form of Mach's equations. Some possible statements of Mach's principle are:[7]

   (a) The inertial properties of a body are determined (or affected by) the distribution of matter in the universe. (probably gravitational)

   (b) The inertia of a body is not intrinsic but due to a gravitational interaction with the stars.

   (c) The inertial properties of matter are not intrinsic, but are due to a gravitational interaction with the rest of matter in the universe.

3. Connection between inertia and kinetic energy and momentum and matter distribution.[8]

   (a) Since kinetic energy and momentum can be considered to be the main manifestations of inertia, it appears very likely that the momentum and kinetic energy (and perhaps rest mass energy) of a body depend not only on its motion but also on the distribution of matter in the universe.

   (b) It follows that the momentum and kinetic energy of a body (being inertial quantities) are really manifestations of a gravitational interaction of that body with the stars.

   (c) Inertial mass.

4. Connection between QM and inertial mass, kinetic energy and momentum, and thus to a gravitational interaction with the matter distributed throughout the universe.

   (a) The wave properties of a particle depend on its momentum and energy.

   (b) Wave mechanics is based completely on the wave properties observed to be associated with momentum (d Broglie wavelength) and kinetic energy ($E = h\nu$). All 3 basic equations of wave mechanics (Schrödinger, Dirac, Klein-Gordon) are set up to give the correct (de Broglie) wavelength and frequency when applied to a free particle. See chapter 4.

   (c) Inertial mass plays a fundamental though obscure role in the equations governing quantum processes. (The 3 equations all contain inertial mass.)

5. Conclusions: - ca April 1974

   (a) Either the wave properties of particles depend on the distribution of matter the universe or Planck's constant depends on the distribution of matter in the universe, or some combination of both.

   (b) Wave mechanics should have a strong theoretical connection with gravitation.

   (c) A theoretical relation should exist between gravitation and quantum theory. Quantum theory should explicitly contain the gravitational interaction between a particle with the rest of matter in the universe that determines the inertia of that particle. If the inertia of a particle is not intrinsic, but determined by a gravitational interaction with the rest of matter in the universe, then quantum mechanics should explicitly contain that interaction.

[7] ca April 1974

[8] ca April 1974

20.4 Gravitation-Mach: Effects of cosmology on local physics through the gravitational interaction

- Although Einstein's theory of gravitation has Machian properties, it is not itself consistent with Mach's principle[9].

- In the equations of Sciama, Waylen, and Gilman[16], for the integral formulation of Einstein's field equations, the integral obtained is actually an integral equation, since the integrand depends on the metric. The question of the first piece of matter contributing a lot because of having a different metric doesn't occur because the metric used in the integrand is ther final solution to the equation. If we can guess beforehand, what that metric is, then we can treat the integral equation as a simple integral, using our correctly guessed metric in the integrand.

- Sciama's calculation of inertia[11] implies a $1/r$ force between Sun & Earth without matter in the universe.

  Compare with M. V. Hayes's[17] $1/r$ solution of his field equations as the only time independent solution.

- Carry out Sciama's[11] calculation of the origin of inertia for General Relativity.

  G should be a function of the mass distribution in the universe.

- Energy and momentum are the sources of a gravitational field. But they are dependent on gravitational interactions with the stars because of the induced inertia. It could be confusing.[10]

- Perhaps rest mass is not intrinsic to particles, but is instead induced by a gravitational interaction with the rest of matter in the universe.

  Maybe there is an intrinsic rest mass, but that the observed rest mass includes the intrinsic rest mass plus that induced by the rest of matter in the universe, like mass renormalization.

- The relationship $E^2 = c^2p^2 + m^2c^4$ may not be a universal relation, but may depend on a gravitational interaction with the stars.

  The speed of light may also not be a universal constant, but may be determined by a gravitational interaction with the stars.

- What constitutes an acceptable form for the geodesic equation which exhibits Mach's principle explicitly and explicitly exhibits that the motion is independent of the choice of the frame of the observer?[11]

- The form of the Lienard-Wiechert potentials comes from the Green's function solutions to $\nabla^2 A_\mu - \frac{1}{c^2}\frac{\partial^2 A_\mu}{\partial t^2} = \delta^4(\mathbf{x})$ and gives as solutions that information from sources travels in straight lines with velocity c in inertial frames.

  Alternatively, in arbitrary frames, a corresponding equation has a Green's function that has photons traveling along geodesics.

  To incorporate Mach's principle, we would need to allow photons to propagate arbitrarily in an arbitrary reference frame and then interact gravitationally with the rest of matter in the universe. The most probable path of such a photon after having scattered from the curvature

[9] ca 1974
[10] ca April 1971. See chapter 9.
[11] Any frame, because it is a tensor equation. - October 1979

of space would then be along a geodesic. However, we need the explicit interaction to do the calculation and show Mach's principle explicitly.[12]

- Photons travel in curved paths in non-inertial frames. Thus, the Green's functions will show that the information on currents follows such curved paths.

- Is the speed of light constant only in inertial frames? If so, then the stars must determine the speed of light, since it is defined only in inertial frames, and they [the stars] determine the inertial frames.[13]

- D and H are directly electromagnetic quantities since they are defined in terms of charges and currents. E and B, however, are defined in terms of forces and acceleration of charges. E and B are therefore dependent on the inertia of an electron and therefore on the density of stars in the universe. Therefore, ϵ_0 and μ_0 are constants that are not universal, but depend on the mean density of matter in the universe. ϵ_0 is proportional to ρ_0 and μ_0 is proportional to $1/\rho_0$. Thus, the speed of light is independent of ρ_0. However, I don't really believe this.[14]

- For the speed of light to depend on the mean density of matter in the universe, it may be necessary for the ratio of the inertia parallel to the path to that perpendicular to the path to also depend on the mean density of matter in the universe. This is to get the product of ϵ_0 and μ_0 to change, by changing the effective forces of E and B fields on moving charges.[15]

- We need to recognize the contribution of the stars when Maxwell's equations are written in a non-inertial frame[16]. [17]

- It seems to be an inertial property that bodies with finite rest mass cannot reach the speed of light and that light can travel only at the speed of light.[18]

- If light propagates through a medium that is accelerating relative to an inertial frame, then is the propagation velocity of the light relative to the inertial frame, or to the medium or some combination?

  The inertial frame is also a medium. We need to consider an explicit interaction with the inertial frame (or its sources) in considering the propagation of light described in a non-inertial frame.[19]

- If inertia is due to an inductive interaction with the rest of matter in the universe, then maybe we shouldn't consider conservation of 4-momentum as being absolutely true, but only to the extent that the universe is much heavier than our test particle.

  This is in agreement with the result that massive particles deviate from exact geodesics.

20.5 E&M-Mach: Effects of cosmology on local physics through the electromagnetic interaction

- Analogy of dispersion relation for free masses with that for radio waves in the ionosphere.

[12] We can do it with the metric just as well. - October 1979
[13] ca April 1971. See chapter 9.
[14] ca April 1971. See chapter 9.
[15] ca April 1971. See chapter 9.
[16] ca April 1971. See chapter 9.
[17] This is taken into account in the action. - October 1979
[18] ca 1971-1972. See chapter 11.
[19] It is a combination; both effects must be considered. - October 1979

Apparently, the ionosphere gives an effective rest mass to the photon.

In fact, for radio waves in the ionosphere, the medium can be moving. The effective plasma frequency is the plasma frequency at rest. For plane waves, this can be seen by a Lorentz transformation. I'm not sure about an accelerating frame.

- Suppose the universe were a plasma. Then a photon would appear as a particle with mass.

- Magnetic fields will tend to get "frozen in" to material media through interactions with the particles in the media. This is especially true in media having free or partly free charges such as metals or plasmas.

  Thus, the Earth's magnetic field probably moves with the Earth rather than remaining in an inertial frame, and the field of a bar magnet may rotate with the magnetic rather than remain in an inertial frame.[20]

- A rotating bar magnet or current should drag inertial frames by the electric induction. (Sort of an electromagnetic Lense-Thirring effect) I don't think this effect is correctly taken into account in General Relativity.[21]

- It is very significant, that an inertial frame, (that is, the frame determined by the distribution of distant matter in the Machian sense) is the same for all particles.

  But, is it the same for all particles? Are there experiments that would reveal a difference? Or, are there existing observations that could be explained on the basis of different particles sometimes having different inertial frames?[22]

- Electromagnetic Mach's principle - moving charges tend to produce electromagnetic inertial frames for charged particles.

  Photons travel in curved paths in non-(gravitational)inertial frames.[23]

- Is there an electromagnetic Mach's principle? That is, does the presence of charged particles create an "inertial" frame for other charged particles?[24] [25]

- Redo the calculation of motion of charges in a charged universe by Hoyle and Narlikar.

- The application of Maxwell's equations to a positively charged universe. See chapter 5.

  Extend the calculation to the case where the mass-induced inertia is not negligible compared to the charged-induced inertia, and apply the result to the charge density suggested by Hoyle and Narlikar.

  Possible explanation of Lorentz contraction?

- Check out my theory of Lorentz contraction in a charged universe.[26]

[20] A magnetic field can always be considered to be fixed in the frame of reference. - October 1979

[21] I think it is OK in terms of my quantum basis for Mach's principle.

[22] If the universe were charged, there would be a difference. - October 1979

[23] ca 1967

[24] ca 1967

[25] Yes, I know how to correctly do this calculation now in terms of the action and taking a classical limit. - October 1979

[26] October 1979

20.6 QM-Mach: Effects of cosmology on local quantum physics

- Suppose the mean density of matter in the universe were 1/2 what it is now. Then there are several possibilities:[27]

  1. $\hbar$, α, and c remain the same, and the radius of the first Bohr orbit a_0 doubles.
  2. a_0, $\hbar$, and c remain the same, and α doubles.
  3. a_0, $\hbar$, and α remain the same, and c doubles.
  4. a_0, α, and c remain the same, and $\hbar$, is halved.
  5. Something more complicated like ϵ_0 and μ_0 changing.

  I suspect that c would increase. I suspect that α would change.[28]

- $S = 10^{120} \pm 1$ radian for our universe = radius of our universe measured in terms of the Compton wavelength of our universe.[29]

  Same ratio for electron = $\alpha = 1/137$ if we take the classical electron radius as a measure of the size of the electron. The electron is more complicated because it has both mass and charge; we are then measuring the ratio of 2 kinds of things.

- Try to develop a QED that explicitly contains the gravitational (inertial) interaction.[30]

  The photon propagator must already implicitly contain the gravitational (inertial) interaction.

  Also the Fermion propagator.

- Some sort of correspondence principle should apply so that for a universe of constant density (with time), usual quantum mechanics applies.

- Is there an observable effect related to the possibility (or conjecture) that Planck's constant might be proportional to the mean density of matter in the universe?

- Do the size of Bohr orbits and the energy levels in the hydrogen atom depend on the distribution of matter and mean density of the universe?

  Is it possible that as the universe evolves that the spectra of atoms are changing so that "old" radiation (coming from distant parts of the universe) emitted when the universe was probably denser shows a red or blue shift? Clearly, the observed red shift would be a combination of that effect plus that due to expansion. Is there an observation that would separate these 2 effects?[31]

- In a coordinate system fixed with the particle, wave properties disappear (wavelength equals infinity) and Mach's principle is satisfied.

  October 1979: This is an illusion. Quantum properties still exist because of a quantum superposition of states, and Mach's principle depends on the metric's being determined by the matter.

  The classical limit doesn't occur for infinite wavelength, but for zero wavelength.

[27] See chapter 10.
[28] August 1971
[29] October 1979
[30] October 1979: Yes, we do it through the metric.
[31] ca April 1974

20.7 Gravitation

- Physical constants - values and definitions of Planck length, time, and mass
- What is wrong with scalar, spinor, vector, etc. theories? (Why is gravity a spin 2 theory?)
- Work to do: Read up on Dicke's scalar-tensor theory. It reduces to Einstein's theory when $\omega = \infty$.
- Work to do: bending of light around the sun - disign a satellite experiment to get bending to 1/2%.
- Gravitational radiation (can we detect it?)
- Work to do: Photons in an expanding balloon lose energy and drop in temperature. Suppose the balloon is the universe. What are the photons pushing against? Can energy have gone into gravitational energy of universe? Try at least 2 expressions for energy density and check agreement.
- Many fingered time.
- Make up a table that shows what equations result from what operations in General Relativity.
- In 3 dimensions, a rotation can be defined by a vector.

  In 4 dimensions, a rotation is defined by giving the 2 axes that are held fixed or by giving the 2 axes that move.
- Compare conformal invariance with quaternion regularity.
- Is it possible the radiation from quasars is so intense because it comes from a contracting phase of the universe?
- Velocities don't add. Maybe energies don't add either.
- If a spinning body is moving in a straight line in one Lorentz frame, does it accelerate in another Lorentz frame?

  Tom Martin says that in the unprimed frame if you have a body spinning with its axis along the y axis, and view that body from a primed frame moving along the x axis, then it will appear that the spinning body is accelerating in the z direction.
- Only particles of infinitesimal mass follow geodesics. Real particles deviate from exact geodesics.
- Use of super potentials for gauge independent calculations?
- If c is thought of as the scaling factor between the length axes and the time axis, then it would be reasonable to set it to one.
- Sources of the gravitational field are m_g, $m_g\mathbf{v}$, & $\frac{1}{2}m_g v^2$.
- Computer program to calculate change in gravitational field and particle positions given initial conditions - ca 1967

  See chapter 3 and [18, Weinberg, pp. 163-165]

- If we wish to not make a geometrical interpretation for gravitation, then we are stuck with the fact that covariant differentiation sometimes introduces gravitational forces.

- What is the correct method for including potential energy in calculations?

  October 1979: By putting it correctly in the Lagrangian.

- An accelerating reference frame corresponds to an angular velocity of the $x-t$ axes. Calculate the corresponding centrifugal force and Coriolis force.

- $g_{\mu\nu}$ gives the distance between points. We need something analogous that gives the relative direction between points. (maybe)

- In the rest frame of a body we need only $g_{0\mu}$ to write the force law. Thus, we need calculate only those 4 quantities from the apparent motion of the sources in the universe; we don't need the other 3 independent terms in the metric tensor.

  Is it possible the relation of $g_{0\mu}$ to the sources takes on a simple form as observed from a geodesic?

- Wheeler's conjecture, that we can derive Einstein's equations from purely intrinsic geometric quantities is wrong because in superspace to get the path connecting two 3-geometries we have to use a Lagrangian that depends on extrinsic parameters.[32]

- Maybe physical law is only approximately invariant under Lorentz transformations.

  Is this related to the phenomenon that massive bodies only approximately follow geodesics?

- The question of the effect of sources in determining inertial frames is directly tied up with how the geodesic equation is determined because gravitation travels on geodesics.

  Actually, the above is not quite true. Gravitational waves may travel on geodesics, but the Green's function transferring information on the metric may not. I think this is analogous to comparing Green's functions for electromagnetism. The static field, the induction field, and the radiation field do not travel on the same path in spacetime.

- Einstein-Cartan formulation - puts mass and spin on the same footing.

- October 1979 - If the Robertson-Walker metric is conformally flat, then it should be a solution for empty space because the field equations are conformally invariant when there is no matter.

- Work to do: Calculate $g_{\mu\nu}$ for a spinning mass (Lense-Thirring effect)

- October 1979 - Is there an unnormalized form for the field equations where $g^{\mu}_{\nu} \neq \delta^{\mu}_{\nu} =$ unit tensor?

- Should there be a G in Einstein's field equations in light of Mach's principle suggesting that G is not a universal constant but really a constant that depends on the mean density of matter in the universe?

  Yes, because the masses that go into the energy-momentum tensor are inertial masses. The product GT is then independent of the inertia that has been induced into the sources. The product GT depends only on gravitational mass.

- Inertial mass is equivalent to <u>passive</u> gravitational mass.

[32]From Derek Raine, summer 1975.

- Define gravitational mass and gravitational mass density to distinguish them from ordinary (inertial) mass and mass density. Gravitational mass is intrinsic. Define gravitational mass M_G and m_G in gravs such that the force in dynes between the two masses separated by r cm is $F = M_G m_G / r^2$. One grav is one cm dyne$^{\frac{1}{2}}$ or one cm$^{3/2}$gm$^{-\frac{1}{2}}$ sec^{-1}. For the same body, $m = m_G/\sqrt{G}$. Thus, a body having a gravitational mass of one grav has an inertial mass of 3.88 kilograms.[33]

- Try to reformulate Einstein's field equations in terms of gravitational mass instead inertial mass so that G does not appear.

- Write Einstein's equations for a small number of bodies.

  October 1979 - This is no longer necessary. We know now from considering the action that a sparse universe has large quantum effects.

- The standard interpretation of inerta nowadays is not that it is an intrinsic property of bodies or particles, but that it is a property of space-time or geometry.

- Conservation of energy means energy is independent of time.

  Conservation of momentum means momentum is independent of time.

  The above applies to a closed system.

  There are no closed systems.

  Angular momentum comes in units.

  Conservation of angular momentum implies conservation of momentum.

- Perhaps our theory should make more of a distinction between the space and time axes. Treating it as 3 space plus one time axis instead of 4 equivalent axes. I think Wheeler mentions this in his paper on quantum geometrodynamics[19, probably].

  October 1979 - Yes, he does. Also in [20, Misner, Thorne, and Wheeler].

20.8 Gravitation-E&M: Interaction between E&M and gravitation, and comparisons between E&M & gravitation

- Compare $T^{\mu\nu}{}_{;\nu} = 0$ with Miles Hayes's field equations.[17]

$$T^{\mu\nu} = F^{\mu}{}_{\alpha} F^{\alpha\nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \tag{20.1}$$

- $T^{\mu}{}_{\nu;\mu} = 0$ gives geodesic equation plus Lorentz force.

- Maybe a particle can tell the difference between a vector and a tensor field by seeing how the fields change with time. The particles producing the field must obey force laws also. Thus the field at our test particle will change according to the way the particles producing the field move.

- Is E & M different from gravitation? Yes.

- Calculate the refractive index for light in a gravitational field.

[33] ca 1973

- Try to eliminate charge and mass in Maxwell's equations & Lorentz force equation.

  I think it can't be done.

- The source of the EM field is the current density, a 4-vecotr having zero divergence.

  The source of the gravitational field is the energy-momentum tensor, having zero covariant derivative. But there is no corresponding conserved quantity as there is in the EM case. (The analogous quantity that would be conserved would be the 4 momentum, but T doesn't include the gravitational energy and momentum.)

- "The major difference between the electromagnetic and the gravitational fields is that the source of the electromagnetic potential A is a conserved current J that does not involve A because the electromagnetic field is not itself charged, whereas the source of the gravitational field h is a conserved 'tensor' τ that must involve h because the gravitational field does carry energy and momentum."[13, Weinberg, p. 171]

  The real difference between E&M and gravitation may be that there are many stars in the universe that have mass but are not charged.[34]

- In writing the geodesic equation including the Lorentz force, we should maybe also include a radiation damping term.

$$m\frac{dv^\mu}{d\tau} = F^\mu_{\text{ext}} + \Gamma^\mu \tag{20.2}$$

$$F^\mu_{\text{ext}} = eF^{\mu\nu}v_\nu \tag{20.3}$$

$$\Gamma^\mu = \frac{2}{3}e^2\left(\frac{d^2v^\mu}{d\tau^2} - v^\mu\frac{dv_\lambda}{d\tau}\frac{dv^\lambda}{d\tau}\right) \tag{20.4}$$

- By writing the geodesic equation in any frame other than the rest frame, we have introduced coordinates and relative velocities that must be irrelevant to the dynamics of the problem.

  On the other hand, the rest frame may be special gravitationally but not electromagnetically.

- That from the point of view of a single particle, the gravitational field looks like a vector not a tensor field suggests that the difference between the gravitational and electromagnetic interactions (tensor versus vector) is not intrinsic, but arises from the large amount of mass present in the universe.

  October 1979: The vector-tensor character of a field has to do with its transformation properties. If we look only in one frame (rest frame) we are not testing the tensor or vector character of the field.

20.9 Gravitation-QM: Quantum gravity equals Quantumgeometrodynamics (QGD), and gravitational effects in quantum mechanics

- Sakharov: "Gravitation is the 'metric elasticity of space'."

[34] ca 1971-1972. See chapter 11.

- Use non-linear antenna theory to calculate propagators for General Relativity.
- The spin of an electron is like a gravitational magnet. There should be gravitational spin-orbit coupling.[35] See chapter 4.
- Gravitational effects in quantum theory: Gravitational effects are extremely important on the quantum level.[36] In off-handed remarks of fuzzy thinking it is usually assumed that gravitational effects are negligible in the realm where quantum effects are important. This thinking refers on one level to the probable negligible effects of the static gravitational forces between particles, and on another level to the separability of gravitational and quantum effects by considering gravitation only in that it provides a background metric for quantum effects to reside in. Although the viewpoint on this second level is valid, it is not the only valid viewpoint. For example, hydrogen atom wave functions result when the electromagnetic interaction between an electron and a proton balance inertial forces on the quantum level. In this sense, considering inertia to be a gravitational interaction, gravitational effects are extremely important on the quantum level.
- There is a problem in unifying quantum mechanics and relativity because of uncertainty.[37] See chapter 4.
- The nature of the stress-energy tensor.
- (7) Gravitational energy and momentum are not included in the energy-momentum tensor as sources of the gravitational field.

  Except that for gravitational waves, if we average the gravitational energy and momentum over several wavelengths, then we may include the averaged value in the energy momentum tensor.

  This is very important. Because if we consider the wave nature of particles, then the energy and momentum we put in the energy-momentum tensor are averaged values over several wavelengths of the particle waves. Thus, it may be that the deciding factor of whether the energy and momentum of something goes into the energy-momentum tensor is not whether it is gravitational, but whether it is an averaged value over several wavelengths.

  Thus, the energy-momentum tensor includes only averages of energy and momentum over several wavelengths.
- (3) What is the stress-energy tensor for a Dirac particle or other non-classical fields? What is the Lagrangian?

  Check Feynman & Hibbs[21]. Also check Leighton[10] & Bethe-Salpeter equation.[22, Schweber, pp. 710-719]

  Probably use $T_0^0 = mc^2\psi^*\psi$ for time average over a cycle and $T_0^0 = mc^2\psi^2$ for instantaneous. Consider spin, charge, magnetic momentum [moment?], angular momentum. Spin 1/2.

  $T = T(u_1, u_2, u_3, u_4)$ (spinor components)

  Spinor components are analogous to E & B.

  Get T in therms of them.

  It seems that T is independent of t & x for a plane wave and spin eigenfunction.

[35] ca 1967
[36] ca April 1974
[37] ca 1967

- (4) Does the existence of a superposition of classical solutions (3-geometries) give quantum mechanics or is this a separate effect?

- Einstein's field equations involve derivatives with respect to coordinates. However, until we have a metric (that is, a geometry), we have no coordinates with which to take derivatives with respect to. Thus, there is no consistent way in which we can think of the solution for the metric as being built up starting from nothing. We can only look for a consistent solution of the field equations in terms of the geometry which is also a result of solving the field equations.

  The same statement applies to my quantum statement of Mach's principle. Without a geometry, there is no way to define a path, and no way to talk about the path dependence of the action.

- Understand Einstein's calculation of geodesic equation from field equations.

  Extend the calculations to wave calculations.

- Quantum properties of space versus geometric properties of space.

- If we assume that each particle is or has a clock, so that it oscillates at a particular frequency, then the wave properties of particles follow, and the term $\partial\tau/\partial x^\mu$ in my gravitational vector potential gives variations of the phase function with position.

  The functions π & τ are then related to the wave function for the particle observed externally.

- g_{44} refers to the rate of a clock at observer. No violet shift implies g_{44} is constant.

- What is the criterion that two particles on different world lines see the same gravitational vector potential in an arbitrary coordinate system?

  October 1979: $g^{\mu\nu}\pi_\mu\frac{\partial}{\partial x^\nu}\left(g^{\alpha\beta}\pi_\alpha\pi_\beta\right) = 0$ implies they have the same wave function. An interaction can separate them (irreversibly).

- Consider:

  1. A wave function represents our knowledge of a system.
  2. Gravitational energy is not localizable. $t_{\mu\nu}$ is not unique.
  3. We cannot observe the detailed shape of a wave function.
  4. We can observe the detailed shape of a radio wave, sound wave, or ocean wave.
  5. We must also distinguish mechanical momentum (the physical momentum transported by the wave) and canonical momentum (the p that goes into Hamilton's equations) for classical waves.
  6. We can define $T_{\mu\nu}$ as an average for gravitational waves.
  7. Poynting's vector $E\times H$ gives the physical momentum carried by an EM wafe; the propagation velocity of a wave packet (group velocity) is related to the canonical momentum. (phase velocity, group velocity?)
  8. What wave mechanics properties will we observe in a universe with only a few particles?

- The canonical momentum p represents a derivative with respect to 4-position. It may thus be related to gravitation because gravitation is a geometric object.

  Other types of waves may be represented by derivatives with respect to other quantities.

  Maybe any conserved quantity can be represented by a derivative with respect to something and by some kind of wave, not just a space wave.

- The total vector potential for gravitation and electromagnetism is very similar to the relation between the wave properties of matter and the gravitational and electromagnetic properties. Since the equation I derived is valid usually for only one particle, it may explain why standard quantum theory is valid for only one particle. In addition, it may point the way to developing a quantum theory valid for many particles without introducing creation and annihilation operators.[38]

- What is the correct interpretation of the gravitational vector potential?[39]

- Gravitational vector potential: Although we can't go to the rest frame of a photon, we can go to the rest frame of a pair of photons or to the rest frame of a set of photons.

- Apply the gravitational vector potential to a hydrogen atom.

- We don't have to choose the same h for quantum geometrodynamics as for quantum mechanics. Or do we?

- Write a short summary of the gravitational vector potential giving the results and leaving out the derivations. Start by summarizing each section of the larger paper.

- Do we get a formula like $(E - \phi)^2 - (p - A)^2 = M^2$ for gravitation?

  October 1979: $g_{\mu\nu}\pi^\mu\pi^\nu = M^2$, where $p_\mu = \pi_\mu + A_\mu$

- Is A equal to the 4-momentum of the EM field in addition to being a 4-potential?

  Is A multivalued?

- Why is quantum mechanics a gravitational phenomenon?

  October 1979: It isn't. Quantum properties are the natural state. Interaction, any kind, reduces the quantum effect and leads toward classical behavior.

20.10 E&M

- $S = E \times H^*$ is not valid for complex frequency. Maybe this explains why S does not agree with packet direction for a lossy medium.[23, 24, 25, 26, Hines, 1951]

- Since we can produce all of the elementary particles by photon-photon scattering giving pair production, all of physics including strong interaction, weak interaction, and gravitation should be contained implicitly in a complete electromagnetic theory.

  Of course, we need to include the cosmology as a background.

- Randy Ott says that the frequency domain is more fundamental than the time domain because ϵ and μ are functions of frequency.

- Apply Feynman's path integral method to try to "derive" Maxwell's equations from geometrical optics.

- Kline & Kay say that the integral representation of Maxwell's equations is more fundamental than the differential form.

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[38] October 1979: What is the general formula for a tensor force?

[39] October 1979

- Meaning and usefulness of the combination $B - iE$, where B is the magnetic field, and E is the electric field.

- Local shift in charge on an accelerating electron[40]

- Maybe EM also needs a tensor theory.[41] See chapter 4.

- Do we really know the correct form of Maxwell's equations for non-inertial frames?

  See top of page 107 in Adler, Bazin, & Schiffer[27]

  Also see pages 124, 125, and 126 in Weinberg[18], which implies that we do know the correct form.

- October 1979: Maybe the interaction term in the action S_{int} is only a linear approximation.

- $\phi = \int_V \int_t \frac{\rho dV icdt}{\sqrt{r^2 - c^2t^2}} = c \int_V \int_t \frac{\rho dV dt}{\sqrt{c^2t^2 - r^2}}$

  $\mathbf{A} = \int_V \int_t \frac{\rho \mathbf{v} dV icdt}{\sqrt{r^2 - c^2t^2}} = c \int_V \int_t \frac{\rho \mathbf{v} dV dt}{\sqrt{c^2t^2 - r^2}}$

- Is the group velocity really the same as the phase velocity of light in free space? Perhaps light is dispersive. When the velocity of light is measured, which velocity is measured, group or phase?[42] See chapter 4.

  October 1979: Simultaneously measure frequency (from atomic transition) and wavelength (from diffraction or cavity). (to get the phase velocity)

- Maybe it is only an approximation that we can treat the Lagrangian for the electromagnetic field as quadratic.

20.11 E&M-QM: QED=quantum electrodynamics

- The action of polaroid is a microscopic phenomenon. An electromagnetic wave interacts with a single atom.[43]

- Get higher order terms in Maxwell's equations from Professor Dreitlein.

  I did this, but the next higher term was zero for the vacuum.

- Photoelectric effect[10, Leighton, p. 67]

  On the basis of black-body radiation, Planck hypothesized that the oscillators, but not the radiation field, were quantized.

  On the basis of the photoelectric effect, Einstein proposed that the radiation field itself might be quantized. The quantitative verification of his equation established quite firmly the discrete nature of the radiation field.

  One of the most difficult things to explain is the localization, in a single electron, of the radiant energy which, on the classical theory of radiation, must have fallen on an area covering some 10^8 atoms.

[40] Douglas Gough, 1967
[41] ca 1967
[42] ca 1967
[43] ca 1969. See chapter 6.

Another difficulty is the fact that the time delay in the emission of photoelectrons does not exceed 3×10^{-9} seconds, whereas on the wave theory of light, time delays varying with the intensity from a few microseconds up to several days would be expected.

Actually, the theoretical treatment of radiation is now considered to be in a rather satisfactory state of perfection, and we shall esamine some of the simpler features of the theory by which both the wavelike and the particle-like properties of radiation are accounted for. This theory is called quantum electrodynamics.

- Can the quantum effects on radiation be completely explained by saying that light travels on more than the classical path? Yes, I think so.

  October 1979. No, I don't think so. We must consider a quantum superposition of classical E&M fields.

- Find out about the guy who uses quaternions for QED from Professor Brittin. Is this Caianiello?

- Calculate a differential equation for QED from Feynman's integral equation for QED.

  October 1979. See pages 227 and 274 in Schwinger's book.[28] These are actually citations to two of Feynman's papers.[29, p. 751] and [30, pp. 456-457]. Calculate a differential equation for QED from Professor Brittin's class notes and compare with the above.

  October 1979. See lectures 27 and 32 for the spring.[44]

- Try to reduce QED to a differential equation and compare it with Miles Hayes's equation.[17]

  See page 274 in Schwinger's book on QED.[28].[30, pp. 456-457].

- I feel that the concept of a photon is a useful mathematical device rather than a reality.[45]

- Calculate $<(\text{energy} - <\text{energy}>)^2>$ for a vacuum using usual second quantization rules.

- Pair production[46]. The card shows a drawing with ct on the vertical axis versus x on the horizontal axis with wavy lines for photons coming in from the bottom at 45-degree angles and meeting at the origin with and electron and a positron emerging from the origin and moving up in the second and third quadrants for positive t.

- Make photoelectric effect calculation for an incident E. M. wave packet without assuming the E. M. field is quantized to show that the assumption of a quantized field is redundant (see Hoyle & Narlikar, Ann. Phys. **54** pp. 207-239, 1969).[31]

  October 1979. It is probably not redundant.

- If quantum theory is applied to light, does it imply that the velocity of light is uncertain?

- It's not enough to quantize just the radiation field.

- Correct way to quantize EM: functional integration - Federov, Popov. The correct citation might be [32, Popov and Faddeev, 1967]

- Is light really quantized, or is this just a result of the quantization of orbits in an atom? Yes, it is quantized in that we can have a quantum superposition of classical EM fields.

[44] I doubt if I still have those notes.
[45] ca April 1974
[46] ca 1973-1974

- "In a Fourier spectrum of an EM wave, if the fundamental is quantized, that does not mean the harmonics have to be quantized."[47]

- If an electron is a wave, then its position is not known exactly, and therefore the electric force it exerts on another electron is also not known exactly.

20.12 QM

- The formula $\lambda = h/p$ holds only because the dispersion relation $E^2 = c^2p^2 + m^2c^4$ holds in free space.

  $p = \hbar k$ in free space.

  In the presence of an interaction, such as E&M or gravitation, it is not obvious how the wave properties should be.

- Maybe Hoyle and Narlikar's argument about spontaneous emission[31, Ann. Phys. 54] can be applied to the harmonic oscillator in general, and then extended to all of quantum theory.

- Minimum principle - Palatini method, QED, Hoyle & Narlikar's method - infinitesimal propagator is:

  $$\frac{1}{\pi}(\gamma\cdot q)\delta'(q^2)\exp\left[\frac{1}{2}im(\gamma\cdot q)\right] = -\frac{1}{\pi}\frac{\delta(q^2)}{(\gamma\cdot q)}\exp\left[\frac{1}{2}im(\gamma\cdot q)\right] = \frac{1}{\pi}\left[(\gamma\cdot q)\delta'(q^2) - \frac{1}{2}im\delta(q^2)\right] , \tag{20.5}$$

  where

  $$\delta'(q^2) = -q^{-2}\delta(q^2) \tag{20.6}$$

  Finite propagator is:

  $$K_0^+(2;1) = \frac{1}{2\pi}\theta(t_2 - t_1)(\nabla\!\!\!/_2 - im)\left[\delta(s^2) - \frac{m}{2s}\theta(s^2)J_1(ms)\right] , \tag{20.7}$$

  where

  $$\nabla\!\!\!/ \equiv \gamma\cdot\nabla \tag{20.8}$$

- $n \to p + e^- + \overline{\nu}_e$ = weak decay of neutron = Baryon scattering plus charge pair production plus Lepton pair production, where

  n = neutron = neutral Baryon,

  p = proton = bound state of Baryon & chargeon,

  e^- = electron = bound state of Lepton & chargeon

  The accompanying drawing represents the Baryon by a solid line, the chargeon by a wiggly line, and the Lepton by a solid line.

- Nonlinear perturbation method - Use altered propagator instead of elementary propagator.

- A particle wants to measure the universe. Passively, it gets hit with a quantum and gets moved. Therefore its measurement of the velocity of the universe (and therefore the velocity of itself relative to the universe) is inaccurate.

  Similarly for an active measurement.[48]

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[47]Dream - 25 March 1974.

[48]March 1979

- A wave packet can be formed out of several electrons at various energies, or out of one electron which through interaction loses some of its energy. For that case, the spectrum is for the same electron at different times.[49]

- Do wave packets occur only as the result of interaction? See chapter 7.

- Irreversibility is an essential part of measurement.

- Is attenuation purely a macroscopic process?

- Momenta add (that is, k add) because in the non-linear effect of amplitude modulation, we get the sum and difference of 2 frequencies.

- Superposition of solutions holds for linear media. Superposition holds if we have no super selection rules. Is there a connection here?

- Spin = multivaluedness.

  When we measure spin, we force a value to be measured - Maybe when we measure dimension, we force a value to be measured.

  What causes superselection rules?

- The action for a path may not be equal to the path length, but only some smooth, slowly varying function of the path length.

- $$\underbrace{E}_{\substack{\text{“energy” of}\\ \text{quantum oscillator}}} = (n+\tfrac{1}{2})\hbar \overbrace{\omega_0}^{\substack{\text{frequency of}\\ \text{classical oscillator}}} = \hbar \underbrace{\omega}_{\substack{\text{“carrier” frequency of}\\ \text{quantum oscillator}}}$$

- What is a coherent state?

  What is a super selection rule?

- Maybe particle wave functions describe not the particle itself, but instead our measuring apparatus.

  I think this point is strongly related to the standard controversy over whether the uncertainty relations are intrinsic or whether they are caused by our measurements.

- Read papers by Schrödinger, Dirac, Heisenberg, etc. around 1920-1930.

- I have always been bothered by the standard method of choosing a Lagrangian, which seems to be roughly that:

  1. It must give the result we are looking for.
  2. It must satisfy some set of philosophic principles.
  3. It must be simple.

  Although I can not think of an improvement, it bothers me to discard an otherwise acceptable Lagrangian on the basis of simplicity.

[49]ca 1969

- Are there currents inside a hadron?

  Can transform between current quark states and constituent quark states by a basis transformation.

- Classical Hamiltonian formulation of WKB solution to the Dirac equation?

  Hamilton's equations in 4 dimensions allow time to go backwards.

- Is there a transformation of bases to go from the second-order 4-component Dirac equation to the second-order 4-component Klein-Gordon equation?

  Conjecture: If the WKB solutions of two equations are the same, then there is a basis transformation connecting the two equations.

- There should be a frequency associated with the rest mass of an electron, but it has not been observed.

  The Schrödinger equation & the Dirac equation give different frequencies for an electron. Which is right?

- Does nature use fast Fourier transforms as a mechanism for quantum mechanics?[50]

- Maybe there is only one electron (that moves very fast).

- The important point of the uncertainty principle is not the point that it is a trivial consequence of Fourier analysis, but that the detailed structure of an electron wave packet can not be measured. Electrons are not divisible.

- Zitterbewegung is like Brownian motion.

- Interpretation of the Dirac equation as the first-order coupled system of equations.

- Relate $\phi(p_\mu)$ to $\psi(x_\mu)$ by a Fourier transform in four dimensions.

- Is it possible that "stable" particles are those that are trapped in a time well which always moves forward in time, while "unstable" particles are those that are not so trapped and are free to escape from our time well?[51]

- Wave packet solutions of Dirac's equation - See Newton & Wigner (1949)[33] and Dirac (1948)[34].

- Correct QM formulas for charge and current densities?

- Classify various particles and phenomena according to the equations that govern them.

  1. electron - Schrödinger equation or Dirac equation
  2. photon - Maxwell's equations or Klein-Gordon equation

- Make tables of quantities (sources of gravitation) versus theories or observation for each quality, and put in the table yes, no, or ? to answer whether that quantity has that quality in that theory or in observation.

- theories:

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[50] October 1979: No!

[51] ca 1962, see chapter 2

  - Dirac
  - Schrödinger
  - Einstein
  - Brans-Dicke
  - Newtonian
  - Hoyle-Narlikar
  - QED
  - Maxwell

- Pair production occurs at a caustic, where ray theory breaks down

- electron transition in a rotating coordinate system: $\omega + \Omega$, $\omega - \Omega$

- October 1979 - What are the major problems facing quantum field theory today?

  1. renormalization
  2. how to quantize gravitation
  3. cosmic censorship

  Read Hawking papers

  See, "quantum field theory" in my collected works for more problems - ca 1974.

- If we avoid using energy, momentum, and mass, and instead use frequency, wave number, and Compton wavelength, then Planck's constant will not arise in equations. This probably has the same effect as setting $\hbar = 1$, but a different psychological effect.

  The uncertainty principle then follows directly from Fourier analysis.

  $e^2/(\hbar c)$ would be replaced by e^2/c.

  We can write a new system of units based on this.[52]

  October 1979: The uncertainty principle occurs because the details of an electron wave packet are not observable.

- The property of spin (multi-valuedness) arises in classical wave propagation in the presence of a gradient in the medium. The dispersion relation is different for each spin component.

20.13 Mach's principle and quantum geometrodynamics: A quantum basis for Mach's principle

- Can we include ${}^{(3)}$G's that are not geometries? Will such ${}^{(3)}$G's be analogous to evanescent waves?

- Non-existence of geometry implies non-existence of dimension, topology, and coordinates.

  See problem 2-6 on page 34 of Feynman & Hibbs[21].

[52] ca 1972. See chapter 12.

- If we make no observation to distinguish alternatives, then we must add amplitudes of alternatives.

  Is that what causes superselection rules?

  Do we make measurements of geometry and dimensions?

  Do we measure the ways in which the geometry got that way?

- According to Wheeler, if we give the 3-geometry at two space-like hypersurfaces, and the distribution of energy-momentum in the 4-space bounded by those two hypersurfaces, then we determine Einstein's equations by applying a minimum principle. That is, we determine how that 3-geometry changes with time to become the second specified 3-geometry. Suppose we say instead, that all possible paths from one 3-geometry to the second actually occur, and the one determined by the minimum principle is just the one that contributes most (or is most likely). I think this is what Misner, Thorne, and Wheeler[20] refer to as "many fingered time."[53]

- Misner, Thorne, and Wheeler[20]: "Pregeometry = calculus of propositions."

- Prove symmetries from "bucket of dust."

- Can we write physical law in such a way that the positions of objects relative to some coordinate frame do not appear?

  Can we show that physical law in such a form combined with some assumed distribution of matter leads to the usual geometrical properties of space?

  Can the geometric properties of space be derived from a form of physical law in which the positions of objects relative to an arbitrary coordinate system do not appear?

- Need to show that adding masses takes away quantum effects.

- A solution from the beginning to the present is part of a path between A & B, where A is less than now is less than B.

  Also, we have no a priori knowledge of the initial conditions.

  Proposition: All initial states are equally likely. Can't prove this, but this assumption leads to observed Mach's principle.[54]

- Mach's principle, that matter determines inertia or affects inertial frames is already in Einstein's equations with initial conditions (geometrodynamics) but that doesn't answer Mach's problem, "Why is there no apparent rotation or acceleration of inertial frames relative to the bulk of matter in the universe?"[55]

- Purpose: Try to answer the question, "Why do inertial frames have no acceleration relative to average matter distribution?"

  In giving this answer, we can only relate it back to another assumption that seems more fundamental.

  = Mach's problem[56]

- The purpose of this paper is to move to a more believable level of assumption why "inertial frames don't seem to rotate or accelerate relative to the average distribution of matter."[57]

[53] October 1979: No, this is quantum geometrodynamics.
[54] March 1979
[55] March 1979
[56] March 1979
[57] March 1979

- The card shows a bunch of dots somewhat uniformly distributed (which presumably represent galaxies or clusters of galaxies) and a pair of axes at right angles to each other (presumably representing an inertial frame) with and ω and an arrow attached to indicate a rotation of the inertial frame relative to the matter.

- Add one light particle to Schwarzschild metric.

- The flat empty space example does not prove Mach's principle, it only shows why flat empty space is no embarrassment for Mach's principle.

  The same is true for asymptotically flat spaces.

- All flat empty spaces contribute according to their initial amplitudes. On the average, we would expect them to contribute equally, but with arbitrary phases. That is, the most likely contribution from flat empty spaces is equal, but with arbitrary phases.

- Phase and amplitude coherence of initial state:

  1. Nonlinearties can cause phase and amplitude coherence of linearized solutions. Examples:
     (a) troichoidal waves
     (b) solitary waves
     (c)
  2. The phase and amplitude of solutions to linear equations depend only on initial conditions. Examples:
     (a) waveguide
     (b) guitar string
     (c) hydrogen atom

- We know enough general properties about QGD to make the Mach's principle calculation.

- Spaces that are equivalent relative to the distribution of matter contribute equally if they have equal initial amplitudes.

- A sufficient mechanism for Mach's principle

  Solutions to Einstein's equations are not the only paths that contribute to the total amplitude, but they are probably the most important for many cases.[58]

- We know the past from measurement. We know the history from many consecutive measurements (bubble chamber). If we have many particles in the universe, then we have many opportunities for interaction and therefore for measurement. With only a few particles, we would have a very quantum world.[59]

- If the Lagrangian depends only on relative coordinates, then Mach would be satisfied.

- It is difficult to express physical law without using a reference frame in which to describe the motions of bodies.[60]

  Similarly, it is difficult to express physical law in such a way that it explicitly depends on only the relative positions of bodies.

[58] March 1979
[59] March 1979
[60] October 1979: It is difficult, but not impossible. There are coordinate-independent representations.

I feel that these two difficulties are related, and that it has been the non-recognition of the full implications of Mach's principle of relativity that has led to the multiple interpretations of Mach's principle of inertia.

- Analogy between Mach's principle and QED

  Only the relative positions of bodies can be relevant in physical law.

  Only the relative energies of states can be measured.

- Path integral representation of the universe in the rest frame of each particle

  Each particle sees the universe in some state. We want to calculate the amplitude that from the viewpoint of a particle the universe changes from state 1 to state 2. This is

  $A(2,1) = \int_\Gamma A_\Gamma(2,1)\mathrm{d}\Gamma$,

  where Γ is a way in which the universe could get from state 1 to 2.

  Can I derive simple quantum mechanics this way?

- Get some correlations between actions and propagators.

- What is the most general "natural" state? (the most general empty space solution)[61]

  How many kinds of configurations are there?

- Either interaction with the rest of matter in the universe gives wave properties to particles, or it gives phase coherence between conceivable trajectories to the wave properties already possessed by the particles.

  October 1979: The latter is true.

- Geometry & pregeometry

  Geometry = light geometry = light metric

  Pregeometry = action independent of path

- Green's function in non-inertial frames

  "natural = straight line" should be changed to:

  "natural state = no particular phase relationships among possible paths connecting two points"

- October 1979 - I showed in my paper (Chapter 29) that in the classical limit the matter determines the geometry, but I didn't find a relation between local inertial mass and the mass of the universe.

- Is all motion relative?

  Consider the amplitude that absolute motion is relevant.

  Or the amplitude that the motion relative to some arbitrary coordinate system is relevant.

  Perhaps it is good enough to say that motion relative to arbitrary coordinate systems have equal amplitude. That would seem to lead to Wheeler's concept of superspace.

[61] October 1979

- Mach's principle - Suppose Mach's principle is not correct. Suppose we really have absolute space.

  Then we need to explain why inertial frames are observed to coincide very accurately with the bulk of matter in the universe.

  October 1979: No, that could be consistent with absolute space.

- Main method of attack:

  1. Find out specifically how our metric depends on cosmology.
  2. We think we know how the wave properties of particles depend on the metric.
  3. Combine the two theoretically.

  October 1979 - The above is backwards. First, a quantum formulation in terms of a quantum superposition of geometries and of a quantum superposition of particle trajectories, then the classical limit to get a classical metric.

- Three ways to picture interactions in the "bucket of dust"

  1. Describe a path by listing in order of with which particle.
  2. Interactions like with QED, but cannot involve geometry or distances or directions.
  3. Interactions defined after appearance of geometry - self-consistent bootstrap method.

  (1. may be the same as 2.)

  October 1979 - The third method is the correct one at least for a first approximation.

- We need something like statistical mechanics to deduce geometry. Depends on having large numbers of particles.

  Geometry is a sort of average behavior due to the interactions of a large number of particles.

- Amplitude for a process.

  All the ways a process can occur.

  Amplitude for each way.

  Amplitude for a way in which absolute motion is relevant.

  Representations in superspace.

  Likelihood.

- Energy and momentum are the sources of the gravitational field. Since frequencies and wavelengths are associated with energy and momentum, waves are associated with the source of the gravitational field. That waves have not been observed to be associated with the sources of any other fields including E&M, suggests the hypothesis that the wave properties of particles is a gravitational phenomenon.[62]

  October 1979 - Wave properties are the natural state. Interactions lead to larger wavenumbers and larger frequencies, and therefore toward a classical limit. (any kind of interaction, gravity, E&M, etc.)

[62] ca 1967

20.14 General

- We would like our laws to make sense when applied to hypothetical situations and/or cosmologies and/or mass distributions.
- Three ways to view physical law:[63]
  1. Physical law is the set of rules by which nature operates. In this view, our equations and methods for making calculations do not constitute physical law, but only approximations to it.
  2. Physical law is a man-made model used to describe, classify, approximate, and make calculations and predictions about the way nature operates.
  3. Physical law is both of the above. This view is inconsistent, but seems to be the most popular view.
- Research Ideas:
  1. Make a rain cloud in a laboratory.
  2. Study an air bubble in free fall.
  3. Shape of flames.
  4. Plasma physics.
  5. Sea surf - patterns, simulate on a computer, measure how far waves come up.
  6. Do sea gulls know when it's high tide?
  7. Making walls or towers withstand surf.
  8. Action of surf on sand.
  9. Using surf as an analog computer.
- Sometimes I wonder whether life as a child is experiencing new things and life as an adult is re-experiencing and reminiscing and wanting to relive an experience. Nostalgia.
- A city becomes big when the number of people working in the city is greater than the number of people living in the city who work.[64]
- Energy and momentum are good to have.[65]
- Energy and momentum are conserved.
- value in writing a computer program to test vague ideas
- A given conjecture may:
  1. Clearly disagree with observation,
  2. predict consequences no different from present theories,
  3. predict consequences differing from those of present theories in only subtle ways, requiring careful experiments to distinguish the two.[66]

[63] ca May-July 1974
[64] April 2014: There is a logical problem with that statement.
[65] For a given amount of energy, more momentum means less entropy. April 2014
[66] ca April 1974

- What are the requirements for a unified theory?
- Fourier analysis is related to translation invariance because the exponential function is related to conservation of momentum.

  Thusit is also related to the autocorrelation function, because that is related to translation invariance.
- Ontology - the science of being or reality; the branch of knowledge that investigates the nature, essential properties, and relations of being. (view of the world)

  Epistemology - the theory or science of the method and grounds of knowledge, especially with reference to its limits and validity. (what you can derive from that view)

  From Julian Barbour, summer 1975
- Leibniz relative time - the independent variable that we use when observing changes in the universe.[67]
- I can't believe that nature knows about branch points.
- Purpose of what I'm doing - to define the problem and point to directions to look for a solution
- Time is related to expansion geography.

  Time is the coordinate where something changes.

  Time is the coordinate which is not symmetric to reversal.
- Existence $\neq$ survival.
- We can consider compound systems as particles.
- Integral equation illustrating the use of the singularity expansion method.

  Given E, solve for J.

  $\int \Gamma \cdot J \mathrm{d}s = E$

  Γ includes radiation condition. Expand around singularities of Γ.
- Generalization of calculation of distances with a metric in spaces with non-integer dimension.[68]

  $\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \equiv \sum_{i=1}^N \sum_{j=1}^N g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$

  $\rightarrow \int_0^N \int_0^N g_{ij}\,\mathrm{d}i\,\mathrm{d}j\,\mathrm{d}x^i\,\mathrm{d}x^j$

  $\rightarrow \int_0^N \int_0^N g(ij)\,\mathrm{d}i\,\mathrm{d}j\,\mathrm{d}x^i\,\mathrm{d}x^j$
- Law of sufficient reason
- It is possible to feel an idea coming the same way we feel something with our fingers. Some ideas and thoughts feel differently in the same way that different things feel differently to the touch.

[67] From Julian Barbour, summer 1975

[68] See paper by Stillinger.

- Consider a physical law - say conservation of energy. One point of view is that it is a law and is always true. In that case, we don't need an explanation for why it is true.

  Second point of view - Suppose that energy is not conserved absolutely, but only that it is usually conserved, or mostly conserved, or that the probability of it's not being conserved in any process is very unlikely. Suppose we observe that it seems to be conserved. Then we need to explain why it appears to be conserved even though it is not absolutely conserved.

- Feynman - The most reasonable possibilities often turn out not to be the situation.

- The most important part of making a decision is to decide upon what to base the decision.

20.15 Ideas

- In the relation $\nabla \cdot U = u_x + v_y + w_z$, the first two terms nearly cancel.

- Meso scale = 10 to 200 km

  Large scale = 200 to 40000 km.

- For the background medium, assume: hydrostatic equilibrium, geostrophic flow, and consider deviations from these conditions in the calculations.

20.16 More Ideas

- October 1979 - Background caused by a massless, charged universe. Background $\equiv A_\mu(x)$.

  Can we get by without a metric? No, but we can get a metric without matter.

  $S = S_{\text{mat}} + S_{\text{geom}} + S_{\text{int}} + S_{\text{field}}$.

  $S_{\text{field}} = g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}$, $S_{\text{mat}} = 0$, and $S_{\text{geom}} = 0$ implies flat space implies we can choose $g^{\mu\nu} = \eta^{\mu\nu}$.

  If J^μ is at rest with respect to $\eta^{\mu\nu}$, then $S_{\text{field}} = 0$. Get a quantum superposition of flat metrics. S_{field} is an even function of relative acceleration and relative rotation rate. Because space is flat, we can use ordinary Maxwell theory in that flat space.

  Setting the variation of S to zero implies $\eta^{\mu\nu}$ fixed with charge distribution.

- 4 November 1979 - We need to develop formulas for product integrals and product differentials just as we have for ordinary integrals and differentials.[35, Hoyle and Narlikar]

- 4 November 1979 - In choosing a Lagrangian, we need to put in sources for the particle fields.

  Also, (or maybe instead), we should consider as particles only quantities that are conserved, such as Leptons and Baryons and chargeons. Then we don't have to consider sources.

20.17 Additional ideas on cards

1. - By specifying the commutation relations of the differential operators in a differential equation, we don't need to specify the equation in a particular representation. We can derive general properties of the equation.

2. - Read [36, Ashtekar, 1980]
   - Read [37, Kaku, 1975]

- Read [38, Woolf, 1980, Some strangeness in the proportion]
- Read [39, Weiss, 1981]
- In defending my paper, I must not attack the referee.[69]
- Check [40, 41, Courant and Hilbert] for "variety invariants" for arbitrariness in eigenvalues of propagation equation.[70]

  Also see if there is a Hamiltonian or Lagrangian for the system. Look for conserved quantities, constants of the motion.[71]
- Check out Weyl quantization for the acoustic-gravity wave problem.[72]
- Read papers by Bohr and Einstein.[73]
- Read [42, Von Neumann's book]
- See paper by Niels Bohr and reply by Einstein in [43, book by Schilpp]
- Read [44, book about Niels Bohr]
- Read [45, Paul Adrien Maurice Dirac on the occasion of his seventieth birthday]
- Read [46, Quantum theory and gravitation, edited by A. R. Marlow]
- Read [47, Philosophical problems concerning the meaning of measurement in physics by Morgenau]

  Makes a distinction between measurement and preparation of a state.

  Maintains that the projection postulate is unnecessary.

  An ensemble of measurements is necessary to define the mean and variance of a measured quantity.
- Read [48, On the path of Albert Einstein]
- Read [49, The conceptual development of quantum mechanics]

  Chapter 9 discusses two basic problems in QM: completeness and measurement. He refers to Von Neumann's book[42]
- Read [50, Matrix mechanics]

  Describes operation of cloud chamber, photographic emulsion, bubble chamber, and counter as measuring devices.

  Makes the distinction between <u>indeterminate</u> and <u>uncertainty</u>. Any physical variable (such as position), that has no definite value unless experimental arrangements are made to measure it, is said to be <u>indeterminate</u>. A physical quantity which is merely <u>uncertain</u> has a definite value, which however, is unknown to the experimenter.

  Examines the standard paradoxes in the light of the above distinction.

  A measurement occurs when

  For Schrödinger's cat, the indeterminacy disappears when the photon interacts, or fails to interact, with the counter.
- Read [51, Quantum Mechanics. V. 1. Fundamentals by Gottfried]

  Volume II was never published.

  See chapter IV for the theory of measurement.

[69] Berlin, August 1981
[70] The phrase "variety invariants" does not appear in the index for either volume.
[71] Jeff Rauch in Berlin, August 1981
[72] Berlin, August 1981
[73] Berlin, August 1981

The idea of a measurement is to make a 1-to-1 correspondence between a macroscopic property of a system and an internal property of a system.

Once we have separated the original wave packet macroscopically (as in the Stern-Gerlach experiment), then we approximate the real density matrix by one in which we neglect cross terms between states in one packet with states in another packet. The idea is that the difference could never be measured because it would take very long to bring the packets back together again.

$\mathrm{Tr}\,\rho^2 = 1$ for a pure state, less for a mixed state.

$\mathrm{Tr}\,\rho^2$ is time invariant under the Schrödinger equation (when the Hamiltonian is Hermitian).

- Read [52, Eddington's 1928 book]

  See pages 252-253: All measurements can be reduced to pointer readings (or the intersection of two world lines).

3.
- Read [34, Dirac's book].
- Read [53, book on measurement].
- Read [54, Bohr's book].
- Read [55, paper by Fulling].

4.
- As Maxwell's equations are silent on whether source-free EM fields exist, so are Einstein's equations silent on whether source-free gravitational fields exist. As we have no strong evidence indicating the absence of source-free fields, it seems appropriate to allow for the possibility of their existence.
- The calculus of variations is for the classical limit. A continuum is a classical approximation.

5.
- Read [56, 57, Robert Parker's papers].

6.
- Why do inertial frames in our universe appear not to rotate relative to the stars?
- Define Mach's principle $\equiv$ inertial frames tied to global distribution of matter.
- Reasons
  (a) Classical
  (b) Quantum
- Mach conjectured that the distant stars determine inertial frames here.
- Point out that I have a different to a solution.

7.
- There is more to gravity than curvature.

8.
- Write a paper: "Mach's principle versus independent degrees of freedom for the geometry" (inertial frames).
- Read [58, Bohr & Rosenfeld's paper]. E&M has to be quantized. Independent degrees of freedom implies quanta.

9.
- We normally think of a classical theory as being complete within itself. But this does not have to be. A correct classical theory must be justified in terms of the classical limit of the correct quantum theory.[74]

[74] Thursday 11 September (probably 1969, probably not 1975 or 1980)

- I need to emphasize application to a general action and general properties of quantum gravity.
- I'm not solving any problems in quantum gravity today.

10.
- I am working on quantum cosmology rather than quantum gravity.
- Quantum cosmology versus quantum gravity.
- Factoring out diffeomorphism group: I need to point out that I am considering cosmologies. For small patches (like our galaxy or solar system, or local things like a few black holes), I would get the usual result.

11.
- Point out that by basing physical law on a Lagrangian that depends only on relative quantities, we satisfy Mach's ideas on general covariance.
- Relative configuration space. Intrinsic configuration space.
- Intrinsic configuration space = Wheeler's superspace.
- Minisuperspace - C. W. Misner
- I need to point out that my formalism satisfies the first and second Mach's principle.
- We only observe relative motion.
- There is no independent time.
- General Relativity Theory has some Machian features.

12.
- (a) Physically equivalent systems must have the same action.
  (b) Physically different systems may have the same action.

13.
- I assume that the initial state is a pure state.
- Hawking - density matrix = operator for partition function.
- Will the wave function over 3-geometries spread quickly without all that matter out there?

14.
- Write a paper: "Measuring the geometry"
  What is the operator that corresponds to measuring the position of the cannonball?
- Is the geometry directly measurable?
  No, we infer it indirectly by observing the trajectories of bodies or particles.
  $< \phi_2, t_2 | \phi_1, t_1 >$ measurable.
- $S(g, \phi)$ already includes our test electron as part of ϕ.
- $< \phi_2, t_2 | \psi >$ requires an integration over g_2.
- Hawking - physical state made up of an integral over final 3-geometries.
- $H(G^{(3)}, \pi) = M(G^{(3)}, i\ \nabla)$ There should be a slash through that gradient.

15.
- Develop physics in a sparse universe.
- If there is a relationship between the stars and local inertial frames, then what would physics in our solar system look like if there were no stars, that is if the sun and planets and other solar system bodies comprised the only matter in the universe?
- Would the solar system behave any differently if there were no stars?
  If so, in what way?

- Define inertial force field.

  Standard meaning of Mach's principle: The inertia goes away with the stars.

  My meaning of Mach's principle: Classical (non-quantum) behavior goes away with the stars.

- How do we define strength of inertial interaction? Is it that as we have fewer and fewer stars, that inertia gets less and less, or that geometry becomes more and more affected by quantum fluctuations?
- Instead of doing an empty space calculation, I should add the Earth plus a well equipped laboratory and maybe the solar system as test particles.[75]
- Calculate action of a single mass in Minkowski space.
- Hawking - Boundary term $\int K ds$ diverges in asymptotically flat case - must subtract off the surface term for flat case - won't work in arbitrary spaces.
- Hawking - Action depends on what asymptotic space it is - difficult to calculate if not asymptotically flat.

16.
- How can I specify the matter distribution independently from the geometry?

  We don't, we simply notice that only certain initial states contribute to classical final state.

- Bertotti: Consider all solutions $g(x)$ to Einstein's equations for all possible matter fields $\phi(x)$. Separate these into equivalent classes. Consider all equivalence class that have the same ϕ.
- Read [59, 60, N. H. Christ & T. D. Lee - Columbia preprint, "Quantum expansion of soliton solutions", 1975].

  Read [61, S. S. Chang - Northeastern: Phys. Rev., "Quantization of Yang-Mills fields by separation of gauge variables", 1979].

  Separate gauge from dynamic degrees of freedom.

- Hawking - Flat space is the only thing we can define that is gauge invariant.
- Hawking - Can't define Green's functions in curved space. Can define diffeomorphism that moves points around in curved space.
- Write paper about including all Minkowski spaces in path integrals.
- Hawking - Says that the matter distribution determines the geometry - that you can't really compare the matter distribution for 2 different geometries.
- Equivalence classes
- Bertotti - $T_{\mu\nu}$ is not sufficient to specify E & B for null fields. ($B^2 = E^2$, $E \cdot B = 0$)
- Anowitt says there are constraints on the initial and final 3-geometries.[76]

17.
- Are there constraints on the 3-geometry?

  No.

- What is Helmholtz's theorem?

  From Wikipedia, the free encyclopedia

  There are several theorems known as the Helmholtz theorem:

  - Helmholtz decomposition, also known as the fundamental theorem of vector calculus

[75] Tuesday 16 September 1969 or 1975

[76] 1980

  - Helmholtz theorem (classical mechanics)
  - Helmholtz's theorems in fluid mechanics
- Are my solutions maxima or minima for S?
  If they are maxima, then I have a tachyon.
  (a) Only one extremum
  (b) Integrant > 0 because $\rho > P$
  Implies extremum is a minimum of S.
- Duff (at Imperial College) criticizes Hawking's space-time foam.
  Get Duff's reprint from Oxford conference in April 1980.
- Arnowitt:
  Instantons are solutions that are regular on Euclidean space but not when continued back to Minkowski spacetime.
  Example: $\frac{1}{x^2} \to \frac{1}{c^2t^2-x^2}$
- Bertotti:
  Measure inertial frames by the angle of star light as a function of t.
  The drawing shows an angle $\theta(t)$ between a line pointing to a star and a line pointing to a quasar.
- Express gravitational vector potential in differential forms.[77]
- Does God play dice?
- Read De Witt's papers on the quantum theory of gravity.[62, 63, 64]
- Read Feynman's paper in Magic Without Magic.[65]
- Read 2 papers by Penrose in GRG vol. 7.[66, 67]
  Self dual Weyl tensor
  Can't get Hilbert space.
- Notes on twisters from Paul Tod in blue notebook.
- Read Feynman's book on statistical mechanics. [68]
- Read papers in Phys. Rev. D ca 1977 by Gibbons & Hawking. [69]
- Ian Moss:
  Cutoff for renormalization requires coupling constant to vary to keep the Green's function constant so that cutoff doesn't make any difference. Coupling constant same for difference.
  Read Collins & Macfarlane, Phys. Rev. D10, pp 1201-1202.[70]
  Read Collins, Phys. Rev. D10, pp 1213-1218. [71]
- Read Advances in twister theory; ed by L P Hughes & R S Ward. [72]
- Read [73, Penrose in Battelle Rencontres (fat blue book)].
- Read paper by Nick Woodhouse (N. M. J. Woodhouse) on Wick rotation.
- Derek Raine recommends reading DeWitt's chapter [74] in Hawking & Israel's book[75].
- Read papers by Abers & Lee, Physics Reports, ca 1974 on Yang-Mills symmetry breaking. [76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99]
- Read paper by Jackiw in Phys. Rev. ca 1974-1976. [100] Calculates spontaneous symmetry breaking in one-loop approximation in Yang-Mills.
- Does my symmetry property hold for all orders?

[77] July 2014. I seem to remember that I have done that. However, the formula cannot be derived using differential forms.

Chapter 21

Mach's Principle[1]

21.1 Energy and momentum are gravitational properties of matter

In one sense, energy and momentum are clearly gravitational quantities, since they form part of the energy-momentum tensor, and are thus sources of the gravitational field. In fact, I think it is significant that the wave properties of particles are apparently associated only with the sources of gravitational interactions and not with the sources of strong, electromagnetic, or weak interactions.

However, I think it is important to establish that energy and momentum are purely gravitational quantities rather that merely having a strong relationship to gravitation. That is, I will argue, using Mach's principle, that energy and momentum, being inertial quantities are really gravitational quantities. At least, I can make the argument for kinetic energy. I'm actually not sure yet whether the argument can be extended to include potential energy, or whether it should be.

In performing an analysis based on Mach's principle, I may not necessarily restrict myself to the conventional interpretations of theoretical results nor to the standard theories for explaining observations because I want to leave as much room as possible in the theoretical structure to accommodate a wide variety of interpretations of the relationship between wave properties and inertial properties. In particular, I want to be careful in the interpretation of the relationships between the macroscopic and the microscopic properties of matter.

21.2 Statements of Mach's principle

There have been many statements of Mach's principle. In fact, some people have said that there are probably as many statements of Mach's principle as there are people who have an interest in Mach or his principle. Some people even doubt that any of these statements should even be called a "principle." So here, I am putting down a small list of various statements of "Mach's Principle."

1. The inertia of a body is not an intrinsic property of the body, but is due to an interaction (probably gravitational) with the rest of matter in the universe.

2. Inertial frames are determined by the matter distribution in the universe.

3. The metric tensor is determined (completely) by the matter distribution in the universe.

4. Inertial reaction forces and "fictitious" inertial forces are real forces,and they are caused by a gravitational interaction with the rest of matter in the universe.

[1] ca 1974

5. Only relative motion between bodies is observable, and therefore, only relative motion between bodies can enter into physical laws. (That is, absolute space does not exist.)

6. Non-inertial frames should be just as valid as inertial frames for describing the motions of bodies.

7. The inertial forces that are required to correctly describe the motions of bodies in non-inertial frames must have a physical origin related to the motions of all matter in the universe relative to that frame, and not be a consequence of having chosen an incorrect frame.

One feature common to most of these is that the particular quantity (inertial frame, metric, or inertia) is completely determined by the distribution of matter in the universe, a considerably stronger statement than that it is merely affected by the distribution of matter.

My position is that although I believe in the nonexistence of absolute space as a philosophical principle, I recognize that there is no a priori physical reason for the nonexistence of absolute space. Until predictions based on either the existence or nonexistence of absolute space conflict with observation, then either philosophical principle must be considered as a valid basis to construct a physical theory. Although I intend to base all of my calculations on the nonexistence of absolute space, I will not argue that one must do so.

Although as Bondi said "If absolute space exists, it is strange that it waits until second derivatives to reveal itself."

The relationship between these various statements is not obvious. Especially, since an inertial frame is not the same in Newtonian theory as in Einstein's theory, and the most obvious definitions of inertia in the two theories do not agree. In Newtonian theory, an inertial frame is one in which Newton's second law holds without having to include inertial forces. In Einstein's theory, an inertial frame is one which follows a geodesic. The active gravitational mass of Newtonian theory is replaced by the energy momentum tensor in Einstein's theory. Passive gravitational mass and inertial mass, present in Newtonian theory, are not present in Einstein's theory because they have been set equal by the equivalence principle. Inertia in Newtonian theory is of two types. First, the dp/dt term in Newton's second law. Second, the inertial forces required to make Newton's second law valid in non-inertial frames. These terms are easily recognizable because they involve the inertial mass of the body whose motion is being considered, and the acceleration of the reference frame relative to an inertial frame. Because it is possible to transform one type of inertial force into another by transformation to a reference frame accelerating with respect to the first frame, both of the two types of inertial forces must be of the same origin. In that case, the Newtonian inertial frame loses its special significance, so that it is useful to adopt the Einstein definition of inertial frame.

21.3 Derek Raine's formulations of Mach's principle

Derek Raine had some different kinds of statements of Mach's principle:

1. General statement of Mach's principle by Mach and Einstein: The resistance of a body to changes in its state of motion arises from its gravitational interaction with all the other matter in the universe.

2. Philosophical statement of Mach's principle: Because motion is relative, inertial frames must be determined by the distribution of matter in the universe.

3. Physical statement of Mach's principle: The equivalence principle strongly suggests that inertial forces are really gravitational forces.

4. Mathematical statement of Mach's principle: That the metric tensor be determined completely by the distribution of matter in the universe.

Chapter 22

Do magnetic field lines rotate with a bar magnet?[1]

To help answer the above question, we consider the following questions about the rotation of an axially symmetric bar magnet about its axis of symmetry.[2]

1. Does a charge move if placed near a rotating bar magnet? Notice that if the magnetic field lines are considered to rotate, then the charge is moving relative to the magnetic field lines.
2. Suppose a charge is caused to revolve around a fixed bar magnet. What will happen? Notice that the charge and the bar magnet have the same relative motion as in the previous question.
3. What will happen to a current in a wire that surrounds a fixed bar magnet?
4. What will happen if a bar magnet rotates inside a fixed loop of wire?
5. What happens in the above cases if motion relative to the inertial frame is considered?

The above question [that is, the title of this chapter] has no meaning, in that the same measurements will result no matter what the answer to the question.

First, assume that the magnetic field lines do not rotate. Then as seen in a non-rotating frame, charges within the magnet will see a radial force directed inward or outward depending on the sign of the charge and whether the rotation is parallel or antiparallel to the magnetic field within the magnet. The charges will move according to the force they see until a charge distribution is set up so as to create a static electric field of the correct magnitude and direction to exactly oppose the magnetic force on charges rotating with the magnet. When equilibrium is reached, there will be a surface charge of one sign and within the magnet will be a volume charge distribution of the opposite sign.

For the case of a long cylindrical bar magnet of radius R and uniform magnetic field B_0 when the magnet is not rotating, we get the following results if we assume the magnet to also be a perfect conductor. In the rest frame:

$$B_z = \gamma^2 B_0 = B_0/(1 - \omega^2 r^2/c^2) \tag{22.1}$$

$$E_r = -vB_z = -\omega r \gamma^2 B_0 = -\omega r B_0/(1 - \omega^2 r^2/c^2) \tag{22.2}$$

[1] 1974. Electromagnetic Lense-Thirring effect

[2] A paper by David Bartlett et al. (1977)[101] has just come to my attention. They get no effect from spinning a bar magnet, but claim that such a result is consistent with assuming either that the magnetic field lines rotate or do not rotate with the magnet. I think that agrees with my analysis.

In a frame rotating with the magnet:

$$B_z = \gamma B_0 = B_0/(1 - \omega^2 r^2/c^2)^{1/2} \tag{22.3}$$

$$E = 0 \tag{22.4}$$

The surface charge density is

$$\sigma = \omega R B_0/(1 - \omega^2 R^2/c^2) \tag{22.5}$$

The volume charge density is

$$\rho = -2\omega B_0/(1 - \omega^2 r^2/c^2)^2 \tag{22.6}$$

One interpretation of these results is that the field lines do rotate with the magnet inside the magnet, but far from the magnet the field would be very nearly the same as when the magnet did not rotate, because the electric field of the charge distribution would be small, and the magnetic field from that moving charge distribution would also be small. Thus outside the magnet, the lines seem to stay fixed in space. This is not really a correct way of looking at the situation. In any frame, we have a magnetic field, and an electric field. Neither field "moves." They just "are."

Consider two identical bar magnets with B equal and parallel rotating about their shared axis of symmetry. Assume the gap separating them is small. Put a test charge in the gap. As observed in a non-rotating frame, the test charge will initially accelerate in the radial direction if it is initially at rest, because it sees an electric field. Opposite charges will accelerate in opposite directions. The electric field is due to the charge distribution within the magnet. An observer rotating with the magnet can also test the electric field within the gap. He will observe that in his frame there is no electric field in the gap. If he puts a negative charge, a positive charge, or a neutral particle in the gap initially at rest, he will observe initially equal outward accelerations on all three kinds of bodies due to the centrifugal force.

Chapter 23

Mach's principle and quantum theory I: A quantum statement of Mach's principle[1]

This is the first in a series of reports on the re-examination of the interpretation of the relationship between the wave and inertial properties of matter. It is not intended to be a finished paper even in the preprint sense.

My purpose in distributing this now is to encourage comments, criticisms, and discussion of the ideas set forth herein. These would be especially helpful because I am working essentially alone in a field I have never worked in before. I am particularly concerned that I may be unaware of some work relevant to this investigation.

I realize that this investigation will not universally be considered relevant to the mainstream of research in physics. I am interested in comments relevant to that point of view also.

Abstract

Standard quantum theory depends strongly on a relationship between the wave and inertial properties of particles. If Mach's principle is correct in stating that the inertia of a particle is not intrinsic, then the standard interpretation that the above relationship is one between (intrinsic) wave properties and (intrinsic) inertial properties needs re-examination.

I assume that the general problem of isolating inertial effects can be reduced to that of inertial effects on particle propagators. Comparison of the Machian view of inertia with a quantum view leads to the following quantum statement of Mach's principle:

The action, which gives the path-integral contribution of a path to the amplitude for propagating from one given point to another, arises from an interaction with the rest of matter in the universe.

Future work will be to develop models (restricted by reality) that give the action as a function of the path and the distribution of matter in the universe.

23.1 Motivation

Mach's principle can be approximately stated: The inertia of a body is not an intrinsic property of a body, but instead arises from an interaction with the rest of the matter in the universe. Mach (1911, pp. 76-77)[102, pp. 76-77] presents the best philosophical arguments I have seen to support his hypothesis.

[1] Essay number 17 in the printed list of essays. 16 April 1974

I was first introduced to Mach's principle in 1967 by a paper "on the origin of inertia" (Sciama, 1953)[11]. During that year I began to wonder if the wave properties of particles might not also arise from this interaction because of the close connection between the wave properties of particles and their inertial properties. I am devoting this year (September 1973 to September 1974) to investigating that possibility.

Standard quantum theory is a formalism strongly dependent on the well known relationships

$$\hbar\omega = E \tag{23.1}$$

connecting frequency with energy, and

$$\hbar\mathbf{k} = \mathbf{p} \tag{23.2}$$

connecting the de Broglie wavelength

$$\lambda = 2\pi/|\mathbf{k}| \tag{23.3}$$

with momentum.

To make the connection with inertia, we notice that inertia is the resistance of a body to a change in its state of uniform motion. In classical mechanics, inertia is represented by the more precisely defined quantities, momentum and kinetic energy. Thus we may consider the momentum and kinetic energy in (23.1) and (23.2) to represent the inertial properties of the particle. Similarly, wavelength and frequency represent the wave properties of particles. Hence (23.1) and (23.2) is a relationship between the wave and inertial properties of particles. In the standard interpretation of (23.1) and (23.2) both the inertial and the wave properties are considered intrinsic to the particle.

If, however, Mach's principle is correct (as I think it is), the inertial properties of a particle are not intrinsic, making the standard interpretation of (23.1) and (23.2) unacceptable. Either the wave properties of a particle are also not intrinsic, or there must exist an interaction connecting the intrinsic wave properties of a particle with the induced inertial properties of a particle. In either case, the interpretation of the relationship (23.1) and (23.2) needs re-examination.

So far, I have managed to reduce the general problem of inertial effects to that of inertial effects on particle propagators. Further, by comparing the Machian view of inertia with a quantum view, I have been able to identify the following quantum statement of Mach's principle:

The action, which gives the path-integral contribution of a path to the amplitude for propagating from one given point to another arises from an interaction with the rest of matter in the universe. In the following pages I develop the quantum statement of Mach's principle.

My goal for the future is to develop several hypothetical models, consistent with observation, that show the dependence of the action on the distribution of matter in the universe. Hopefully, the form of such models will Indicate possible experimental tests of the models.

23.2 Assumption: Inertial effects on all physical phenomena can be attributed to inertial effects on particle propagators.

Inertial effects play a strong role in physical phenomena from celestial mechanics down to elementary particle physics. The direct role played by inertia is often hidden by mathematical techniques. An example is the assumption that an isolated physical process (isolated except for the presence of distant matter in the universe) is invariant under transformations belonging to the Poincaré group (Lorentz group plus translations).

If Mach's principle is correct, not only will such an interaction determine the inertia of each particle in the universe, but it will determine which frames (if any) have the property of being inertial frames, the properties of those frames, and the transformation laws connecting one to another. Thus, if Mach's principle is correct, the properties of the Poincare group arise from an interaction of local matter with the rest of matter in the universe. (Or, if we choose to consider the Poincare group as a mathematically defined object, then Mach's principle suggests the possibility that an interaction with the rest of matter in the universe determines that transformations belonging to the Poincare group connect one inertial frame to another.)

It is important, but difficult, to isolate inertial effects correctly, being careful to include all effects that are truly inertial and to exclude all non-inertial effects. It is tempting to include any effect that seems universal and constant. In addition to the properties of the Poincare group, our whole geometry (and the fact that we have one) and the form for the free particle propagators are probably inertial effects. In fact, the next paragraph suggests that it is probably appropriate to choose the particle propagators as representative of all inertial effects.

The electromagnetic interaction in a general coordinate frame is more complicated than in an inertial frame in flat space time. Part of the complication arises in applying the Lorentz force equation to a charged particle in a general frame. That complication, however, can be attributed directly to the inertial effects on the propagator for the charged particle in question. In addition, there are differences in Maxwell's equations which relate the electromagnetic fields to the source currents, These differences manifest themselves directly as differences in the free space Green's function or photon propagator. The differences in the photon propagator can in turn be attributed directly to inertial effects. For example, a photon does not in general travel in a straight line in an accelerated or rotating frame. Thus we can attribute inertial effects on both aspects of the electromagnetic interaction to inertial effects on particle propagators. I will assume without further justification that this property applies to all physical phenomena, so that to study the effects of inertia on particle propagators is sufficient for a study of the effects of inertia on all physical phenomena.

Notice that to satisfy Mach's principle it is not sufficient merely to have rest mass determined by the distribution of matter in the universe as, say, a mass field as Hoyle and Narlikar (1972b, c)[103, 104] have done because that would not account for the inertia of the photon. To fully satisfy Mach's principle it may be necessary that the form of particle propagators be determined by the distribution of matter in the universe.

23.3 What is inertia?

Classically, the motion of a particle is determined by two types of interactions: those we can account for in terms of recognized sources, and those we cannot account for. Historically, we have called the first type of interaction "forces", and the second "inertia". Inertia is not usually classified as a real (as opposed to ficticious) force because its source has not been positively identified, although Mach's principle suggests that it is in some way due to an interaction with the rest of (matter in the universe.

23.4 Machian interpretation of inertia

Mach's principle would deny the existence of the second type of interaction and instead assign inertia to an interaction with the rest of matter in the universe. Unfortunately, the apparent independence of inertia on space and time defies a direct test of Mach's hypothesis.

To illustrate a possible relation between a Machian view of inertia and a quantum view, I put forth the following question: Why does light travel along a geodesic in free space? Mach's principle answers: Because of an interaction (probably gravitational) with the rest of matter in the universe. The following section answers the same question from a quantum point of view.

23.5 A quantum interpretation of inertia

Consider a path integral calculation (Feynman & Hibbs, 1965)[21] of a free space particle propagator (Green's function). All conceivable paths connecting the two points contribute to the total amplitude. However, those paths whose action is an extremum with regard to variation of the path contribute more than others because paths near the extremal paths interfere constructively, while paths distant from extremal paths interfere destructively with their neighbors because their phases are incoherent. In the classical limit, the extremal paths contribute so overwhelmingly that the contribution of all other paths to the amplitude can be neglected. The extremal paths are then classical trajectories and in the geometric interpretation are geodesics.

Thus, in answer to the question: Why does light travel along a geodesic in free space? quantum theory answers: All paths contribute to the amplitude for the propagation of light from one given point to another, but in the classical limit, only geodesies contribute significantly to the amplitude because they have an action that is an extremum.

23.6 A quantum statement of Mach's principle

Both the Machian and the quantum interpretation of inertia stated in the previous two sections are so compelling that I want to state Mach's principle in a form that embodies the quantum interpretation. I propose the following quantum statement of Mach's principle:

The action, which gives the path-integral contribution of a path to the amplitude for propagating from one given point to another, arises from an interaction with the rest of matter in the universe.[2]

Two alternate physical interpretations of this statement of Mach's principle are possible;

1. Interaction with the rest of matter in the universe gives wave properties to particles;
2. Interaction with the rest of matter in the universe alters the phase relationships among the conceivable paths.

23.7 A quantum interpretation of pre-geometry

Pre-geometry is the term wheeler (1968)[19] uses to describe a situation in which a geometry does not exist. A useful quantum definition of pre-geometry is a situation in which the action is independent of the path. Clearly, in such a situation, classical trajectories would not exist, and we would generally say that geometry does not exist. (Notice that inertia in the usual sense would not exist either.) This quantum interpretation seems to be in philosophical agreement with Wheeler's (1968, p. 294)[19, p.294] statement, "Not first geometry and then quantum principle, but first the quantum principle and then geometry!"

23.8 Future work – Search for an explicit relation giving the action as a function of the path and the distribution of matter in the universe

To continue this investigation, it is necessary to find several models that explicitly express the action as a function of the path and the distribution of matter in the universe. These models must satisfy certain criteria to be acceptable:

[2] Sometimes it is either not useful or not possible to express the path integral contribution of a path to the amplitude in terms of an action (Hoyle and Narlikar, 1972a)[35]. The above quantum statement of Mach's principle can clearly be generalized to apply to such a case.

1. Application of these models for the action must not lead to consequences that are in disagreement with known observations. In particular, the action models must lead to particle propagators that are in observational agreement with the presently known forms of the particle propagators.

2. These models may not assume any a priori preferred coordinate frames.

3. As the average density of matter in the universe approaches zero, the action must (at least in the first approximation) become independent of the path.

One productive line of thought may be the following: Since all paths contribute to the path integral, so do those paths that go out to the stars and other distant matter. We can assume that these virtual paths "know" the distribution of matter because the paths are where the matter is and can determine where the inertial frames are. We might ask, what is the procedure by which one of these virtual paths, by probing distant matter, gives an action that appears to be proportional to its path length?

It is necessary to be careful here. That may not be a valid question to ask because path length may be defined to be equal to the action in the classical limit. After all, we are considering geometry itself to be an inertial effect.

It is likely that this investigation is closely related to the problem of developing a formulation of quantum geometrodynamics (Wheeler, 1968[19]; DeWitt, 1972[105]). The two investigations may, in fact, complement each other.

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Historical note - June 2011

I wrote this during a year that I had taken a leave of absence from the National Oceanic and Atmospheric Administation and had an office in the University of Colorado Physics Department.

During that time Paul Adrian Maurice Dirac was a guest in the Department for a few months. I thought it would be useful to take advantage of his expertise and have a dialog with him about a possible relationship between Mach's Principle and Quantum theory. When I asked him, he replied, "Well, I don't believe in Mach's Principle, so I don't think there would be much point." I was so shocked, could not think of any useful response, so that was the end of it.

Chapter 24

Thoughts for discussion[1]

- Do wave packets occur only as the result of interactions?
- A wave packet can be formed out of several electrons at various energies, or out of one electron which through interactions loses some of its energy. For that case, the spectrum is for the same electron at different times.
- If quantum theory is applied to light, does it imply that the velocity of light is uncertain?
- Is there a local shift in charge on an accelerating electron?
- Is the group velocity really the same as the phase velocity of light in free space? In an experiment, which velocity is actually measured?
- Is quantum mechanics a gravitational or geometrical phenomenon?
- There is a problem in unifying quantum mechanics and relativity because of uncertainty.
- Energy and momentum are the sources of the gravitational field. Since frequencies and wavelengths are associated with energy and momentum, waves are associated with the sources of the gravitational field. That waves have not been observed to be associated with the sources of any other fields including E&M, suggests the hypothesis that the wave properties of particles are gravitational phenomena.
- The spin of an electron is like a gravitational magnet. There should be gravitational spin-orbit coupling.
- The difference between E&M and gravitation may be that there are many stars in the universe that have mass but are not charged.
- An accelerating reference frame corresponds to an angular velocity of the x-t axes. Calculate the corresponding centrifugal force and the Coriolis force.
- Do the size of Bohr orbits and the energy levels in the hydrogen atom depend on the distribution of matter and mean density of the universe?
- We need to recognize the contribution of the stars when Maxwell's equations are written in a non-inertial frame.
- It seems to be an inertial property that bodies with finite rest mass cannot reach the speed of light.

[1] 9 October 1974

- Is attenuation purely a macroscopic process? Zitterbewegung is like Brownian motion.
- In a Fourier spectrum of an EM wave, if the fundamental is quantized, that does not mean the harmonics have to be quantized.

Chapter 25

Discussion of 11 October questions by David Peterson[1]

2) Question: What is the physical picture of electron spin and why does it have a spinor representation. Why are complex numbers required?

Response: Are complex numbers required for describing electron spin?

3) Question: Create a physical model of vacuum fluctuations. (MTW p. 1190): "Of all the remarkable developments of physics since WW II, none is more impressive than the predictions and verifications of the effects of the vacuum fluctuations in the electromagnetic field on the motion of the electron in the hydrogen atom."

Response: Here is my physical picture of vacuum fluctuations. Suppose the electromagnetic field were not intrinsically quantized, but that electromagnetic radiation is observed to be produced and absorbed in lumps because atomic levels are quantized, and therefore, atomic transitions will produce or absorb only lumps of radiation. Consider a closed system containing radiation and matter. (We make the closed system simply by isolating some arbitrary region of space for some length of time.) Consider the initial spectrum of the radiation. If it is not intrinsically quantized, then each Fourier component may have any amplitude, rather than being restricted to $(n+1/2)\hbar\omega$. Suppose the matter then absorbs as much as it can from that Fourier component. What is left is by definition the vacuum state, and that amount is greater than or equal 0 and less than one photon. If we repeat this experiment many times, we will get a different amount left in the vacuum each time, but always in the range between none and one photon. The average value will be 1/2 a photon, the deviation from that for each particular case is vacuum fluctuation. Thus, it is easier to explain vacuum fluctuation by assuming the EM field behaves like a classical wave than it is by assuming it is quantized.

Alternate explanation - The above analysis has the same difficulties that the standard textbook analyses have. We have assumed a finite (so that the Fourier spectrum will be discrete) isolated (so that there will be no energy exchange between the outside and our test radiation field) system. Those conditions are inconsistent. There is no way to confine a radiation field without interacting with it. The standard approach is to consider the system to be in a box with perfectly conducting walls. Thus, although the radiation field interacts with the walls, their only effect is to give periodic boundary conditions to the field. There will be no energy exchange between the walls and the various Fourier components of the radiation field. That is a macroscopic view. Microscopically, there is a lot of energy exchanged between the atoms and the conduction band electrons in the metallic wall, but the average effect is zero. Thus, a radiation field confined in a box is isolated macroscopically, but interacts and fluctuates microscopically. This would lead at least to vacuum

[1] 23 October 1974

fluctuations.

I think the correct way to treat this problem is to somehow think of a system that can be correctly analyzed microscopically without too much difficulty. For example, it is probably possible to analyze the effect of fluctuations caused by interactions with the metal using statistical mechanics.

4) Question: According to my simple calculations, a classical electron (radius 2.8f) would have to have a shell spinning at $v = 375c$ to give its measured magnetic moment. At a diameter of .5f this would give $v = 4000c$. If electron size is less than 10^{-16} cm, the speed is ridiculous. In physical terms, how does the Dirac Theory solve this paradox?

Response: If you put the charge in a ring instead of a shell at the classical electron radius, you need only a velocity of 137c. On the other hand, then you have to redo the calculation for the classical electron radius for a ring instead of a shell. Since the ring packs the charge closer together, it could be at larger radius (I bet a lot larger).

Chapter 26

The uncertainty principle[1]

The important point on the uncertainty principle is not the point that it is a trivial consequence of Fourier analysis, but that the detailed structure of an electron wave packet can not be measured. Electrons are not divisible.

[1] 20 November 1974

Chapter 27

A gravitational vector potential[1]

[1] 18 December 1974

A Gravitational Vector Potential

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First Draft - 18 December 1974

Summary (see page 23)

I suggest that standard quantum theory (without second quantization) is based unknowingly on the simplification of being able to represent an external gravitational potential by a vector potential. If this is correct, then to do second quantization correctly, we not only must consider the mutual and self interactions within the system of particles under consideration, but we must also generalize quantum theory to correctly apply to the case where the external field cannot be represented by a vector potential.

gravitational interaction. In addition, we know that A represents the electromagnetic interaction with the particle. If we now rewrite (2), we have, using (3),

$$i\hbar \frac{\partial}{\partial x^{\mu}} = P_{\mu} = \pi_{\mu} + eA_{\mu} \tag{4}$$

With the above interpretation, we now have the wave properties of matter represented on the left of (4) and the gravitational and electromagnetic properties on the right. Equation (4) suggests that π and A play analogous roles; the former for the gravitational interaction, the latter for the electromagnetic interaction. If that is true, then do both represent momenta, or do both represent vector potentials? I don't know the answer to the first question, but the answer to the second is a qualified yes. the following analysis shows that a quantity related to π can be thought of as a gravitational vector potential. The qualification is that two particles at the same point in space but having different velocities will in general experience a different gravitational vector potential. This difficulty disappears when we require that the gravitational vector potential (GVP) at a particle be irrelevant to physical law unless it is measured in the coordinate frame of that particle. This restriction makes sense on physical grounds. The coordinates of particles relative to some arbitrary observer should be irrelevant to physical law. Only the relative coordinates of the particles should be relevant.

2. Derivation of the gravitational vector potential (GVP)

Equation (4) suggests the possibility that the mechanical 4-momentum can be thought of as a vector potential. We will see below, that with some qualifications, this is nearly true.

The geodesic equation including the electromagnetic interaction for a particle of rest mass m and charge e is

$$m\left(g_{\mu\nu}\ddot{x}^{\nu} + g_{\mu\nu,\alpha}\dot{x}^{\alpha}\dot{x}^{\nu} - \tfrac{1}{2} g_{\alpha\beta,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}\right) + e\left(A_{\mu,\alpha} - A_{\alpha,\mu}\right)\dot{x}^{\alpha} = 0 \quad (5)$$

(where $\cdot$ = d/ds and comma denotes ordinary (not covariant) differentiation). If we now choose a frame moving with the particle (where $\dot{x}^0=1$, $\ddot{x}^{\nu}=0$, $\dot{x}^i=0$), then (5) becomes

$$m\left(g_{\mu 0,0} - \tfrac{1}{2} g_{00,\mu}\right) + e\left(A_{\mu,0} - A_{0,\mu}\right) = 0 \quad (6)$$

The $\mu = 0$ case gives

$$m\, g_{00,0} = 0 \quad (7)$$

while the $\mu = i$ case gives

$$m\left(g_{i0,0} - \tfrac{1}{2} g_{00,i}\right) + e\left(A_{i,0} - A_{0,i}\right) = 0 \quad (8)$$

Let us define a covariant vector g which in the rest frame of the particle has the components

$$g_{\mu} = \left(\tfrac{1}{2} g_{00},\ g_{i0}\right) \quad (9)$$

~~Then substitution of (9) into (6) gives~~
Then

$$m\left(g_{\mu,0} - g_{0,\mu}\right) + e\left(A_{\mu,0} - A_{0,\mu}\right) = 0 \tag{10}$$

is the same as (8). We still need to require (7) as an additional condition. That the geodesic equation has the same form as the Lorentz force equation in the rest frame of the particle has been pointed out by Einstein (1946) & Pandres (1957).

Now notice that the frame independent equation

speculation

$$m\left(g_{\mu,\nu} - g_{\nu,\mu}\right)\dot{x}^{\nu} + e\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\dot{x}^{\nu} = 0 \tag{11}$$

reduces to (10) in the rest frame of the particle. Thus, (11) should be equivalent to the geodesic ~~e~~equation in an arbitrary coordinate system. To check this, we notice that in the rest frame of the particle,

Now we want to calculate a formula for g_μ in an arbitrary ~~frame~~

$$\pi_\mu \equiv m\, g_{\mu\nu}\dot{x}^{\nu} = m\left(g_{00},\, g_{i0}\right) \tag{12}$$

and

$$m\, g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = m\left(g_{00},\, 0,\, 0,\, 0\right) \tag{13}$$

Thus, if we define a covariant vector D which in the rest frame of the particle has the components

$$D_\mu = (1, 0, 0, 0) \tag{14}$$

then from (9) we get

$$g_\mu = \frac{\pi_\mu}{m} - \frac{1}{2}\, g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} D_\mu \tag{15}$$

Transforming (14) from the rest frame to an arbitrary frame gives

$$D_\mu = \frac{\partial \tau}{\partial x^\mu} \tag{16}$$

where τ is the proper time of the particle. Substituting (16) into (15) gives

$$g_\mu = \frac{\pi_\mu}{m} - \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \frac{\partial \tau}{\partial x^\mu} \tag{17}$$

Substituting (17) into (11), and using the identities

$$\frac{\partial \tau}{\partial x^\mu} \dot{x}^\mu = 1 \tag{18}$$

and

$$\ddot{x}^\alpha = \frac{\partial \dot{x}^\alpha}{\partial x^\mu} \dot{x}^\mu \equiv \dot{x}^\alpha_{,\mu} \dot{x}^\mu \tag{19}$$

μ

gives

$$m\left(g_{\mu\alpha} \ddot{x}^\alpha + g_{\mu\alpha,\nu} \dot{x}^\alpha \dot{x}^\nu - \frac{1}{2} g_{\alpha\beta,\nu} \dot{x}^\alpha \dot{x}^\beta + \right.$$
$$\left. - \frac{1}{2} \frac{\partial \tau}{\partial x^\mu} \dot{x}^\nu \frac{\partial}{\partial x^\nu} \left(g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \right) \right) +$$
$$+ e\left(A_{\mu,\nu} - A_{\nu,\mu} \right) \dot{x}^\nu = 0 \tag{20}$$

(See Appendix A for the details of this calculation.) In the rest frame of the particle, we have

$$\dot{x}^\nu \frac{\partial}{\partial x^\nu} \left(g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \right) = \frac{\partial g_{00}}{\partial \tau} = g_{00,0} = 0 \tag{21}$$

because of (7). Since the quantity in (21) is a scalar, it must be zero in all frames, so that (20) is seen to be identical with the geodesic equation (5) including the electromagnetic interaction. Thus, (11) is equivalent to the geodesic equation including electromagnetic interaction.

Using the definition of the mechanical 4-momentum π given in (12), the gravitational vector potential in (17) can be rewritten.

$$g_\mu = \frac{\pi_\mu}{m} - \frac{1}{2m^2} g^{\alpha\beta} \pi_\alpha \pi_\beta \frac{\partial \tau}{\partial x^\mu} \tag{22}$$

We are now considering π to be more than just the mechanical 4-momentum of the particle. We are considering it to be a vector field defined over a finite region of space which includes the world line of the particle and which is equal to the mechanical 4-momentum on the world line. We are now also considering τ to be a scalar field defined over a finite region of space which includes the world line of the particle and which is equal to the proper time of the particle on the world line of the particle.

By itself, (11) is not equivalent to (5), but we must require (21) in addition as a constraint. Equation (21) can be rewritten using the definition of π in (12) as

$$g^{\mu\nu} \pi_\mu \frac{\partial}{\partial x^\nu} \left(g^{\alpha\beta} \pi_\alpha \pi_\beta \right) = 0 \tag{23}$$

Let us define a covariant vector

$$q_\mu = m g_\mu + e A_\mu$$

$$= \pi_\mu - \frac{1}{2m} g^{\alpha\beta}\pi_\alpha \pi_\beta \frac{\partial \tau}{\partial x^\mu} + e A_\mu \qquad (24)$$

so that (11) becomes

$$\left(q_{\mu,\nu} - q_{\nu,\mu}\right)\dot{x}^\nu = 0 \qquad (25)$$

We can think of q as the total (gravitational plus electromagnetic) vector potential, since (25) with the constraint (23) is equivalent to the geodesic equation (5) with the electromagnetic interaction.

We have now achieved our goal, since the combination π + eA is part of the total vector potential, just as suspected from equation (4). However, there is an extra term in (24), and we might wonder about its significance, since it involves a gradient, just like the term on the left of (4).

3. The geodesic as the characteristic in the solution for the GVP

Direct calculation from the geodesic equation shows that the gravitational interaction can be written as arising from the following 4-vector potential:

$$m\, g_\mu = \pi_\mu - \frac{1}{2m}\, g^{\alpha\beta}\pi_\alpha \pi_\beta\, \tau_{,\mu} \tag{3-1}$$

where $\pi_\mu(x^\mu)$ is a vector field satisfying the constraint

$$\left(g^{\alpha\beta}\pi_\alpha \pi_\beta\right)_{,\nu}\, g^{\mu\nu}\pi_\mu = 0\,. \tag{3-2}$$

and $\tau(x^\mu)$ is a scalar field satisfying the constraint

$$\tau_{,\beta}\quad g^{\alpha\beta}\pi_\alpha = m \tag{3-3}$$

$g^{\alpha\beta}$ is the metric tensor, and m is the mass of the particle (a constant).

Direct calculation from (3-1) gives (see appendix B)

$$\begin{aligned}(g_{\mu,\nu} - g_{\nu,\mu})\, g^{\nu\lambda}\pi_\lambda &= \frac{1}{m}\,\pi_{\mu,\nu}\, g^{\nu\lambda}\pi_\lambda\; + \\ &+ \frac{1}{2m^2}\, g^{\alpha\beta}{}_{,\mu}\,\pi_\alpha\pi_\beta\,\tau_{,\nu}\, g^{\lambda\nu}\pi_\lambda\; + \\ &- \frac{1}{m}\, g^{\alpha\beta}\pi_{\alpha,\mu}\pi_\beta\left(1 - \frac{1}{m}\tau_{,\nu}\, g^{\lambda\nu}\pi_\lambda\right) - \frac{1}{2m^2}\left(g^{\alpha\beta}\pi_\alpha\pi_\beta\right)_{,\nu} g^{\lambda\nu}\pi_\lambda\,\tau_{,\mu}\end{aligned} \tag{3-4}$$

Substitution of the constraints (3-2) and (3-3) into (3-4) gives

$$(g_{\mu,\nu} - g_{\nu,\mu})\, g^{\nu\lambda}\pi_\lambda = \frac{1}{m}\,\pi_{\mu,\nu}\, g^{\nu\lambda}\pi_\lambda + \frac{1}{2m}\, g^{\alpha\beta}{}_{,\mu}\,\pi_\alpha\pi_\beta \tag{3-5}$$

Setting (3-5) equal to zero, it can be solved for Π by the method of characteristics to give

$$\frac{dx^\alpha}{ds} = \frac{1}{m} g^{\alpha\beta} \pi_\beta \qquad \text{(a)}$$

$$\text{(3-6)}$$

$$\frac{d\pi_\mu}{ds} + \frac{1}{2m} g^{\alpha\beta}{}_{,\mu} \pi_\alpha \pi_\beta = 0 \qquad \text{(b)}$$

Substituting (3-6a) into (3-3) gives

$$m \frac{\partial \tau}{\partial x^\beta} \frac{dx^\beta}{ds} = m \qquad \text{(3-7)}$$

or,

$$\frac{\partial \tau}{\partial x^\beta} \frac{dx^\beta}{ds} = \frac{d\tau}{ds} = 1 \qquad \text{(3-8)}$$

Thus, we may take $s = \tau$. Denoting derivatives with respect to s or τ by a dot, and substituting (3-6a) into (3-5) and setting (3-5) to zero gives

$$m(g_{\mu,\nu} - g_{\nu,\mu})\dot{x}^\nu = \pi_{\mu,\nu} \dot{x}^\nu + \frac{1}{2m} g^{\alpha\beta}{}_{,\mu} \pi_\alpha \pi_\beta = 0 \qquad \text{(3-9)}$$

or,

$$m(g_{\mu,\nu} - g_{\nu,\mu})\dot{x}^\nu = \dot{\pi}_\mu + \frac{1}{2m} g^{\alpha\beta}{}_{,\mu} \pi_\alpha \pi_\beta = 0 \qquad \text{(3-10)}$$

which agrees with (3-6b).

Equation (3-6) is the Hamiltonian form of the geodesic equation. Thus, the characteristics of (3-5) correspond to classical trajectories, with $s=\tau$ as the proper time and m for the mass of the particle, and π as the mechanical 4-momentum of the particle. That (3-6b) is the same as (3-10), shows that our geodesic equation resulted from a force equation of the same form as that for an electromagnetic interaction. In fact, we could have chosen a vector potential

$$g_{\mu} = m\, g_{\mu} + e\, A_{\mu} \tag{3-11}$$

that combines the gravitational and electromagnetic vector potentials weighted by the mass and charge of the particle. Then substitution of (3-11) into the vector force equation

$$\left(g_{\mu,\nu} - g_{\nu,\mu}\right)\frac{1}{m}\, g^{\nu\lambda}\pi_{\lambda} = \left(g_{\mu,\nu} - g_{\nu,\mu}\right)\dot{x}^{\nu} = 0 \tag{3-12}$$

would yield the usual geodesic equation combined with the Lorentz force equation.

$$\dot{\pi}_{\mu} + \frac{1}{2m}\, g^{\alpha\beta}{}_{,\mu}\,\pi_{\alpha}\pi_{\beta} + e\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\dot{x}^{\nu} = 0 \tag{3-13}$$

From (3-6a) it follows that

$$\pi_{\mu} = m\, g_{\mu\nu}\,\dot{x}^{\nu} \tag{3-14}$$

thus verifying that along a trajectory π corresponds to the mechanical 4-momentum of the particle. Substituting (3-14) into (3-13) gives the more usual form for the geodesic equation with Lorentz force

$$m\left(g_{\mu\nu}\ddot{x}^{\nu} + g_{\mu\nu,\alpha}\dot{x}^{\alpha}\dot{x}^{\nu} - \frac{1}{2} g_{\alpha\beta,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}\right) + e\left(A_{\mu,\nu} - A_{\nu,\mu}\right)\dot{x}^{\nu} = 0 \quad (3\text{-}15)$$

We now see from (3-1), that the gravitational 4-potential contains a term in addition to the mechanical 4-momentum.

We see that the gravitational vector potential is determined by the metric and by the vector and scalar fields π and τ. Given a metric, we have some freedom in choosing π and τ, subject only to the constraints (3-2) and (3-3). Once we have chosen π and τ, the trajectories are determined to be the characteristics of (3-5). Thus, once we have determined a particular vector potential of the form (3-1), the trajectories are determined to belong to a particular family of trajectories, and these trajectories are a subset of allowed geodesics for that metric.

Substituting (3-6a) into (3-2) gives

$$\left(g^{\alpha\beta}\pi_{\alpha}\pi_{\beta}\right)_{,\nu} m\frac{dx^{\nu}}{ds} = 0 \quad (3\text{-}16)$$

or,

$$m\frac{d}{ds}\left(g^{\alpha\beta}\pi_{\alpha}\pi_{\beta}\right) = 0 \quad (3\text{-}17)$$

Thus, using (3-14),

$$g^{\alpha\beta}\pi_\alpha\pi_\beta = m^2 g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \tag{3-18}$$

is constant along a trajectory. In the rest frame, of the body (3-18) becomes

$$m^2 g_{00} \tag{3-19}$$

Direct calculation using (3-6a) shows that

$$\frac{d}{ds}\left(g^{\alpha\beta}\pi_\alpha\pi_\beta\right) = 2m\dot{x}^\mu\left(\frac{d\pi_\mu}{ds} + \frac{1}{2m}g^{\alpha\beta}{}_{,\mu}\pi_\alpha\pi_\beta\right) \tag{3-20}$$

Thus, (3-6b) guarantees that (3-18) will be constant along a trajectory, and taking s = τ, (3-8) guarantees that (3-3) will be satisfied.

The equations (3-6) are the general form of a geodesic. Thus, all geodesics may be written in the form (3-6). Since (3-6) guarantees that (3-2) and (3-3) will be satisfied, all geodesics may be written in terms of a vector potential of the form (3-1). Further, we have shown that all characteristics of (3-1) are geodesics.

Thus, we have a method for classifying geodesics of a particular metric into families, each family resulting from a particular vector potential of the form (3-1).

I have been wondering whether the standard wave equations are unwittingly based on this classification of geodesics into families by the vector potential (3-1).

4. Equations of motion for a system of particles in terms of the GVP seen by each particle in the system

For describing the motion of a system of particles, it is useful to choose some arbitrary coordinate frame for the coordinate frame of the observer. This method allows the motion to be well defined and the description of the motion to be reasonably objective. In addition to describing the kinematics from the point of view of an arbitrary coordinate frame, we also use such a frame to describe the dynamics of the particles, that is, the laws that govern their behavior. As far as we can tell, writing the laws of motion in terms of an arbitrary coordinate frame has led to no difficulties.

However, one interpretation of Mach's principle is that the coordinates of objects relative to some arbitrary observer should be irrelevant to physical law, that only the relative coordinates should even enter physical law. One method for trying to implement this belief is to try to express physical law in terms of relative coordinates while still using an arbitrary observer to express the coordinates of the objects and to calculate the relative coordinates. It is not obvious that such a technique will guarantee that only/truly relative coordinates will enter into physical law.

A more promising approach is to write the laws of motion for each particle in its own rest frame. Such an approach has the advantage of expressing the laws governing the motion of a particle in terms of what we imagine to be the environment of the particle. It has two disadvantages. First, the method will have to be altered for massless particles, which have no rest frame. Second, it will be difficult to combine and compare the resulting equations without taking the chance of introducing quantities which should be irrelevant to physical law.

Still, it is worth trying for the insight that we might gain. In addition, it may be that we are required to use that method to correctly describe nature. Consider the geodesic equation with Lorentz force written in the rest frame of the particle from (6)

$$m\left(g_{\mu 0,0}-\tfrac{1}{2}g_{00,\mu}\right)+e\left(A_{\mu,0}-A_{0,\mu}\right)=0 \qquad \text{(4-1)}$$

Consider what this equation means. It is supposed to describe the motion of the particle, and yet, neither the position, velocity, nor acceleration of the particle appear explicitly in the equation. They can't of course, because all of these quantities are identically zero in the rest frame of the particle. The equation (4-1) determines the motion of the particle in the following way: Both the metric g and the EM 4-potential A depend on the ~~relative~~ motion of the other particles in the universe relative our frame of observation (in this case, the rest frame of the particle). Thus, the potentials seen by the particle (g and A) will depend on how the particle moves relative to the rest of the universe. Equation (4-1) requires the particle to move in just such a way that the gradients of the potentials seen by the particle just satisfy (4-1). We can assume that particles are thus able to sense not only the potentials at a point, but at least over a distance large enough to recognize a gradient in potential.

path integral, formulation

To describe a system of particles, we would need one equation for each particle of the type (4+1). Such a system of equations would of course be equivalent to the geodesic equations we now use to describe such a system if we keep everything else the same. One of the requirements now present in General Relativity, for instance, is that equations be covariant. That they say the same thing when written in another coordinate frame. If, however, ~~will~~ we write the laws of physics only in the coordinate frames of particles, then we can relax that restriction some. It's not obvious that once we have relaxed that restriction, that our new system of equations will still be equivalent to the old.

5. What experiments can be done to show that each particle sees a different GVP?

Section 3 showed that a system of particles or potential particles could be classified into families by using the GVP. In general, different particles having different velocities, but otherwise seeing the same environment (the same metric, the same A, and the same sources producing those potentials) will see a different gravitational vector potential (GVP) because the GVP involves the velocity of the particle.

This is a different situation from the way we normally think of a potential. Normally, the usefulness of a potential is that once we have chosen a coordinate frame, we can calculate the potential in that frame once we know the positions and velocities of all of the sources in that frame. That same potential will then allow us to calculate the motion of any particle put into that environment. We can use the same potential for all particles.

As we have noticed, the GVP does not satisfy this criterion. Given a coordinate frame and a collection of gravitational sources in that frame, there is no way to calculate the GVP. The reason is, of course, that gravitation is not a vector field. It is not possible to characterize it in terms of just a vector potential. What is happening, is that for a given particle, or family of particles, it is possible to find a vector potential such that application of a force equation of the same form as the Lorentz force gives the correct motion of the particle (the geodesic equation). Nothing more. Such a vector potential has no simple relation to the sources.

Now that we have recognized that our vector potential is not really a vector potential at all, we want to know, "Can we tell the difference?" The obvious way to tell the difference is to take a collection of gravitational sources and show that they lead to different GVP's for two different particles. Unfortunately, in practice, the only gravitational sources we have been able to establish experimentally are sources of static gravitational fields such as the sun and the earth. Thus, comparison with the sources is so far not a method that can be used to experimentally show that we don't really have a vector field. In fact, except for local sources of a static gravitational field, the metric is determined experimentally by local experiments, not by reference to sources.

There is a second method. Section 3 showed that particles having trajectories belonging the same family saw the same GVP (and its spatial dependence), but that particles having trajectories belonging to different families saw different spatial dependence of the GVP. All we have to do then, is to take a given reference frame in which some gravitational sources have produced a given metric field, and compare the motions of two particles taken one at a time, and show by direct calculation from their observed trajectories that they must have seen different GVP's. However, there is a difference between what we can see as equation calculators, and what the particles can see. Can a given particle tell that it is acting under the influence of a different vector potential than another? Classically, at least, the answer must be no. The only way that two such particles might have a chance of telling the difference would be if they were at the same place or near the same place. But under those conditions, each particle would see the effect of the other, and each would in fact see a different metric, so that they would see different vector potentials anyway.

I think we can safely say then, that at least classically, we can think of particles as being acted on by a GVP, at least in the restricted sense mentioned above. In fact, not only can the particles not tell that the gravitational field is not from a vector potential, but they should not even be able to tell the difference between the usual 4-potential which has an EM origin and the GVP. The particle will simply see the sum of the two weighted by its charge and mass respectively, as given in equation (24).

Maybe there is a chance to tell the difference on the quantum level. We can imagine a good possibility, because in the path integral formulation of quantum mechanics, we think of a particle as taking all possible paths in going from one point to another (or even if it stays in the same place). Thus, we can imagine a particle as being able to sense the state of the entire universe, thus even in a sense occupying the whole universe. Thus, we can imagine the possibility of a particle realizing (although I can't think of a particular mechanism) that another particle does not see the same GVP as it does. We might imagine that the particles having trajectories belonging to the same family might all be in phase. A family of trajectories looks something like a bunch of parallel curves; it is very easy to think of them as having phase coherence. We might even imagine that interference effects result from the interaction of particles whose trajectories belong to different families.

6. Possible relation to second quantization

Section 5 raised the possibility that perhaps quantum effects might be used to show that a GVP is inadequate to describe the gravitational interaction. In fact, I now raise the possibility that the quantum effects that we observe are the result of trying to describe the gravitational interaction (a tensor field) in terms of a vector interaction (the GVP).

First, I want to argue that standard quantum theory (only first quantization) treats the gravitational interaction as a vector interaction, and that the standard wave equation results from that point of view.

Consider the operator equation (4)

$$P_\mu = i\hbar \frac{\partial}{\partial x^\mu} = \pi_\mu + eA_\mu \tag{6-1}$$

and let it operate on the wave function

$$\psi = e^{-\frac{i}{\hbar}\frac{1}{2m} g^{\alpha\beta}\pi_\alpha \pi_\beta \tau} \tag{6-2}$$

← pulsating particle

where we will restrict the covariant vector field π in (6-2) to be such that not only does the constraint (23) apply, but in addition, we will require that the scalar field (3-18) be constant everywhere, not only on each trajectory. This simply restricts our consideration to a single species of particle in effect. In addition, we should notice, that not only does the scalar field τ satisfy (3-3) (same as (18)), but in addition, its directial derivative perpendicular to the trajectory is zero.

only if medium is isotropic

With these considerations, operating on (6-2) with (6-1) gives

$$P_\mu \psi = \frac{1}{2m} g^{\alpha\beta} \pi_\alpha \pi_\beta \frac{\partial \tau}{\partial x^\mu} \psi = (\pi_\mu + eA_\mu)\psi \qquad (6\text{-}3)$$

or,

$$\left(\pi_\mu + eA_\mu - \frac{1}{2m} g^{\alpha\beta} \pi_\alpha \pi_\beta \frac{\partial \tau}{\partial x^\mu}\right)\psi = 0 \qquad (6\text{-}4)$$

Now for comparison, from (24)

$$g_\mu = \pi_\mu + eA_\mu - \frac{1}{2m} g^{\alpha\beta} \pi_\alpha \pi_\beta \frac{\partial \tau}{\partial x^\mu} \qquad (6\text{-}5)$$

Using (6-5), (6-4) can be rewritten

$$g_\mu \psi = 0 \qquad (6\text{-}6)$$

where (6-6) is not an operator equation as was (6-3), but indicates purely multiplication.

Clearly, I have contrived the constants in the wave function (6-2) to make the agreement between (6-4) and (6-5) come out right. However, the wave function is a reasonable one, being a wave that progresses along the trajectory of the particles. My purpose here was to show the similarities between (6-1) and (6-6). The example merely emphasized the similarities. In particular, I wanted to demonstrate the appearance of the gradient term in the two equations. The gradient term in (6-1) represents the basis of wave mechanics. The gradient term in (6-6) comes from trying to represent the gravitational interaction in terms of a vector potential. I am suggesting that somehow the gradient term in (6-1) really results (I don't know how) from trying to treat the gravitational interaction as a vector potential.

We know that standard quantum theory (without second quantization) can be applied in only a limited number of cases. I am now suggesting that those cases correspond to situations where the particles cannot tell that the GVP is not a good representation of the gravitation. First consider the following cases:

1) A single particle in an external gravitational and electromagnetic field. This case can be handled by ordinary quantum theory. In addition, because there is only one particle, its trajectory will belong to a single family.

2) Same as above, but include self interaction; that is, part of the field seen by the particle has been produced by the particle itself. Standard quantum theory no longer handles this case; at least QED is required. In addition, the field seen by the particle no longer corresponds to a single vector potential, because part of the field was produced by the particle itself.

3) A beam of non-interacting particles in an external gravitational and electromagnetic field. This case can be handled by ordinary quantum theory. In addition, because we have a beam, all of the particles have trajectories that belong to the same family.

4) A system of non-interacting particles in an external gravitational and electromagnetic field. This case can be handled by standard quantum theory. In addition, although the particles may have trajectories belonging to many families, there is no way for the particles to be aware of this and act

7. Difficulties with applying the same analysis to the Dirac equation

The motivation to look for a gravitational vector potential (GVP) came from looking at the Klein-Gordon equation. The Schroedinger equation would have worked just as well for the non-relativistic case with less interesting results. The Dirac equation would have also worked just as well in flat space-time, but to then apply the results to a general Riemannian manifold would have been inconsistent.

I am assuming that the equations given by Brill and Wheeler (1957) represent the accepted method to write the Dirac equation in a general Riemannian manifold. With some change in convention, Dirac's equation is:

$$g^{\alpha\beta}\gamma_\alpha \pi_\beta \psi = m\psi \tag{7-1}$$

where the mechanical 4-momentum π is given by

$$\pi_\mu = p_\mu - eA_\mu \tag{7-2}$$

A is the electromagnetic 4-potential, and p is the canonical 4-momentum given by

$$p_\mu = i\hbar\left(\frac{\partial}{\partial x^\mu} - \Gamma_\mu\right) \tag{7-3}$$

where Γ is the spinor connection.

If we now rewrite (7-2) using (7-3) we have

$$i\hbar\left(\frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu}\right) = p_{\mu} = \pi_{\mu} + eA_{\mu} \qquad (7\text{-}4)$$

Actually, this still gives the same motivation to see whether π represents a gravitational vector potential as did (4), but we now have the extra term of the spinor connection.

Reference

Brill, Dieter and John Archibald Wheeler, Interaction of neutrinos and gravitational fields, Rev. Mod. Phys. 29, pp 465-479, 1957.

Appendix A. Verification of the GVP

We want to show that [redacted] the gravitational vector potential (GVP) from (17)

$$g_\mu = g_{\mu\alpha}\dot{x}^\alpha - \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \tau_{,\mu} \qquad \text{(A-1)}$$

when substituted into a Lorentz-type force equation

$$(g_{\mu,\nu} - g_{\nu,\mu})\dot{x}^\nu = 0 \qquad \text{(A-2)}$$

gives the geodesic equation

$$g_{\mu\alpha}\ddot{x}^\alpha + \left(g_{\mu\alpha,\beta} - \frac{1}{2} g_{\alpha\beta,\mu}\right)\dot{x}^\alpha \dot{x}^\beta = 0 \qquad \text{(A-3)}$$

when evaluated on the geodesic.

Here, we are treating $\dot{x}$ as a contravariant vector field that has the same value as the particle velocity on the geodesic. Also, τ is a scalar field that ~~has~~ is equal to the proper time on the geodesic.

Taking the derivative of (A-1) gives

$$g_{\mu,\nu} = g_{\mu\alpha,\nu}\dot{x}^\alpha + g_{\mu\alpha}\dot{x}^\alpha_{,\nu} + \qquad \text{(A-4)}$$

$$- \frac{1}{2}\left(g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right)_{,\nu} \tau_{,\mu} - \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \tau_{,\mu,\nu}$$

Interchanging the indices in (A-4) and expanding gives

$$g_{\nu,\mu} = g_{\nu\alpha,\mu}\dot{x}^{\alpha} + g_{\nu\alpha}\dot{x}^{\alpha}_{,\mu} - \frac{1}{2} g_{\alpha\beta,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}\tau_{,\nu} +$$

$$- g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}_{,\mu}\tau_{,\nu} - \frac{1}{2} g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\tau_{,\nu,\mu} \qquad \text{(A-5)}$$

Subtracting (A-5) from (A-4), and using

$$\tau_{,\mu,\nu} = \tau_{,\nu,\mu} \qquad \text{(A-6)}$$

gives

$$g_{\mu,\nu} - g_{\nu,\mu} = g_{\mu\alpha,\nu}\dot{x}^{\alpha} - g_{\nu\alpha,\mu}\dot{x}^{\alpha} + g_{\mu\alpha}\dot{x}^{\alpha}_{,\nu} +$$

$$- g_{\nu\alpha}\dot{x}^{\alpha}_{,\mu} - \frac{1}{2}\left(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\right)_{,\nu}\tau_{,\mu} +$$

$$+ \frac{1}{2} g_{\alpha\beta,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}\tau_{,\nu} + g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}_{,\mu}\tau_{,\nu} \qquad \text{(A-7)}$$

Multiplying by ẋ gives

$$\left(g_{\mu,\nu} - g_{\nu,\mu}\right)\dot{x}^{\nu} = \dot{x}^{\alpha} g_{\mu\alpha,\nu}\dot{x}^{\nu} - g_{\nu\alpha,\mu}\dot{x}^{\alpha}\dot{x}^{\nu} +$$

$$+ g_{\mu\alpha}\dot{x}^{\alpha}_{,\nu}\dot{x}^{\nu} - g_{\nu\alpha}\dot{x}^{\alpha}_{,\mu}\dot{x}^{\nu} - \frac{1}{2}\left(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\right)_{,\nu}\dot{x}^{\nu}\tau_{,\mu} +$$

$$+ \left(\frac{1}{2} g_{\alpha\beta,\mu}\dot{x}^{\beta} + g_{\alpha\beta}\dot{x}^{\beta}_{,\mu}\right)\dot{x}^{\alpha}\dot{x}^{\nu}\tau_{,\nu} \qquad \text{(A-8)}$$

or

$$\left(g_{\mu,\nu} - g_{\nu,\mu}\right)\dot{x}^{\nu} = \left(g_{\mu\alpha}\dot{x}^{\alpha}\right)_{,\nu}\dot{x}^{\nu} - g_{\nu\alpha,\mu}\dot{x}^{\alpha}\dot{x}^{\nu} +$$

$$+ \frac{1}{2} g_{\alpha\beta,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}\dot{x}^{\nu}\tau_{,\nu} - g_{\nu\alpha}\dot{x}^{\alpha}_{,\mu}\dot{x}^{\nu} +$$

$$+ g_{\alpha\beta}\dot{x}^{\alpha}_{,\mu}\dot{x}^{\beta}\dot{x}^{\nu}\tau_{,\nu} - \frac{1}{2}\left(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\right)_{,\nu}\dot{x}^{\nu}\tau_{,\mu} \qquad \text{(A-9)}$$

Changing dummy indices gives

$$(g_{\mu,\nu}-g_{\nu,\mu})\dot{x}^{\nu}=(g_{\mu\alpha}\dot{x}^{\alpha})_{,\nu}\dot{x}^{\nu}-g_{\beta\alpha,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}+$$
$$+\tfrac{1}{2}g_{\alpha\beta,\mu}\dot{x}^{\alpha}\dot{x}^{\beta}\dot{x}^{\nu}\tau_{,\nu}-g_{\beta\alpha}\dot{x}^{\alpha}_{,\mu}\dot{x}^{\beta}+$$
$$+g_{\alpha\beta}\dot{x}^{\alpha}_{,\mu}\dot{x}^{\beta}\dot{x}^{\nu}\tau_{,\nu}-\tfrac{1}{2}(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})_{,\nu}\dot{x}^{\nu}\tau_{,\mu} \qquad \text{(A-10)}$$

or

$$(g_{\mu,\nu}-g_{\nu,\mu})\dot{x}^{\nu}=g_{\mu\alpha}\dot{x}^{\alpha}_{,\nu}\dot{x}^{\nu}+g_{\mu\alpha,\nu}\dot{x}^{\alpha}\dot{x}^{\nu}+$$
$$-g_{\alpha\beta,\mu}\left(1-\tfrac{1}{2}\tau_{,\nu}\dot{x}^{\nu}\right)\dot{x}^{\alpha}\dot{x}^{\beta}-g_{\alpha\beta}\dot{x}^{\alpha}_{,\mu}\dot{x}^{\beta}(1-\tau_{,\nu}\dot{x}^{\nu})+$$
$$-\tfrac{1}{2}(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})_{,\nu}\dot{x}^{\nu}\tau_{,\mu} \qquad \text{(A-11)}$$

or

$$(g_{\mu,\nu}-g_{\nu,\mu})\dot{x}^{\nu}$$
$$=g_{\mu\alpha}\dot{x}^{\alpha}_{,\nu}\dot{x}^{\nu}+\left(g_{\mu\alpha,\beta}-g_{\alpha\beta,\mu}\left(1-\tfrac{1}{2}\tau_{,\nu}\dot{x}^{\nu}\right)\right)\dot{x}^{\alpha}\dot{x}^{\beta}+$$
$$-g_{\alpha\beta}\dot{x}^{\alpha}_{,\mu}\dot{x}^{\beta}(1-\tau_{,\nu}\dot{x}^{\nu})-\tfrac{1}{2}(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})_{,\nu}\dot{x}^{\nu}\tau_{,\mu} \qquad \text{(A-12)}$$

We have the following identity

$$\ddot{x}^{\alpha}=\frac{\partial\dot{x}^{\alpha}}{\partial x^{\nu}}\dot{x}^{\nu}=\dot{x}^{\alpha}_{,\nu}\dot{x}^{\nu} \qquad \text{(A-13)}$$

Also, on the geodesic, the following is an identity

$$\tau_{,\nu}\dot{x}^{\nu}=\frac{\partial\tau}{\partial x^{\nu}}\dot{x}^{\nu}=1 \qquad \text{(A-14)}$$

Substituting (A-13) and (A-14) into (A-12) gives

$$(g_{\mu,\nu} - g_{\nu,\mu})\dot{x}^{\nu} = g_{\mu\alpha}\ddot{x}^{\alpha} + \left(g_{\mu\alpha,\beta} - \tfrac{1}{2} g_{\alpha\beta,\mu}\right)\dot{x}^{\alpha}\dot{x}^{\beta}$$

$$- \tfrac{1}{2}\left(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\right)_{,\nu}\dot{x}^{\nu}\tau_{,\mu} \qquad \text{(A-15)}$$

which leads to the result in equation (20) in the text.

In addition, we have from (21),

$$\dot{x}^{\nu}\frac{\partial}{\partial x^{\nu}}\left(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\right) = 0 \qquad \text{(A-16)}$$

so that (A-15) becomes

$$(g_{\mu,\nu} - g_{\nu,\mu})\dot{x}^{\nu} = g_{\mu\alpha}\ddot{x}^{\alpha} + \left(g_{\mu\alpha,\beta} - \tfrac{1}{2} g_{\alpha\beta,\mu}\right)\dot{x}^{\alpha}\dot{x}^{\beta} \qquad \text{(A-17)}$$

which shows that (A-2) is equivalent to the geodesic equation (A-3) when the supplementary condition (A-16) is used.

Appendix B. Alternate verification of the GVP

We want to show that the gravitational vector potential (GVP) from (22) or from (3-1)

$$g_\mu = \frac{\pi_\mu}{m} - \frac{1}{2m^2} g^{\alpha\beta}\pi_\alpha \pi_\beta \frac{\partial \tau}{\partial x^\mu} \tag{B-1}$$

where $\pi_\mu (x^\mu)$ is a vector field satisfying the constraint

$$\left(g^{\alpha\beta}\pi_\alpha \pi_\beta\right)_{,\nu} g^{\mu\nu}\pi_\mu = 0 \tag{B-2}$$

and τ (x^μ) is a scalar field satisfying the constraint

$$\tau_{,\beta}\, g^{\alpha\beta}\pi_\alpha = m \tag{B-3}$$

when substituted into a Lorentz-type force equation

$$\left(g_{\mu,\nu} - g_{\nu,\mu}\right) g^{\nu\lambda}\pi_\lambda = 0 \tag{B-4}$$

gives an equation which leads to the Hamiltonian form of the geodesic equation.

Taking the derivative of (B-1) gives

$$\begin{aligned} m\, g_{\mu,\nu} = \pi_{\mu,\nu} - \frac{1}{2m}\left(g^{\alpha\beta}\pi_\alpha \pi_\beta\right)_{,\nu} \tau_{,\mu} \, + \\ - \frac{1}{2m} g^{\alpha\beta}\pi_\alpha \pi_\beta \, \tau_{,\mu,\nu} \end{aligned} \tag{B-5}$$

Exchanging the free indices on (B-5) and expanding gives

$$m\, g_{\nu,\mu} = \pi_{\nu,\mu} - \frac{1}{2m} g^{\alpha\beta}{}_{,\mu} \pi_\alpha \pi_\beta \tau_{,\nu} + - \frac{1}{m} g^{\alpha\beta} \pi_{\alpha,\mu} \pi_\beta \tau_{,\nu} - \frac{1}{2m} g^{\alpha\beta} \pi_\alpha \pi_\beta \tau_{,\nu,\mu} \tag{B-6}$$

Subtracting (B-6) from (B-5) and using

$$\tau_{,\mu,\nu} = \tau_{,\nu,\mu} \tag{B-7}$$

gives

$$m(g_{\mu,\nu} - g_{\nu,\mu}) = \pi_{\mu,\nu} - \frac{1}{2m}\left(g^{\alpha\beta}\pi_\alpha\pi_\beta\right)_{,\nu} \tau_{,\mu} + - \pi_{\nu,\mu} + \frac{1}{2m} g^{\alpha\beta}{}_{,\mu} \pi_\alpha \pi_\beta \tau_{,\nu} + \frac{1}{m} g^{\alpha\beta} \pi_{\alpha,\mu} \pi_\beta \tau_{,\nu} \tag{B-8}$$

Multiplying by $g^{\nu\lambda}\pi_\lambda / m$ gives

$$(g_{\mu,\nu} - g_{\nu,\mu}) g^{\nu\lambda}\pi_\lambda = \frac{1}{m}\pi_{\mu,\nu} g^{\nu\lambda}\pi_\lambda + - \frac{1}{2m^2}\left(g^{\alpha\beta}\pi_\alpha\pi_\beta\right)_{,\nu} \tau_{,\mu} g^{\nu\lambda}\pi_\lambda - \frac{1}{m}\pi_{\nu,\mu} g^{\nu\lambda}\pi_\lambda + \tag{B-9}$$

$$+ \frac{1}{2m^2} g^{\alpha\beta}{}_{,\mu} \pi_\alpha \pi_\beta \tau_{,\nu} g^{\nu\lambda}\pi_\lambda + \frac{1}{m^2} g^{\alpha\beta} \pi_{\alpha,\mu} \pi_\beta \tau_{,\nu} g^{\nu\lambda}\pi_\lambda$$

or, changing dummy indices and collecting terms,

$$\left(g_{\mu,\nu} - g_{\nu,\mu}\right) g^{\nu\lambda}\pi_\lambda = \frac{1}{m}\pi_{\mu,\nu} g^{\nu\lambda}\pi_\lambda + + \frac{1}{2m^2} g^{\alpha\beta}{}_{,\mu}\pi_\alpha\pi_\beta\tau_{,\nu} g^{\lambda\nu}\pi_\lambda + - \frac{1}{m} g^{\alpha\beta}\pi_{\alpha,\mu}\pi_\beta\left(1 - \frac{1}{m}\tau_{,\nu} g^{\lambda\nu}\pi_\lambda\right) + - \frac{1}{2m^2}\left(g^{\alpha\beta}\pi_\alpha\pi_\beta\right)_{,\nu} g^{\lambda\nu}\pi_\lambda\tau_{,\mu} \tag{B-10}$$

Substituting the constraints (B-2) and (B-3) into (B-10) gives

$$\left(g_{\mu,\nu} - g_{\nu,\mu}\right) g^{\nu\lambda} \pi_\lambda = \frac{1}{m} \pi_{\mu,\nu}\, g^{\nu\lambda} \pi_\lambda + \frac{1}{2m}\, g^{\alpha\beta}{}_{,\mu}\, \pi_\alpha \pi_\beta \qquad \text{(B-11)}$$

which will lead to the Hamiltonian form of the geodesic equation when set to zero and solved by the method of characteristics.

I had handwritten some notes on the back page of the above manuscript. These are:

- Apply quantum theory to interacting particles?
- Superposition?
- Superselection rules?

The reference on page 106 to "page 23" is to page 127.

27.1 Afterthoughts - 2009

The interpretaion of the meaning of the gravitational vector potential is not easy, nor is the interpretation of the geodesic equation and the Lorentz force. In the frame of the particle, (6) gives the geodesic equation as the gradient of a tensor field, in the same way that the Lorentz force is the gradient of a vector field.

However, the similarity between the form of the gravitational force and the Lorentz force in the frame of the particle (6) is closer than expressed in the preceding paragraph. For each interaction, there are two terms. The first gives the induction force, while the second term gives the static part of the field. It shows that, except for the connection to the sources, D. W. Sciama's 1953 paper, "On the origin of inertia",[11] was correct in characterizing the gravitational interaction as being analogous to the electromagnetic interaction when looked at in the frame of the body.

The interpretation of the geodesic equation plus Lorentz force in the frame of the body (6) is that the body will move (that is, will adjust its frame) so that the equation (6) is satisfied. In an arbitrary frame, the interpretation is quite different. In an arbitrary frame, A and g are fixed. It is then the $\ddot{x}$ term that is determined by evaluating the rest of the equation for a given $\dot{x}$.

Thus, when we use the gravitational vector potential in an arbitrary frame, it does not have the same meaning as the electromagnetic vector potential, even though it appears to be used in the geodesic equation in a similar way. The difference is that the electromagnetic vector potential is fixed by choosing the frame, but finding the gravitational vector potential requires actually solving the geodesic equation. That is, it still has the same meaning as the terms in the geodesic equation when written in the frame of the body. (The body will choose its frame to satisfy the equation.)

In a discussion of the gravitational vector potential with Ronald Adler on 14 April 2009, he pointed out that a family of freely falling bodies would be called a "timelike geodesic congruence" in the absence of the electromagnetic interaction. I am not sure what it is called when we include the electromagnetic interactions, maybe a timelike trajectory congruence.

This chapter had some speculation that wave mechanics might be an artifact of the gravitational vector potential. This speculation is not valid, however, because the wave properties of particles are not artifacts; they are facts because a particle is an oscillator oscillating at its rest frequency (in its own frame).[18]

[18]Note added in September 2010: Actually, a particle does not oscillate at its rest frequency. That is simply its characteristic frequency in the same way that the plasma frequency is the characteristic frequency of a plasma. The particle will actually oscillate at its driven frequency, which is true for any oscillator. Thus, we may speculate that the rest frequency of a particle is determined by the distribution of matter in the universe. We may further speculate that the universe is a medium like a plasma that has a characterestic frequency. However, there are so many different particles, each with its own characteristic mass and therefore each with its own characteristic rest frequency. So, the medium that is the universe has many characteristic frequencies. How does that work? Also, in a plasma, the electrons in the plasma oscillate with the EM wave. What oscillates in the medium that is the universe, and why are there so many characteristic frequencies? Could it be something like an atom, that has many energy levels, and therefore many characteristic frequencies?

The real significance of the gravitational vector potential, however, (as is expressed in the chapter) is that the form of the wave equation for wave mechanics is based (unknowingly) on the gravitational vector potential. The quantum condition is

$$\tau_{,\mu} \to \frac{2im\hbar}{g^{\alpha\beta}\pi_\alpha\pi_\beta}\frac{\partial}{\partial x^\mu} \ . \tag{27.1}$$

However, because the gravitational vector potential is valid only for a single congruence, the wave mechanics based on it is valid only for a single particle or a congruence of non-interacting particles.

It is possible to write the geodesic equation (including the Lorentz force) in an explicitly covariant form.

$$mU^\alpha_{;\beta}\dot{x}^\beta + eg^{\alpha\mu}(A_{\mu;\beta} - A_{\beta;\mu})\dot{x}^\beta = 0 \ , \tag{27.2}$$

where $U^\alpha = \pi^\alpha/m$ is a vector field as defined in this chapter, and m and e are the mass and charge of the particle in question.

Here, U plays a role similar to that of a vector potential. The tensor character of the geodesic equation is explicit here, so we can raise and lower indices at will. Thus, we may write (27.2) as

$$mU_{\alpha;\beta}\dot{x}^\beta + e(A_{\alpha;\beta} - A_{\beta;\alpha})\dot{x}^\beta = 0 \ . \tag{27.3}$$

The combination $e\dot{x}^\beta$ is proportional to the current 4-vector, and is a source of the electromagetic field, as well as being the quantity upon which the field acts in the Lorentz force. Similarly the quantity $m\dot{x}^\beta\dot{x}^\gamma$ is a source of curvature, and is acted on by curvature. Equation (27.3) suggests that $m\dot{x}^\beta$ is a source of gravitational fields, as well as the quantity upon which the gravitational field acts in the geodesic equation. For both gravitation and electromagnetism, we can convert from mass and charge to mass and charge density and write the force equation as

$$U_{\alpha;\beta}j^\beta_{\mathrm{grav}} + (A_{\alpha;\beta} - A_{\beta;\alpha})j^\beta = 0 \ , \tag{27.4}$$

where $j^\beta_{\mathrm{grav}} \equiv \rho\dot{x}^\beta$.

Another possibility is

$$mF_{\alpha\beta\gamma}\dot{x}^\beta\dot{x}^\gamma + e(A_{\alpha;\beta} - A_{\beta;\alpha})\dot{x}^\beta = 0 \ , \tag{27.5}$$

where

$$F_{\alpha\beta\gamma} \equiv U_{\alpha;(\beta}\tau_{;\gamma)} = \frac{1}{2}\left(U_{\alpha;\beta}\tau_{;\gamma} + U_{\alpha;\gamma}\tau_{;\beta}\right) \ , \tag{27.6}$$

and τ is a scalar field as defined in this chapter. Similarly,

$$F_{\alpha\beta\gamma}T^{\beta\gamma} + (A_{\alpha;\beta} - A_{\beta;\alpha})j^\beta = 0 \ . \tag{27.7}$$

Chapter 28

The quantum basis for Mach's principle[1]

abstract

A modern interpretation of Mach's principle that requires no new theories is that the matter distribution determines the geometry. There are many reasons, however, that the geometry probably has independent degrees of freedom.

It is possible to reconcile this conflict through a coherent quantum superposition of classical time histories.

28.1 Independent degrees of freedom for the geometry?

Both the original interpretation of Mach's principle that matter determines inertial frames and a more modern interpretation that matter determines geometry are inconsistent with independent degrees of freedom for the geometry. However, a weaker interpretation will not explain why inertial frames appear not to rotate relative to the stars unless absolute space exists.

However, a careful reading of Mach shows that his philosophical arguments were based on a principle of general relativity. If we drop the strong interpretation, keep the principle of general relativity, and allow independent degrees of freedom for the geometry on the quantum level, then we obtain the strong result in the classical limit.

28.2 Mach's principle of relativity

Mach espoused at least two related principles. The first is the general principle of relativity (in which all frames of reference are equally valid for expressing physical law)[2]. The second (and the one that bears his name) is that inertia is not an intrinsic property of a body, but instead results from an interaction with the rest of matter in the universe, or, as we say now, matter determines geometry. This second principle Mach derived from the first and from the observation that inertial frames appear not to rotate relative to the stars.

The second principle, however, is inconsistent with the existence of independent degrees of freedom for the geometry, a property we believe the geometry to have from independent considerations.

[1]Essay number 25 in the printed list of essays. October 1979. Early draft of the paper submitted to General Relativity and Gravitation in October 1979

[2]Note added in 2008 - His actual viewpoint was that only relative motions of bodies are observable, and therefore only relative motions of bodies should enter physical law.

But if we drop Mach's second principle, how do we explain why inertial frames appear not to rotate relative to the stars? The answer lies in quantum theory, and in particular in quantum geometrodynamics, in which we allow a quantum superposition of classical states. Most important, we allow both the initial and final states to be quantum superpositions of classical states. We then recover Mach's second principle in the classical limit (in which...).

Mach's principle really is elusive. On the one hand, it seems to be the only way to explain why inertial frames do not rotate relative to the stars without invoking absolute space. But if it is really valid, why has no one been able to find an explicit mechanism to automatically insure it in all these years? Why is the best that we have been able to do to use <u>it</u> to try to make boundary conditions for the field equations? Are we saying that physical law is the field equations plus Mach's principle? (No, field equations plus boundary conditions)

There has always been an uneasiness about Mach's principle. I think that is because Mach's principle is inconsistent with allowing independent degrees of freedom for the geometry.

Chapter 29

The quantum basis for Mach's principle[1]

[1]Submitted to General Relativity and Gravitation 15 October 1979

The quantum basis for Mach's principle

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abstract

Mach's principle seems to require that geometry arise from material sources alone, whereas other considerations require that the geometry have independent degrees of freedom through initial conditions.

We can resolve that apparent inconsistency within a quantum framework if we allow arbitrary initial states which we represent by a quantum superposition of classical states. On the quantum level, the geometry has independent degrees of freedom. In the classical limit (in which we require the action to be stationary with respect to variation of the initial conditions on the free-field part of the geometry in addition to the usual criteria), we recover only selected solutions of Einstein's field equations.

When that selection is unique, we recover Mach's principle automatically. A quantum superposition of spatially homogeneous perfect fluid space-times yields the Robertson-Walker space-time in the classical limit in agreement with a previous Machian classification.

A classical limit in the above sense does not exist for a quantum superposition of empty space-times or a matter distribution consisting only of singularities because, there, the action is independent of the initial conditions. While a classical limit in the above sense may exist for a quantum superposition of asymptotic space-times, the dependence of the action on the geometry in the asymptotic regions is so weak as to leave a quantum superposition of significantly different geometries in those regions.

1. Introduction: Mach's principle versus some independence for the geometry

Mach's principle [1,2,3,4,5,6,7,8] says roughly that the distribution of matter in the universe determines the inertia of every body in the universe. Mach formulated this point of view to avoid using an absolute space to explain why inertial frames appear not to rotate relative to the stars [1,2].

While Einstein's general relativity theory contains Machian effects, in that matter affects inertial frames, inertia in Einstein's theory depends not only on the distribution of matter, but also on initial and boundary conditions. The arbitrariness represented by the initial and boundary conditions can take the place of the arbitrariness of the absolute space that Mach tried to avoid. Empty space-times and asymptotic space-times[1] represent extreme examples of arbitrariness in which initial and boundary conditions completely or nearly completely determine inertia.

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1 On the one hand, using these hypothetical examples to test our physical laws is suspect because it depends on our intuition rather than comparison with observation. On the other hand, we would hope that our laws would be general enough to apply to situations with arbitrary initial conditions.

Sciama and his followers [9,10,11,12] tried to avoid that arbitrariness by using Mach's principle to choose appropriate boundary conditions for Einstein's field equations. In particular, Raine [12] used in part the criterion that there be no source-free component to the initial free-field part (the Weyl tensor) of the gravitational field. His results, that the Robertson-Walker space-time is Machian and that asymptotically-flat space-times are non-Machian agree with our intuition. His criterion that the gravitational field not be source free is equivalent to requiring that the gravitational field have no independent degrees of freedom.

I doubt that the gravitational field has no independent degrees of freedom, however. First, all other fields (the electromagnetic field, for example) have independent degrees of freedom. Second, there is a symmetry between matter and the gravitational field in that matter is a source of gravitational fields, and a strong gravitational field is a source of matter [13,14,15,16,17]. We could logically believe, for example, that matter is not source-free, but it is more symmetric to have independent degrees of freedom for both the matter and the field. Third, the quantization of gravity requires independent degrees of freedom for the gravitational field. Finally, Wheeler's [4,18,19,20] formulation of geometrodynamics in which the gravitational field has independent degrees of freedom through initial conditions seems to be a correct interpretation of the field equations.

But if the gravitational field has independent degrees of freedom then we can no longer explain why inertial frames appear not to rotate relative to the stars. The initial conditions to obtain our universe in which inertial frames seem not to rotate relative to the stars would have been very improbable.

Einstein [21] and Wheeler [22] proposed rejecting open solutions to the field equations to remove the awkward flat empty space-times and the asymptotically flat space-times. That, however, does not give a mechanism to explain why inertial frames appear not to rotate relative to the stars, and rejecting open solutions seems difficult to justify otherwise (See Appendix A).

Thus, Mach's principle seems inconsistent with allowing independent degrees of freedom to the gravitational field. The present paper shows how to resolve that apparent inconsistency by considering phase interference in a quantum superposition of classical states. In particular, section 2 justifies such a quantum superposition of space-times as an initial state, and section 3 shows under what conditions that quantum superposition has a classical limit of a single space-time. Section 4 proposes that those space-times found as a classical limit of the above quantum superposition correspond to Machian cosmologies.

2. Quantum geometrodynamics[2]

In quantum geometrodynamics [19], we do not have a single classical state (3-geometry) at each time, but rather a quantum superposition of 3-geometries (if we take the Everett-Wheeler interpretation [23] of quantum theory seriously). Our universe now is nearly classical, but we believe that quantum effects dominated processes near the initial singularity. Classical states were probably strongly coupled during that time. After the coupling among classical states became negligible,

[2] While we don't yet have a fully satisfactory quantum theory of gravity, we know enough general properties that such a theory must have to see how it relates to Mach's principle.

our universe was probably left as a quantum superposition of classical states. Given that quantum superposition as an initial state, we want to calculate the amplitude for the occurrence of various geometries now.

The path integral method [24] seems most suited to make that calculation. We calculate that amplitude as the sum of the amplitudes for all possible ways of getting from the initial to the final state. If the initial and final states were pure classical states, then each way could be represented by a path in superspace[3]. Where the initial state is a quantum superposition of classical states, however, we must represent each way by the appropriate quantum superpostion of paths in superspace.

We calculate the amplitude for each path in terms of the action associated with that path and we calculate the action as the integral of a Lagrangian[4,5] along that path.

3. The classical limit

Normally, taking the classical limit of a quantum superposition of paths in superspace, in which the endpoints are pure classical states, requires that the action be stationary with respect to variation of the path in superspace while keeping the endpoints fixed. This process

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[3] Superspace is a space in which each point is a 3-geometry.

[4] Although the concepts presented here are in terms of a Lagrangian formulation, they are obviously more general, and would apply, for example, to a product integral [25] formulation as well.

[5] To insure Mach's ideas of relativity [1, 2], the Lagrangian must depend explicitly on only the relative motions of the particles. Barbour [26] generalizes this concept of relativity in a more modern way in terms of relative configurations.

yields a classical time history[6] with the given endpoints. We can consider such a classical time history to be associated with a definite classical initial condition[7].

If the initial state is an arbitrary quantum state (which we may take to be a quantum superposition of classical states), then the above process leads to a quantum superposition of classical time histories, rather than to a true classical limit. The process of finding a classical limit (if one exists) for that situation is not yet finished.

The existence of such a classical limit requires that the initial quantum state be coherent[8]. Incoherent initial states will not be considered further because they are not relevant to the main topic of this paper[9].

To finish finding the classical limit, we require that the action (which is now a functional of the classical initial conditions on the free-field part of the geometry) be stationary with respect to variation of the classical initial conditions[10,11].

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6 By a classical time history, I mean a solution of Einstein's field equations.

7 A classical initial condition is a 3-geometry plus lapse and shift functions [27], or a 3-geometry plus extrinsic curvature with the appropriate constraints.

8 By a coherent state, I mean one in which the quantum amplitude is a continuous function of the classical initial conditions in some parameterization.

9 Unless our concepts of quantum geometrodynamics are wrong, our world would not be as we observe it without such a coherent initial state. I can imagine a mechanism in terms of strong nonlinear coupling of classical states in the distant past for producing such a coherent state.

10 We assume here that the amplitude for the path dominates over the amplitudes associated with the initial state in determining the classical limit.

11 From here on, the term, "classical limit" will be used in this sense.

Such a classical limit will exist only if the action depends strongly enough on the classical initial conditions. Table I classifies space-times according to how strongly the action depends on the classical initial conditions.

Column (a) of Table I includes those space-times for which the action is independent of the classical initial conditions. These include empty space-times[12] and space-times whose only matter is a collection of singularities[13], the latter being asymptotic space-times. Clearly there can be no classical limit for a quantum superposition of such cosmologies, in agreement with a Machian intuition about such space-times.

Column (b) includes those space-times for which the action depends so weakly on the classical initial conditions that space-times which are significantly different would still exhibit constructive interference among themselves. If S_c is the action for a classical time history for which the action is stationary with respect to variation of the classical initial conditions, then all classical time histories having an action S such that

$$|S - S_c| \lesssim \hbar \tag{1}$$

will reinforce constructively, and can thus exist as a coherent quantum state. If these have significantly different geometries in some regions

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12 In empty space-times, there is no matter, so that $L_{matter} = 0$ and also that $T^{\mu\nu} = 0$. $T^{\mu\nu} = 0$ implies $R^{\mu\nu} = 0$ which implies $R = 0$, so that from (B4), $L_{geom} = 0$ for a solution to the field equations. Therefore, from (B3) we get that $L = 0$, and therefore the action S from (B2) is also zero for all empty space-times.

13 See Appendix C.

of space-time, then there will be no classical limit in those regions. Asymptotic space-times are in this category (the non-classical regions will be away from the matter distributions), as are some sparse cosmologies, where the non-classical regions can cover all of space-time, or all of space at some time. Appendix D discusses these space-times in more detail.

The extent to which geometries are "significantly" different is somewhat arbitrary. Our own universe is not purely classical, but has some quantum effects.

The concept that part of a space-time can be nearly classical while large regions might not be is probably new.

Column (c) includes those space-times for which the action depends strongly enough on the classical initial conditions that the geometry is not significantly different for all of those cosmologies for which (1) is valid. These include perfect fluids (if the pressure is not equal to the density and the matter is not too sparse).

A perfect fluid is probably a good approximation for the average large scale matter distribution in our own universe, so it is reasonable to consider perfect fluids further. Appendix E shows that a reasonable action for classical time histories in perfect fluid cosmologies is

$$S = \frac{1}{2}\int (-g(x))^{1/2} \left(\rho(x) - p(x) \right) d^4x \tag{2}$$

where ρ is the density, p is the pressure, and g is the determinant of the metric. The spatial integration is over the whole 3-space. The time integration is from the initial to the final time.

For application to our own universe, it is useful to consider spatially homogeneous perfect fluids. Solutions are known to exist for such matter distributions even when there is a relative rotation of inertial frames and matter distribution [28,29,30,31]. These solutions have pressure, density, expansion rate, shear, and rotation rate that are functions of time and depend on the initial conditions.

Appendix F demonstrates in detail that the classical limit of a quantum superposition of a class of homogeneous space-times is the Robertson-Walker space-time. That result depends on the action's being an even function of a parameter that determines one of the initial conditions on the free field part of the geometry, because then the action is stationary with respect to variation of that parameter when that parameter is zero. That corresponds to the Robertson-Walker space-time, which we normally consider as the best description of our own cosmology[14].

Appendix F argues further from symmetry that if there is a class of homogeneous space-times parameterized by a parameter that corresponds to an initial relative rotation of inertial frames with matter, then the action must be an even function of that parameter. Requiring that the action be stationary with respect to variation of that parameter leads to a value of zero for that parameter, which again corresponds to the Robertson-Walker space-time.

The high symmetry of the Robertson-Walker space-time suggests that the Robertson-Walker space-time may be a classical limit for a quantum superposition of an even more general class of space-times.

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[14] There is, however, no condition that the universe be closed (see Appendix A).

4. Mach's principle

We have seen from the previous sections that a quantum superposition of some spatially homogeneous cosmologies yields a classical limit (the Robertson-Walker space-time) that is one of the Machian cosmologies found by Raine [12]. Further, we saw that those quantum superpositions of space-times having no classical limit (empty space-times, space-times having as matter only singularities, and asymptotic space-times) are non-Machian according to Raine's prescription.

I do not know if that correspondence will always hold. At first we might not expect the two prescriptions to always agree because mine depends on the initial states being a coherent quantum superposition of classical states, whereas Raine's depends on having a classical initial state. However, my prescription is independent of the details of the initial quantum state, and it selects space-times that lack arbitrariness (as measured by the action) with respect to initial and boundary conditions, as does Raine's.

That two such diverse approaches agree for our own cosmology and for the empty and asymptotic space-times is encouraging. Maybe we finally know why inertial frames appear not to rotate relative to the stars. While the answer is probably far from Mach's expectations, it is at least consistent with his ideas of relative motion.

Appendix A

Open universes

To avoid the apparently non-Machian asymptotic solutions to the field equations, Einstein [21] and Wheeler [22] tried to justify as a boundary condition the requirement that allowable space-times must be closed. Because such a requirement would reject open Robertson-Walker space-times (and our own universe with it if the present evidence favoring an open universe holds up), their requirement may be too restrictive. Further, I should point out that the procedure of the present work automatically rejects asymptotic and empty space-times, so that rejecting open solutions is no longer necessary.

First, they point out that boundary conditions are simpler for a closed universe than for an open universe [21,22]. While that may be true, we should reject a space-time only if it is impossible to apply the boundary conditions consistently.

Their second argument is Machian and points out that inertia depends on independent properties of space in some open space-times [21,22]. However, their argument applies only to asymptotic space-times rather than to open spaces in general. So far, there is no proof connnecting Mach's principle with the spatial finiteness of the universe [32].

Third, Einstein [21] argues from a sum-of-inertia Machian type calculation that an infinite universe is possible only if the mean density of matter in the universe vanishes. The open Robertson-Walker space-times, which are apparently Machian [12], seem to contradict Einstein's statement. I suspect that including horizon effects would reconcible the contradiction.

Fourth, Wheeler [22] points out that the action integral may not be finite and well defined on an open space. The total action does not have to be finite because only relative values of the action are important in making calculations, but an infinite value for the action may make those calculations more difficult. However, there are many open space-times with a well-defined action (flat, empty space-times, for example).

Appendix B

Lagrangian formulation[15]

The quantum amplitude of a time history is proportional to

$$\exp(iS/\hbar) \tag{B1}$$

where $\hbar = h/2\pi$, and h is Planck's constant.

S is the action given by

$$S = \int (-g(x))^{1/2} L(x)\, d^4x \tag{B2}$$

where g is the determinant of the metric, and

$$L(x) = L_{geom}(x) + L_{matter}(x) \tag{B3}$$

is the Lagrangian[16].

The Lagrangian for the geometry is

$$L_{geom}(x) = R(x)/16\pi, \tag{B4}$$

where

$$R(x) = R^{\mu\nu}(x)\, g_{\mu\nu}(x) \tag{B5}$$

is the 4-dimensional scalar curvature.

---

[15] These calculations use the conventions of Misner, Thorne, and Wheeler [33].

[16] I do not include spin or fields other than gravitation here because they are not necessary for the main point of this paper.

We vary the action in the usual way to determine classical time histories of the geometry and geodesics along which the matter flows. After having done that, we can evaluate the action for those classical time histories to give the action as a functional of the classical time history (or equivalently as a functional of the classical initial conditions.)

The appendices to follow evaluate the action[17] as a functional of the classical time history for specific matter distributions. In so doing, they freely use[18]

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1 \tag{B6}$$

because it is valid for all geodesics in space-times that are solutions of the field equations.

---

17 We could then use that action either to finish finding a classical limit or to estimate the relative quantum amplitudes of classical time histories in a path integral calculation.

18 The signature of the metric is (- + + +). τ is the proper time. $d\tau^2 = -ds^2$, where s is proper distance.

Appendix C

Matter distributions that contains only singularities[19]

For a matter distribution of singularities whose masses are m_n and whose trajectories are given by x_n^μ, the energy momentum tensor is [34]

$$T^{\mu\nu}(x) = (-g(x))^{-\frac{1}{2}} \sum_n m_n \int d\tau_n \frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n} \delta^4(x - x_n) \tag{C1}$$

Therefore, from Einstein's quations,

$$R^{\mu\nu} \equiv G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} G = 8\pi \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right) \tag{C2}$$

we have

$$\begin{aligned} R^{\mu\nu}(x) = 8\pi(-g(x))^{-\frac{1}{2}} \sum_n m_n \Big[\int d\tau_n \frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n} \delta^4(x - x_n) \\ - \frac{1}{2} g^{\mu\nu}(x) g_{\alpha\beta}(x) \int d\tau_n \frac{dx_n^\alpha}{d\tau_n} \frac{dx_n^\beta}{d\tau_n} \delta^4(x - x_n) \Big] . \end{aligned} \tag{C3}$$

Therefore, (B4), (B5), and (C3) give

$$L_{geom}(x) = -\frac{1}{2} g_{\mu\nu}(x)(-g(x))^{-\frac{1}{2}} \sum_n m_n \int d\tau_n \frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n} \delta^4(x - x_n) . \tag{C4}$$

as the Lagrangian for the geometry for solutions to the field equations.

---

19 When we have only one singularity, and we apply asymptotically flat boundary conditions, we get the Schwarzschild space-time. Although that space-time gives the same external field as a spherically symmetric distribution of mass such as for a star, the action for the two is the not the same. Thus, the calculations of this appendix do not apply to a distribution of stars.

Equations (B2) and (C4) give

$$S_{geom} = -\frac{1}{2}\sum_n m_n \int d\tau_n \frac{dx_n^\mu}{d\tau_n}\frac{dx_n^\nu}{d\tau_n} g_{\mu\nu}(x_n) \tag{C5}$$

as the action for the geometry.

The action for the matter is [35]

$$S_{matter} = \frac{1}{2}\sum_n m_n \int d\tau_n \frac{dx_n^\mu}{d\tau_n}\frac{dx_n^\nu}{d\tau_n} g_{\mu\nu}(x_n) \tag{C6}$$

The total action is the sum of (C5) and (C6), which is identically zero for solutions to Einstein's equations, independent of the initial conditions. Therefore, just as for empty space-times, the action is independent of the space-time for a whole class of space-times. Thus, a quantum superposition of such space-times will have no classical limit.

As the number of singularities considered in these space-times is finite, all such open space-times will be asymptotically determined by the boundary conditions. That no classical limit exists for a quantum superposition of these space-times agrees with independent considerations of Mach's principle [12].

Appendix D

Asymptotic space-times

We saw from appendix C that, for a matter distribution that consisted only of a collection of singularities, the action is always zero for solutions to the field equations, independent of any of the other details of the matter distribution or the geometry, and independent of the initial and boundary conditions. If the number of singularities is finite, then these are asymptotic space-times. We will see here that the action for asymptotic space-times will always show some degree of independence of the geometry in the asymptotic regions, and therefore of the initial or boundary conditions.

In asymptotic space-times, the matter is confined to some region (say M) of space. The geometry outside of M (especially in the asymptotic regions) depends mostly on the boundary conditions, and only near M does it depend strongly on the matter within M. The geometry within M depends mostly on the matter within M (unless M is too sparse) and only near the boundary of M does it depend much on the boundary conditions.

The empty space outside of M makes no contribution to the action[12]. The action depends completely on the matter and geometry within M. Thus the geometry in the asymptotic regions influences the action only indirectly and only to the extent that the geometry inside M must adjust to fit with the geometry outside of M through the field equations.

Suppose there is a space-time for which the action is stationary with respect to variation of the initial conditions on the free-field part of the geometry. Let S_c be the action for that space-time. Then all other space-times for the same initial matter distribution whose action S is such that

$$\left| S - S_c \right| \lesssim \hbar \qquad \text{(D1)}$$

will reinforce constructively with each other. Because the action depends only weakly (and indirectly) on the geometry in the asymptotic regions, the space-times allowed by (D1) can include some whose geometries in the asymptotic regions are significantly different. Therefore a coherent quantum superposition of space-times can exist which includes significantly different geometries in the asymptotic regions.

Thus, a true classical limit will not exist in the asymptotic regions, although a classical limit may exist outside of the asymptotic regions (especially within M) if the matter distribution within M is not too sparse.

Appendix E

Perfect fluid models

For a perfect fluid, the energy-momentum tensor is [38]

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \tag{E1}$$

where ρ is the density and p is the pressure. From (E1) and the field equations (C2), we have

$$R^{\mu\nu} = 8\pi\left[(\rho + p) u^\mu u^\nu - \frac{1}{2} g^{\mu\nu}\left((\rho + p) g_{\alpha\beta} u^\alpha u^\beta + 2p\right)\right]. \tag{E2}$$

Therefore, (E2), (B4), and (B5) give the Lagrangian for the geometry for solutions to the field equations as

$$L_{geom} = -\frac{1}{2}(\rho + p) g_{\mu\nu} u^\mu u^\nu - 2p. \tag{E3}$$

We can take

$$L_{matter} = p + \frac{1}{2}(\rho + p)(1 + g_{\mu\nu} u^\mu u^\nu) \tag{E4}$$

as the Lagrangian for the matter, because [39]

$$T^{\mu\nu} = 2\frac{\delta L_{matter}}{\delta g_{\mu\nu}} + g^{\mu\nu} L_{matter} \tag{E5}$$

gives

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} + \frac{1}{2}(\rho + p) g^{\mu\nu}(1 + g_{\alpha\beta} u^\alpha u^\beta) \tag{E6}$$

for the energy-momentum tensor, which is equivalent to (E1) for solutions to the field equations because of (B6).

Adding (E3) and (E4) gives

$$L = L_{geom} + L_{matter} = \frac{1}{2}(\rho - p) \tag{E7}$$

for the total Lagrangian. From (B2) and (E7), we have

$$S = \frac{1}{2}\int (-g(x))^{\frac{1}{2}} \left(\rho(x) - p(x)\right) d^4x \tag{E8}$$

for the total action. Unlike the case where the only matter was a collection of singularities, the action here is not identically zero unless the density and pressure are equal. The reason for the difference is that there, mass was a scalar and here density is a scalar.

The spatial limits of integration in (E8) are over the whole 3-space. The time integration is from the initial to the final time.

Appendix F

Spatially homogeneous perfect fluid models

Equation (E8) gives the action for perfect fluid models. For comparison with the calculations of Raine [12], we take

$$p = (\gamma - 1)\rho \tag{F1}$$

for the equation of state, where γ is a constant. Thus, the action is

$$S = \frac{1}{2}(2-\gamma)\int \rho(x)(-g(x))^{\frac{1}{2}} d^4x = \frac{1}{2}(2-\gamma)\int_{t_1}^{t_2}\iiint \rho(x,t)(-g(x,t))^{\frac{1}{2}} d^3x\, dt \tag{F2}$$

where the spatial integration is over the whole of the three-space, and t_1 and t_2 are the initial and final times, respectively.

~~Specific~~ solutions to the field equations ~~are scarce. Some~~ are available if the fluid flows normal to the surfaces of homogeneity. Ellis, MacCallum, and Stewart [28,29,30,31] have classified those solutions rather thoroughly. Specifying the following parameters on the initial spatial hypersurface determines the solution.

1) the expansion rates θ_i,
2) the shear 3-tensor σ_{ij} and its time derivative,
3) the local angular velocity, in the rest frame of an observer moving with the fluid, of a set of Fermi-propagated axes with respect to a particular inertial triad Ω^i, and
4) the 3-tensor n_{ij} and the 3-vector a_i, that determine the structure of the 3-geometry.

The above parameters can not be specified independently, but are subject to constraints.

Ellis and MacCallum [30] give the metric for a Bianchi-Behr group type VI_h model (their group class Bbii). Their solution gives

$$\begin{aligned}
&\Omega_1 = \Omega_2 = \sigma_{13} = \sigma_{23} = 0 \\
&\Omega_3 = \sigma_{12} = b/(Y^2 Z) \\
&n_{23} = n_{32} = q_0 / X \\
&\text{all other } n_{ij} = 0 \\
&a_1 = a_0 / X \\
&a_2 = a_3 = 0 \\
&q_0 = -3a_0
\end{aligned} \tag{F3}$$

and

$$\rho = \rho(a_0^2, b^2, \rho(t_1), \theta_i(t_1), t) \tag{F4}$$

Equation (6.6) of reference [30] gives the metric. From it, we get

$$\sqrt{-g} = \sqrt{X^2 - 3Y^2 b^2 F^2}\; Y Z \; \exp(-2a_0 x^1) \tag{F5}$$

and

$$d^3x = X Y Z \; e^{-2a_0 x^1} dx^1 dx^2 dx^3 \tag{F6}$$

where

$$
\begin{aligned}
X &= X(a_o^2, b^2, \rho(t_1), \theta_i(t_1), t) \\
Y &= Y(a_o^2, b^2, \rho(t_1), \theta_i(t_1), t) \\
Z &= Z(a_o^2, b^2, \rho(t_1), \theta_i(t_1), t) \\
F &= F(a_o^2, b^2, \rho(t_1), \theta_i(t_1), t).
\end{aligned}
\tag{F7}
$$

The solution is determined by the constants a_o and b, and by the initial conditions on the matter. The functions depend on a_o and b only through a_o^2 and b^2 because the differential equations (6.7) of reference [30] contain those constants only in terms of their squares.

Substituting (F4), (F5), and (F6) into (F2) gives

$$
S = \int_{t_1}^{t_2} f(a_o^2, b^2, t)\,dt \iiint e^{-4a_o x^1}\,dx^1 dx^2 dx^3 \tag{F8}
$$

where

$$
f(a_o^2, b^2, t) = \tfrac{1}{2}(2-\gamma)\,\rho\, X\, Y^2 Z^2 \sqrt{X^2 - 3Y^2 b^2 F^2} \tag{F9}
$$

and I have suppressed explicit dependence on the initial conditions of the matter.

The dependence of the action in (F8) on the initial conditions of the matter is a dependence on the initial density and the initial expansion rates (the expansion can be anisotropic). The dependence of the action in (F8) on the constants a_o and b is a dependence of the action on initial conditions on the free-field part of the geometry. For the classical limit, we require that the action be stationary with respect to variation of a_o and b. This gives

$$0 = \frac{\partial S}{\partial b} = 2b\int_{t_1}^{t_2} \frac{\partial f}{\partial b^2}\, dt \iiint e^{-4a_0 x^1}\, dx^1 dx^2 dx^3 \tag{F10}$$

and

$$0 = \frac{\partial S}{\partial a_0} = 2a_0\int_{t_1}^{t_2} \frac{\partial f}{\partial a_0^2} \iiint e^{-4a_0 x^1}\, dx^1 dx^2 dx^3$$
$$- 4\int_{t_1}^{t_2} f\, dt \iiint x^1 e^{-4a_0 x^1}\, dx^1 dx^2 dx^3 \tag{F11}$$

We do not need here the result of (F11).

Equation (F10) implies that

$$b = 0 \tag{F12}$$

which from (F3) gives

$$\Omega_3 = \sigma_{12} = 0 \tag{F13}$$

which implies a Robertson-Walker space-time.

The above results depended only on having the action be an even function of the parameter b. We can use this same principle plus symmetry to get a similar result for other classes of homogeneous space-times without making detailed calculations.

For example, we consider the class of homogeneous space-times where inertial frames rotate relative to the matter. Let the initial rotation rate be characterized by Ω. That solution will be the mirror image of that for which the initial rotation rate is characterized by $-\Omega$. By symmetry, the action must be the same for the two cases, that is

$$S(\Omega) = S(-\Omega) \quad . \tag{F14}$$

Thus, the action is an even function of Ω, and thus one solution of

$$\frac{\partial S}{\partial \Omega} = 0 \tag{F15}$$

is

$$\Omega = 0, \tag{F16}$$

which gives the Robertson-Walker metric as before.

The generality of this principle is clear.

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Table I

Classification of space-times according to how strongly the action depends on the initial conditions

	(a)	(b)	(c)
Does the action depend on classical initial conditions?	NO	Yes, weakly (geometry significantly different for $\lvert S-S_c \rvert \lesssim \hbar$)	Yes, strongly (geometry not significantly different for $\lvert S-S_c \rvert \lesssim \hbar$)
Physical interpretation	All classical time histories are equivalent with respect to the action	Some significantly different classical time histories are nearly equivalent with respect to the action	The various classical time histories are not equivalent with respect to the action
Some of the space-times included	All empty space-times[12] Some space-times whose only matter is a collection of singularities (Appendix C)	Most asymptotic space-times (Appendix D)	perfect fluid models (Appendix E)
Does a classical limit exist?	No	sometimes in some regions of space-time	sometimes
Condition for classical limit	No classical limit	1) The action is stationary with respect to variation of the classical initial conditions, & 2) exactly one classical time history fulfils the above condition, & 3) a classical limit exists only in the regions of space-time in which the geometry is not significantly different for $\lvert S - S_c \rvert \lesssim \hbar$.	1) The action is stationary with respect to variation of the classical initial conditions, & 2) exactly one classical time history fulfils the above condition.

S is the action for a classical time history.
S_c is the action for a classical time history for which the action is stationary with respect to variation of the classical initial conditions.

29.1 Referee's report

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Dear Mike,

Enclosed is our referee's report on your paper

'The Quantum Basis for Mach's Principle'

If you think that you can rewrite your paper to satisfy him, please feel free to resubmit.

Best regards.

Encl.

This paper is rather confusing. There are a number of points I don't understand.

1. I don't know what a quantum superposition of classical states is. Does he mean simply a wave function over three geometries? Also, I don't know how classical states can be strongly coupled - by what interaction?

2. The usual sum over histories gives the propagator $G(x,x')$. In this case one would presumably regard x and x' as three geometries. The wave function now is

$$\psi(x) = \int \psi(x')\, G(x',x)\, dx \qquad (1)$$

His argument seems to be something like the following:

write $$G(x',x) = \int e^{iS(\gamma(x,x'))} d\gamma \qquad (2)$$

where γ is a curve from x' to x in superspace. Then, the dominant contributions to $\psi(x)$ come from those paths where the action is an extremal. This classical path through superspace is $\tilde{\gamma}$.

If we fix x then we have

$$\psi(x) \approx \int \psi(x')\, e^{iS(\tilde{\gamma}(x',x))}\, C(\tilde{\gamma})\, dx' \qquad (3)$$

where $C(\tilde{\gamma})$ is some constant arising from the integral over the γ near $\tilde{\gamma}$.

Now, if $\psi(x')$ is some smooth, slowly varying function of x' and so is $C(\tilde{\gamma})$, then the dominant contribution to the above integral comes from those values of x' where the action is an extremum, i.e.,

$$\Psi(x) \approx \Psi(\tilde{x}')\, C(\tilde{\gamma}(\tilde{x}',x))\, D(\tilde{x}')\, e^{iS(\tilde{\gamma}(\tilde{x}',x))} \qquad (4)$$

if this $\tilde{x}'$ is unique. Such cases where $\tilde{x}'$ is unique he defines as the cases with a classical limit. However, we must note that x' will depend on x, the final geometry under consideration.

I am now at a loss as how to continue. Some of the confusion arises because I don't understand what he means by his definitions on page six. I don't know what a state which is a continuous function of classical initial conditions is (footnote 8). The initial wave function $\psi(x')$ is a function of three geometries, not of classical initial conditions (three geometries plus extrinsic curvature). His definition of coherent state would correspond in one particle quantum mechanics to an initial wave function which is a function of both x and p, $\psi(x,p)$, which is impossible in any representation.

3. In appendix C, he obtains an action of 0 for a collection of point singularities. I don't know what a point singularity is. If it's a black hole, his result is in contradiction with Hawking (Phys. Rev. D. 18, page 1749 (1978)). If not then I don't know what they are.

The above queries are enough to suggest that the paper at least needs extensive rewriting to make his meaning clear.

29.2 My response

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Dear Alan,

Thank your for your letter and the referee's report on my paper 1078, "The Quantum Basis for Mach's Principle". His comments were very good, and caused me to examine some points where my thinking had been cloudy.

Before I rewrite the paper, though, I would like to come to some agreement with the referee on what will make an acceptable paper. My remarks below are in response to the referee's three comments.

1. By a quantum superposition of classical states I sometimes meant a wave function over three geometries and sometimes a wave function over four geometries. (Sometimes a wave function over four geometries exists, see below).Which I meant in each case is clear form the context, but I will change my notation in my rewrite to be clearer. I found one place where I made a mistake, though. In the second paragraph on page 4, I wrote " In particular, section 2 justifies such a quantum superposition of space-times as an initial state,...". I should have used " a wave function over three geometries" there.

About how classical states can be strongly coupled. I think I must have chosen a poor expression, but I will try to explain what I mean. I will use the referee's equations to explain. I have enclosed a copy of his comments in

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ZUM SCHREIBEN AN: Dr. Alan Held BLATT: 2

which I have numbered his equations from (I) through (4). First I will explain what I mean if classical states are not strongly coupled. If the referee's equation (3) is a good approximation for the wave function, then the classical states are not strongly coupled in the space-time region whose time boundaries are the initial space-like hypersurface x' and the final space-like hypersurface x. When his equation (3) is not a good approximation, then the classical states were strongly coupled somewhere in the space-time region. Maybe the referee can suggest a clearer terminology. Physically what I meant by the classical states' being uncoupled (in a space-time region) is that the wave function can be represented in that space-time region by a wave function over four geometries where each four geometry is a solution of Einstein's field equations.

The simplest interaction I can think of is for the curvature to be large enough to create some virtual pairs that interact non-classically. If that won't work, I'm sure we can think of an example. In any case, we need a term in the Lagrangian connecting two or more classical solutions of Einstein's field equations.

2. I agree with all 4 of the referee's equations, including their interpretation, except for the interpretation of equation(4). Equation (4) gives an approximation to the wave function over three geometries in the final state. If $\tilde{x}'$ is not unique, then equation (4) must involve a sum over contributions for all $\tilde{x}'$ when the $\tilde{x}'$ are not too close together.

Even if $\tilde{x}'$ is unique, equation (4) does not represent a purely classical final state. I meant to require more. Let the action in (4) be written

$$S\,(\tilde{\gamma}\,(\tilde{x}',\, x) = S\,(\tilde{\gamma}_x(x'_x, x)), \tag{5}$$

where I have indicated by subscripts that $\tilde{\gamma}$ and $\tilde{x}'$ depend on x. To have a purely classical final state, I require not only that $\tilde{\gamma}_x$ and $\tilde{x}'_x$ be unique, but also that

$$\delta_x\; S\,(\,\tilde{\gamma}_x(\tilde{x}'_x, x)) = 0 \tag{6}$$

In the paragraph that begins on page 1749 and ends on page 1750, Hawking argues that my formula (C5'') is the correct action for ~~[illegible]~~ the geometry for a black hole. Thus, we are in complete agreement.

On re-reading my letter of 7 February, I see that I was careless in the development of some of my equations. I give a more careful development below.

In the referee's 4th equation, the classical path $\tilde{\gamma}$ is represented as a function of the three-geometries $\tilde{x}'_x$ and x at the endpoints of the path. Because $\tilde{\gamma}$ is a classical path, it can also be represented as a function of the initial three-geometry $\tilde{x}'_x$ and the initial direction of the path (the lapse and shift functions, which I will represent here as β_x). Thus

$$\tilde{\gamma} = \tilde{\gamma}(\tilde{x}'_x, \beta_x) .$$

Thus, I should have written equation (5) in my letter as

$$S = S(\tilde{\gamma}(\tilde{x}'_x, \beta_x)) . \qquad (5')$$

The explanations in my letter were correct, so that (6) should be written

$$\delta_x \; S(\tilde{\gamma}(\tilde{x}'_x, \beta_x)) = 0 . \qquad (6')$$

In varying the final geometry x in (6'), we can do that either by varying the initial 3-geometry $\tilde{x}'_x$ or the initial lapse and shift functions β_x . Thus, we have

$$\delta_{\tilde{x}'_x} \; S(\tilde{\gamma}(\tilde{x}'_x, \beta_x)) = 0$$

and (7')

$$\delta_{\beta_x} \; S(\tilde{\gamma}(\tilde{x}'_x, \beta_x)) = 0 ,$$

which is equivalent to (6'). Equation (7') states that the action is stationary with respect to the classical initial conditions, which is the result I gave in my paper.

Where we keep the time coordinates on the initial and final space-like hypersurfaces fixed for the variation in (6). But $\tilde{x}'_x$ and $\tilde{\gamma}_x$ are the classical initial conditions for the four geometry that has x as the "final" three geometry. Thus x is a function of $\tilde{x}'_x$ and $\tilde{\gamma}_x$ and the initial and final time. Thus (6) is equivalent to

$$\delta_{\tilde{\gamma}_x} S(\tilde{\gamma}_x(\tilde{x}'_x, x(\tilde{x}'_x, \tilde{\gamma}_x))) = 0$$

and (7)

$$\delta_{\tilde{x}'_x} S(\tilde{\gamma}_x(\tilde{x}'_x, x(\tilde{x}'_x \tilde{\gamma}_x))) = 0$$

which is the result I stated in words in my paper.

The physical interpretation of (7) is as follows. First, it determines not only a unique x, but also a unique $\tilde{x}'_x$ and a unique $\tilde{\gamma}_x$, and thus a unique solution of Einsteins equations. Such a solution has the property that similar solutions have nearly the same action and therefore nearly the same phase (providing other criteria are met, which I discuss in my paper), and therefore interfere constructively.

I realize now that I had not been clear in my paper, and indeed, my thinking had been a little fuzzy. I appreciate the help in clarifying my thinking. In fact, I had not clearly made the distinction either in my paper or in my mind between the extremal condition leading to equation (4) and that in equation (7).

I made a mistake in footnote 8. The initial wave function is a function of only the three geometries.

3. Yes, a singularity is a black hole. I had already been worried that my calculation of the action seemed to disagree with that of Hawking. I have had difficulty understanding his paper. At the top of page 1749, he seems to imply that the calculation of the action depends on the topology. It may be that my calculation of the action for that case is wrong and that I have incorrectly classified black holes. I will try to resolve the disagreement while I am waiting for the answer to my comments.

Thank you again. I appreciate the help.

Michael Jones

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IHR ZEICHEN | IHRE NACHRICHT VOM | UNSER ZEICHEN | DURCHWAHL (05556) 41 | DATUM 28 Feb 1980

Dear Alan,

Concerning my paper number 1078, "The quantum basis for Mach's principle", I said in my letter of 7 February that I would try to resolve the apparent disagreement between my calculation of the action for a black hole and that of Hawking.

I have done that now and find that our calculations are in complete agreement. I would appreciate it if you would forward these comments to the referee.

Hawking calculates the action for the geometry only, whereas I calculate the action for both the geometry and the matter. His calculation should therefore be compared ~~to~~ with my equation (C5). For a solution~~s~~ to the field equations (which Hawking refers to as the background metric), my equation (B6) is valid. Substituting (B6) into (C5) gives

$$S_{geom} = +\frac{1}{2}\sum_n m_n \int_{\tau_1}^{\tau_2} d\tau_n \qquad \text{(C5')}$$

where I have explicitly indicated the limits in proper time at the initial and final spacelike hypersurfaces. For a single black hole, (C5') gives

$$S_{geom} = \frac{1}{2} m (\tau_2 - \tau_1) . \qquad \text{(C5'')}$$

Thank you very much. I hope to hear the referee's comments soon.

Sincerely,

Michael Jones

Michael Jones

29.3 Afterthoughts - 2008

I did not handle this very well.

29.4 Afterthoughts - 2009

Newton noticed that his laws of motion applied in a frame that does not rotate relative to the "fixed stars," and concluded the existence of an "absolute space" with the stars fixed in that absolute space (what we now refer to as an "inertial frame").

Mach denied the existence of absolute space, pointing out that only relative motions are observable. Instead of absolute space, he postulated that inertia is caused by an interaction with the rest of the universe. He suggested that what we now refer to as an inertial frame is caused by the rest of matter in the universe (now referred to as frame dragging).

Einstein based his General Relativity partly on Mach's ideas, and coined the term "Mach's principle" for the idea that inertial frames should be determined by the matter distribution. The term "Mach's principle" now means many things to many people, and there is no general agreement on what it means or on the validity of the various versions.

Although Einstein's General Relativity does include frame dragging, frame dragging is not complete. There are many solutions of Einstein's field equations in which there is inertia in the absence of matter or where there is rotation of inertial frames relative to the average matter distribution. This is because initial and boundary conditions also contribute to inertia.

Some researchers have proposed that boundary and initial conditions be chosen so that inertia and/or the metric be determined by the matter distribution. Some researchers have proposed that Mach's principle be used as a selection principle to discard solutions that have relative rotation of matter and inertial frames or have inertia without matter.

I have argued that if the metric (gravitation) is determined solely by sources (the matter), with no dependence on initial or boundary conditions, then gravitation would be very different from electromagnetic theory, in which the electromagnetic field has its own degrees of freedom, and can have arbitrary initial and boundary conditions. Thus, I have rejected Mach's principle in such a restricted form.

That leaves us still with the problem of explaining why we observe no relative rotation of inertial frames and the visible matter in the universe. I suggested that if we consider quantum gravity, that we could allow all of the solutions to the field equations, and that the non-physical solutions (those with relative rotation of inertial frames and matter) would cancel each other out if we look at the change in the action with variation of the initial and boundary conditions on the gravitational field.

It was at this point that I made a mistake in the present manuscript. I suggested that we vary the initial and boundary conditions on the gravitational field while holding the matter distribution constant. It was pointed out that there is no general procedure for doing that because it is not possible to specify the matter distribution independent of the geometry.

What I should do instead (and what I actually did in my calculations) is consider a family of solutions to the field equations (such as one of the Bianchi cosmologies) in which there is an initial condition that determines the amount of relative rotation of inertial frames and matter, and consider the change in action as that parameter is varied. In this way, we have no problems of the kind I mentioned above. Then, a saddlepoint approximation to the path integral for that parameter will give the solution for zero relative rotation.

Chapter 30

Machian cosmologies as wave packets[1]

From the point of view of quantum gravity, our cosmology is not a solution of Einstein's equations[2], but rather a wave function over 3-geometriea that evolves In time. That our universe appears classical to a first approximation means that it is a wave packet, and the spread of that wave packet is small.

That our cosmology is a wave packet means that the wave function over 3-geometries is stationary with respect to variation of the 3-geometry.

We can think of the time history of such a wave packet as a 4-geometry with a spread, where the spread Indicates the deviation from a classical 4-geometry.

The Initial state determines the characteristics of a such a wave packet. However, the relative Importance of the Initial wave function over 3-geometries and the initial wave function over the matter distribution in determining those characteristics may change with tine.

If the Initial wave function over 3-geometrles were a sharp wave packet, that wave packet would disperse because of the uncertainty relations. For the same reason, If there were a zero amplitude for some 3-geometry In the Initial state. It could not remain so.

These considerations suggest that the characteristics of such wave packets may be nearly independent of the wave function over 3-geometrles In the distant past. It may not, however, be Independent of the wave function over the Initial matter distribution.

We might speculate that after sufficient time a given matter distribution would determine a unique solution to the field equations as the 4-geoaetry for the core of a wave packet. If so, then it would be appropriate to identify such solutions to the field equations as Machian cosmologies. As we shall see, the above speculation Is true. However, the wave packets will be narrow only If the matter is not too sparse.

We often suppose that Machian cosmologies should have the same symmetries as the matter distribution. This turns out to be true, and that symmetry property follows directly from the uniqueness property of Machian solutions.

Methods: I calculate the wave function over 3-geometries g_2 and matter fields μ_2 at time t_2 as an integral over 3-geometries and matter fields of the initial state

$< g_2, \mu_2, t_2|\psi >= \int D(g_1)D(\mu_1) < g_2, \mu_2, t_2|g_1, \mu_1, t_1 >< g_1, \mu_1, t_1|\psi >$

where

$< g_2, \mu_2, t_2|g_1, \mu_1, t_1 >= \int D(g)D(\mu) \exp iS(g, \mu)/\hbar$

is the path integral representation of the amplitude to go from a state with metric g_1 and matter field mu_1 at time t_1 to a state with metric g_2 and matter fields μ_2 at time t_2. $D(g)$ is a measure

[1] July 1980. Presented at the Ninth International Conference on General Relativity and Gravitation, 14-19th July 1980, Friedrich Schiller University, Jena, DDR, on pages 483-484 in Volume 2 of the Abstracts of Contributed Papers

[2] Or whatever classical set of equations turns out to be correct in the classical limit.

on the space of all metrics, $D(\mu)$ is a measure on the space of all matter fields, S is the action, and the integral is taken over all field configurations with the given initial and final values[106].

The greatest contribution to the second integral above occurs where the action is stationary with respect to variation[3] of the 4-geometry g and the matter fields μ.

This restricts 4-geometries to solutions of Einstein's equations, and the matter to follow geodesics[4].

The greatest contribution to the first integral above occurs where the action is stationary with respect to variation[5] of the initial 3-geometry g_1 and the initial matter field μ_1.

Together, these stationarity conditions allow an approximate calculation of the wave function over 3-geometries at the time t_2. In that approximation, only one 3-geometry $\tilde{g}_1$ in the initial state and only one 4-geometry $\tilde{g}$ contributes to the amplitude for a given 3-geometry g_2 at the time t_2. Thus, we can choose a single set of parameters to specify g_2, $\tilde{g}$, and $\tilde{g}_1$. The action S then depends on those parameters.

If the characteristics of wave packets over 3-geometrlea are nearly independent of the wave function over 3-geometriea in the distant past, then the condition that determines the 4-geometry core of such a wave packet is that the action be stationary with respect to variation of all the parameters that specify the 4-geometry. This gives a number of equations equal to the number of parameters and thus for a given matter distribution determines those parameters whenever those equations lead to a unique solution.

Whenever the matter distribution determines such a unique 4-geometry for the core of a wave packet, we can associate that 4-geomecry with a Machian cosmology.

Whenever the matter distribution leads to a unique solution to the field equations (as above), then that solution satisfies the following symmetry property. If the matter and the form of the action share a common symmetry, then the geometry will also have that symmetry.

There are more formal proofs of the above symmetry property, but the following is sufficient. Suppose there were a particular symmetry shared by the matter distribution and the form of the action, but not by the geometry. Then there would exist a transformation that left the matter distribution and the form of the action invariant, but not the geometry. But the transformed geometry must also be a solution of the field equations for that matter distribution. (The form of the action determines uniquely the field equations.) Our original condition was that the geometry be unique. Therefore, the transformed geometry must be the same as the original geometry. Therefore, the geometry satisfies the common symmetry property.

Results:

1. Our universe is nearly classical because it is a wave packet over 3-geometries. Only certain solutions to the field equations can form the core of the time history of such a wave packet. They are the solutions that satisfy the following stationarity condition: the action of the 4-geonetry is stationary with respect to variation of all of the free parameters[6] that specify the 4-geometry.[7] We could choose the free-field initial conditions[8] as those parameters, for example.

[3] The end points are held fixed for this variation.

[4] For simplicity I consider only those cases where the stationarity condition chooses a unique cosmology $\tilde{g}(g_1, g_2, \mu_1, \mu_2)$ and $\tilde{\mu}(g_1, g_2, \mu_1, \mu_2)$.

[5] For this variation, g_2 and μ_2 are held fixed, and $S(\tilde{g}, \tilde{\mu})$ is used for the action.

[6] The above symmetry property implies that the Robertson-Walker metric is the Machian cosmology corresponding to a homogeneous, isotropic matter distribution,

[7] Any 4-geonetry and matter distribution has an action associated with it. When that action is stationary with respect to variation of the geometry and the matter, then we have 4-geometries that satisfy the field equations and matter distributions that follow geodesics. Each of those "classical" cosmologies has an action associated with it.

[8] The above symmetry property implies that the Robertson-Walker metric is the Machian cosmology corresponding to a homogeneous, isotropic matter distribution,

2. Normally the above stationarity condition will determine a discrete set of 4-geometries for a given matter distribution because the number of conditions equals the number of free parameters that specify the 4-geometry.

3. When the above stationarity condition determines a unique 4-geometry for a given matter distribution, then it is appropriate to call such a 4-geometry a Machian cosmology.

4. When the above stationarity condition determines a unique 4-geometry for a given matter distribution, and if the matter distribution and the form of the action have some symmetry in common, then that 4-geometry will also have that symmetry.

5. The above symmetry property implies that the Robertson-Walker metric is the Machian cosmology corresponding to a homogeneous, isotropic matter distribution.

6. Empty spaces and spaces that have only black holes for matter cannot form the core of such wave packets over 3-geometries because the action for those cosmologies is independent of some of the parameters specifying the 4-geometry.[107]

7. In the asymptotic regions of asymptotic space-times, the matter is so sparse that the wave packet over 3-geometries is broad enough to Include significantly different geometries in those regions. An example is an asymptotically flat space such as the Schwarzschild metric.[107]

Discussion:

As far as I know, this is the first attempt to relate quantum gravity with Mach's principle. However, Barbour[108] suggests seeking a variational principle that makes Machian cosmologies extremal compared with neighboring 4-geometries.

Einstein's General Relativity has some Machian effects in that matter contributes to inertia. However, the presence of inertia without matter and the non-uniqueness of solutions of Einstein's equations have led many people to search for a mechanism for Mach's principle that would automatically require all inertia to have matter as a source.

One such line of development has been to use Mach's principle to assign boundary conditions to Einstein's equations[16, 109]. The premise that all gravitational fields must have matter as a source has led to the classification of some cosmologies as Machian and some as non-Machian[109].

For several reasons[107], I do not believe that all gravitational fields must have matter as a source. Instead, as my development here shows, the matter distribution determines uniquely the 4-geometry only when that 4-geometry forms the core of the time history of a wave packet over 3-geometries. My classification of Machian cosmologies (those that can form the 4-geometry core of a wave packet) agrees with Raine's[109] for all cases where I have compared the two.

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Chapter 31

The quantum basis for Mach's principle[1]

[1]Submitted to Foundations of Physics 7 August 1980

The quantum basis for Mach's principle

Submitted to
Foundations of Physics
7 August 1980

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Abstract

While Mach's principle (that the matter distribution determines the geometry) explains satisfactorily why inertial frames seem not to rotate relative to the stars, it is difficult to justify theoretically.

However, by starting with a quantum description and requiring wave packets over 3-geometries in the classical limit, we can justify a limited form of Mach's principle in which the matter distribution sometimes determines the 4-geometry that forms the core of the time history of such a wave packet. In regions where the matter is too sparse, however, the wave packet may be too broad to approximate a classical space-time.

Some types of matter distributions (such as perfect fluids) can determine the geometry, others (vacuum, black holes) cannot. A matter distribution $\tilde{\Phi}$ determines a particular 4-geometry $\tilde{g}(\bar{b},\tilde{\Phi})$ from among all the solutions $\tilde{g}(b, \tilde{\Phi})$ to the field equations if the solution to

$$\delta_{b_i} S(\tilde{g}(b,\tilde{\Phi}), \tilde{\Phi}) = 0 \tag{1}$$

$$\text{for all } b_i \text{ in } b$$

is unique, where b is a set of parameters that specifies a solution to the field equations, and $S(\tilde{g}, \tilde{\Phi})$ is the total action (matter plus geometry).

It is appropriate to call a matter distribution Machian when the solution of (1) is unique, and to call the resulting 4-geometry a Machian cosmology. A homogeneous, isotropic matter distribution is Machian and determines the Robertson-Walker metric as the corresponding Machian cosmology.

The quantum basis for Mach's principle

1. Introduction - Mach's principle versus some independence for the geometry

We observe with high precision that inertial frames generally do not rotate relative to the matter distribution in our universe[(1)]. I know of only three classical explanations for this phenomenon.

1) It is pure coincidence.
2) We have an absolute space-time.
3) The matter distribution determines completely the gravitational field[1] (including the inertial field).

Although all three of these explanations are logically possible, none is satisfactory. The probability of the first explanation is so small, that we need not consider it further. The philosophical arguments against the second are well known, and I do not need to repeat them here.

The third explanation (a modern statement of Mach's[(2,3)] explanation of the above phenomenon, now usually called Mach's principle[(4-10)]) is difficult to justify theoretically.

In the past, most of the effort concerning Mach's principle has concentrated on finding a mechanism for it. Einstein based his General Relativity Theory in part on Mach's principle, but his field equations by themselves certainly do not incorporate Mach's principle. While General Relativity contains Machian effects, in that matter affects inertial frames, inertia in Einstein's theory depends not only on the distribution of matter, but also on initial and boundary conditions. The arbitrariness represented by the initial and boundary conditions can take the place of the arbitrariness of the absolute space that Mach tried to avoid. Empty space-times and asymptotic space-times represent extreme examples of arbitrariness in which initial and boundary conditions completely or nearly completely determine inertia.

My purpose here, however, is not to look into General Relativity's failure to incorporate a mechanism for Mach's principle, but rather to consider the theoretical consequences if it or any other theory did include such a mechanism.

---

[1] The geometry in a metric theory.

If the matter distribution really determined the gravitational field, then the gravitational field would have no independent degrees of freedom, and would thus not be a real field at all. We would have an action at a distance theory[2] rather than a field theory. The important point here is not the definition of terms, but the big difference between a theory in which the field has independent degrees of freedom and a theory in which the field does not.

It is more likely that gravitation is correctly described as a field theory. In particular, Wheeler's(5,11,12,13) formulation of geometrodynamics in which the gravitational field has independent degrees of freedom through initial conditions is probably an appropriate interpretation of the field equations. As with the electromagnetic field, for example, we allow arbitrary initial conditions for both the field and the sources.

Two more arguments favor a field theory for gravitation, but they are not classical arguments. 1) We can quantize a field theory more easily than an action at a distance theory. 2) A symmetry exists between matter and the gravitational field in that matter is a source of gravitational fields and a strong gravitational field is a source of matter(14-18). It would be just as logical for all matter to have a gravitational field as its source as the reverse. It is more likely, however, that both the matter and the field have independent degrees of freedom.

However, Mach's principle has been applied very successfully. Sciama and others(19-21) have used Mach's principle to apply boundary conditions to Einstein's field equations or to select solutions as acceptable cosmologies. The results agree with our intuition. Asymptotically flat spaces (such as the Schwarzschild metric) are non-Machian, whereas conformally flat spaces (such as the Robertson-Walker metric) are Machian(21).

We have some clues to help solve this dilemma between the difficulty of justifying Mach's principle and the lack of an alternative explanation of why

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[2] I will use the term, "field theory", for a theory in which the field has independent degrees of freedom, and the term, "action as a distance theory", for a theory in which there is no independence to the field. I think this is the generally (though not universally) accepted distinction between the two.

inertial frames appear not to rotate relative to the stars. We notice that two of the arguments in favor of a field theory are quantum arguments. Suppose that within a quantum description of gravitation we allow independent degrees of freedom for the gravitational field, but that only Machian cosmologies would have significant amplitudes in the classical limit. Barbour[(22)] hinted in this direction when he suggested seeking a variational principle that makes Machian cosmologies extremal compared with neighboring 4-geometries. I show in the following pages that the above supposition is true for our Universe, but not in general. Although I use quantum geometrodynamics (QGD)[(12)] as an example, the arguments apply to any formulation of quantum gravity[3] based on a Lagrangian[4,5]. On the quantum level, we represent a state as a wave function over 3-geometries. In the classical limit, we require the state to be a wave packet over 3-geometries. Only certain classical space-times[6] can form the core of the time history of such a wave packet. When the matter distribution determines such a space-time uniquely through a variational principle, we can call that space-time a Machian cosmology.

2. Notation

$S(g,\phi)$	functional dependence of the action S on the 4-geometry g and the matter fields ϕ
g	any 4-geometry

[3] While we don't yet have a fully satisfactory quantum theory of gravity, we know enough general properties that such a theory must have to see how it relates to Mach's principle.

[4] Although the concepts presented here are in terms of a Lagrangian formulation, they are obviously more general, and would apply, for example, to a path ordered integral (sometimes called a product integral[(23)]) formulation as well.

[5] To insure Mach's ideas of relativity[(2,3)], the Lagrangian must depend explicitly on only the relative motions of the particles. Barbour[(22)] generalizes this concept of relativity in a more modern way in terms of relative configurations.

[6] By a classical 4-geometry (or space-time), I mean a solution of the field equations.

ϕ	any set of matter fields
$\tilde{g}$	a classical 4-geometry (a solution of the field equations, a solution of $\delta_g S(g,\phi) = 0$, where the 3-geometries at the endpoints are held fixed for the variation)
$\tilde{\phi}$	a classical matter distribution (a solution of $\delta_\phi\, S(g,\phi) = 0$)
$\tilde{g}(b,\tilde{\phi})$	a solution of the field equations for the matter distribution $\tilde{\phi}$
b	a set of parameters that specifies a 4-geometry $\tilde{g}(b,\tilde{\phi})$ that is a solution of the field equations for the matter distribution $\tilde{\phi}$
b_i	one of those parameters
$\tilde{b}$	a set of such parameters that is a solution of $\delta_{b_i}\, S(\tilde{g}(b,\tilde{\phi}), \tilde{\phi}) = 0$ all $b_i\, \varepsilon b$
ε	contained in the set of
$\tilde{S}$	$S(\tilde{g}(\tilde{b},\tilde{\phi}), \tilde{\phi})$

3. Wave packets over 3-geometries

From the point of view of quantum gravity, our cosmology is not a solution of Einstein's equations[7], but rather a wave function over 3-geometries that evolves in time. That our universe appears classical to a first approximation means that it is a wave packet, and that the spread of that wave packet is small.

We can think of the time history of such a wave packet as a 4-geometry with a spread, where the spread indicates the deviation from a classical 4-geometry. We usually assume that any solution of Einstein's equations is

[7] Or whatever classical set of equations turns out to be corrent in the classical limit.

an appropriate space-time for a classical cosmology. However, only those 4-geometries that can form the core of the time history of a wave packet over 3-geometries is an appropriate space-time for a classical cosmology. Appendix A shows that to form the core of such a wave packet, a 4-geometry must satisfy

$$\delta_{b_i} S(\tilde{g}(b,\tilde{\Phi}), \tilde{\Phi}) = 0 \quad \text{all } b_i \in b \tag{1}$$

where $\tilde{g}(b,\tilde{\Phi})$ is a solution to the field equations for the matter distribution $\tilde{\Phi}$. The set of parameters b includes all of the free parameters[8] that specify the 4-geometry $\tilde{g}$. $S(\tilde{g},\tilde{\Phi})$ gives the dependence of the action on the 4-geometry $\tilde{g}$ and matter distribution $\tilde{\Phi}$. Of course, valid solutions of (1) require that the action actually depend on all of the parameters varied.

If $\bar{b}$ is a solution of (1), then $\tilde{g}(\bar{b},\tilde{\Phi})$ is a 4-geometry that is a candidate for a classical cosmology. However, $\tilde{g}(\bar{b},\tilde{\Phi})$ will correspond to a classical cosmology only if the solution to (1) is unique and if the corresponding wave packet over 3-geometries is not too broad. The solution can be unique because the number of parameters equals the number of equations.

When is a wave packet too broad? The time history of the wave packet over 3-geometries will also include all those classical space-times[6] $\tilde{g}(b,\tilde{\Phi})$ for which

$$|S(\tilde{g}(b,\tilde{\Phi}), \tilde{\Phi}) - S(\tilde{g}(\bar{b},\tilde{\Phi}), \tilde{\Phi})| \lesssim \hbar \tag{2}$$

If the matter distribution $\tilde{\Phi}$ is such that $s(\tilde{g}(b,\tilde{\Phi}),\tilde{\Phi})$ depends only weakly on some of the b_i in b, then the time history of the wave packet could include some 4-geometries that are significantly different in some regions. It is appropriate to call such regions non-classical regions. To find out which matter distributions can form narrow wave packets we have to know how strongly the action depends on the parameters that specify the 4-geometry.

8 Those not constrained by the matter distribution

Table I classifies matter distributions according to how strongly the total action (matter plus geometry) depends on the parameters that specify solutions to the field equations for the given matter distribution.

Column (a) of Table I includes those matter distributions for which the action is independent of some of the parameters that specify the 4-geometry. These include the vacuum[9] and matter consisting of singularities (black holes)[10]. For such matter distributions there is no variation of phase to form lasting wave packets over 3-geometries by constructive interference. Equation (1) indicates this in that it has no valid solutions for such matter distributions.

While it is possible to form a wave packet (that approximates a single flat empty 3-space) for a short time, the uncertainty relations will cause such a wave packet to spread. Only solutions of (1) will lead to a wave packet over 3-geometries that persists.

Column (b) includes those matter distributions for which the action depends so weakly on some of the parameters that specify the 4-geometry that the time history of the wave packet would include space-times which are significantly different is some regions. Asymptotic space-times (including asymptotically flat space times) are in this category (the non-classical regions will be away from the matter distributions), as are some sparse cosmologies, where the non-classical regions can cover all of space-time, or all of space at some time. Appendix D discusses these space-times in more detail.

The extent to which geometries are "significantly" different is somewhat arbitrary. Our own universe is not purely classical.

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[9] In empty space-times, there is no matter, so that $L_{matter} = 0$ and also that $T^{\mu\upsilon} = 0$. $T^{\mu\upsilon} = 0$ implies $R^{\mu\upsilon} = 0$ which implies $R = 0$, so that from (B4), $L_{geom} = 0$ for a solution to the field equations. Therefore, from (B3) we get that $L = 0$, and therefore the action S from (B2) is also zero for all empty space-times. To be more accurate, we need to consider empty space as a limit of a space with a sparse matter distribution. The existence of any matter (that isn't too symmetrical) would destroy the apparent equivalence among the various vacuum solutions.

[10] See Appendix C.

Column (c) includes those matter distributions for which the action depends strongly enough on the parameters that specify the space-time that the geometry is not significantly different for all of those cosmologies for which (2) is valid. These include perfect fluids (if the pressure is not equal to the density and the matter is not too sparse).

Because a perfect fluid is probably a good approximation for the average large scale matter distribution in our own universe, it is reasonable to consider perfect fluids further. Appendix E shows that a reasonable total action for classical space-times in perfect fluid cosmologies is

$$S = \frac{3}{2} \int (-g(x))^{1/2} (\rho(x) - p(x))\, d^4x + \text{surface terms}^{(29,38)} \quad (3)$$

where ρ is the density, p is the pressure, and g is the determinant of the metric. The spatial integration is over the whole 3-space.

Unlike the matter distributions in column (a) of table I, the action here depends on the 4-geometry. Thus we can expect (1) to yield valid solutions that determine the 4-geometry core of the time history of a wave packet over 3-geometries.

Appendix F shows that such solutions will always have those same symmetries that are shared by the matter distribution and the form of the action.

A homogeneous, isotropic matter distribution has the most practical significance for our universe. Appendix F shows that the only solution of (1) for that matter distribution is the Robertson-Walker metric. Thus only the Robertson-Walker metric can form the 4-geometry core of the time history of a wave packet over 3-geometries for that matter distribution.

4. The meaning and limitations of Mach's principle

I have considered so far the classical limit of a quantum geometry. In particular, how a wave function over 3-geometries can become a wave packet over 3-geometries. We found that some types of matter distributions (perfect fluids, for example) can form wave packets, while others (vacuum, a collection of black holes) cannot.

Equation (1) determines those 4-geometries from among the solutions to the field equations for a given matter distribution that can form the core of the time history of a wave packet over 3-geometries. When (1) yields a unique 4-geometry for a given matter distribution, it may be appropriate to call that 4-geometry a Machian cosmology. For example, (1) gives the Robertson-Walker metric as the unique solution for a homogeneous, isotropic matter distribution. The other solutions to the field equations for that matter distribution (for example some where inertial frames rotate relative to the matter distribution [24-27]) do not satisfy (1).

Thus, for our universe, the matter distribution determines the geometry in the classical limit. That allows us to explain why inertial frames appear not to rotate relative to the stars. But to generalize this principle to all matter distributions is not valid. Rather, only a limited form of Mach's principle is valid. Some matter distributions determine the geometry, while some do not. It is more appropriate to speak of Machian matter distributions (those that determine the geometry) than of Machian cosmologies. Machian matter distributions are those which give unique solutions to (1).

We have also seen that even when (1) gives a unique solution, the wave packet over 3-geometries may be so broad in some regions of space-time that it does not approximate a classical geometry. The asymptotic regions far from the matter distributions or in sparse regions are such "non-classical" regions. Thus, the asymptotic regions of asymptotically flat space could be called "non-Machian" because the wave packet over 3-geometries is so broad.

In the introduction I pointed out that Raine's [21] results on Mach's principle seemed reasonable, even though I found it hard to justify Mach's principle on theoretical grounds. Now that I have justified in the classical limit a kind of limited Mach's principle, it is appropriate to compare the results. Essentially our methods agree if we interpret them appropriately.

Raine[(21)] classified Minkowski space, asymptotically flat space-times, and vacuum solutions as non-Machian. This agrees with my analysis in that these are solutions to the field equations for matter distributions that I have classified as non-Machian. Raine also classified spatially homogeneous cosmological models containing perfect fluids in which there is anisotropic expansion or rotation as non-Machian. This agrees with my analysis in that such solutions do not satisfy (1) for the given matter distribution. We both agree that the Robertson-Walker metric is Machian. I suspect that the two methods will always agree although the two criteria appear on the surface to be mathematically quite different.

Appendix A.

Classical limit of a wave function over 3-geometries

I calculate the wave function over 3-geometries g_2 and matter fields ϕ_2 at time t_2 as an integral over 3-geometries and matter fields of the initial state

$$\langle g_2,\phi_2,t_2|\psi\rangle = \int D(g_1)D(\phi_1) \langle g_2, \phi_2, t_2|g_1, \phi_1, t_1\rangle \langle g_1, \phi_1, t_1|\psi\rangle \qquad (A1)$$

where

$$\langle g_2, \phi_2, t_2|\, g_1, \phi_1, t_1\rangle = \int D(g)D(\phi)\exp(iS(g,\phi)/\hbar) \qquad (A2)$$

is the path integral(28) representation of the amplitude to go from a state with metric g_1 and matter fields ϕ_1 at time t_1 to a state with metric g_2 and matter fields ϕ_2 at time t_2. D(g) is a measure on the space of all metrics, D(ϕ) is a measure on the space of all matter fields, S is the action, and the integral is taken over all field configurations with the given initial and final values(29).

The greatest contribution to (A2) occurs where the action is stationary with respect to variation[11] of the 4-geometry g and the matter field ϕ. That is where

$$\delta_g\, S(g,\phi) = 0 \qquad (A3)$$

and

$$\delta_\phi\, S(g,\phi) = 0 \qquad (A4)$$

We are interested here in the classical limit. Therefore, we consider initially only those cases where (A3) and (A4) have unique solutions. That is, where

$$\tilde{g} = \tilde{g}(g_1, g_2, \phi_1, \phi_2) \qquad (A5)$$

and

$$\tilde{\phi} = \tilde{\phi}(g_1, g_2, \phi_1, \phi_2) \qquad (A6)$$

---

[11] The endpoints are held fixed for this variation.

The tilde in (A5) and (A6) indicates restriction to a 4-geometry and a matter field that satisfy (A3) and (A4). The arguments indicate the dependence on the endpoints of the path. As is well known, (A3) restricts the 4-geometry in (A5) to be a solution of the field equations, and (A4) restricts the matter in (A6) to be classical and follow geodesics.

The approximate integration of (A2) then gives

$$\langle g_2,\phi_2,t_2|g_1,\phi_1,t_1\rangle \approx f_1(\tilde{g},\tilde{\phi})\exp(iS(\tilde{g},\tilde{\phi})/\hbar) \tag{A7}$$

where $f_1(\tilde{g},\tilde{\phi})$ comes from the contribution of g and ϕ near $\tilde{g}$ and $\tilde{\phi}$ in the integration. We can substitute (A7) into (A1) to give

$$\langle g_2,\phi_2,t_2|\psi\rangle \approx \int D(g_1)D(\phi_1)f_1(\tilde{g},\tilde{\phi})\exp(iS(\tilde{g},\tilde{\phi})/\hbar)\langle g_1,\phi_1,t_1|\psi\rangle \tag{A8}$$

The substance of (A8) is that the main contribution to the wave function over 3-geometries g_2 and matter fields ϕ_2 at time t_2 is from 4-geometries $\tilde{g}$ that are solutions of the field equations and classical matter distributions $\tilde{\phi}$ that follow goedesics. The next approximation will show that particular classical 4-geometries $\tilde{g}$ and particular matter fields contribute more than others.

If the action depends strongly on the integration variables, the greatest contribution to the integral in (A8) occurs where the action is stationary with respect to variation[12] of g_1 and ϕ_1. That is, where

$$\delta_{g_1} S(\tilde{g},\tilde{\phi}) = 0 \tag{A9}$$

and

$$\delta_{\phi_1} S(\tilde{g},\tilde{\phi}) = 0 \tag{A10}$$

we will consider later the result if the action does not depend strongly on g_1. Again, as we are interested in classical limits, we will consider initially only those cases where (A9) and (A10) have unique solutions. That is, where

[12] For this variation, g_2 and $\emptyset_2$ are held fixed, and $S(\tilde{g},\emptyset)$ is used for the action. Only the part of g_1 not constrained by the matter distribution is varied.

$$\tilde{g}_1 = \tilde{g}_1(g_2, \phi_2) \tag{A11}$$

and

$$\tilde{\phi}_1 = \tilde{\phi}_1(g_2, \phi_2) \quad . \tag{A12}$$

The tilde indicates solutions of (A9) and (A10). Thus, an approximate integration of (A8) (and thus of (A1)) gives

$$\langle g_2,\phi_2,t_2|\psi\rangle \approx f_2(\tilde{g}_1,\tilde{\phi}_1)f_1(\tilde{g},\tilde{\phi})\exp(iS(\tilde{g},\tilde{\phi})/\hbar)\langle \tilde{g}_1,\tilde{\phi}_1,t_1|\psi\rangle \tag{A13}$$

where $f_2(\tilde{g}_1,\tilde{\phi}_1)$ comes from the contribution of g_1 and ϕ_1 near $\tilde{g}_1$ and $\tilde{\phi}_1$ in the integration.

The interpretation of (A13) is that only one 3-geometry $\tilde{g}_1$ at the time t_1 and only one 4-geometry $\tilde{g}$ contribute significantly to the amplitude that the 3-geometry at the time t_2 is g_2. The approximations made so far do not give us a classical 3-geometry at time t_2, but only a simpler way to calculate the wave function over 3-geometries at time t_2.

The condition that we have a nearly classical cosmology (that is, that the wave function over 3-geometries at time t_2 is a wave packet over 3-geometries) is that the wave function over 3-geometries is stationary with respect to variation of the 3-geometry.

If the action in (A13) depends strongly on g_2 and ϕ_2, then the action will dominate in determining when the wave function over 3-geometries is stationary with respect to variation of the 3-geometry. I will assume that to be true here. We will consider later the consequences of the alternative.

The condition that we have a classical cosmology (a wave packet over 3-geometries) is that the action be stationary with respect to variation[13] of

[13] For this variation, (A3), (A4), (A9), and (A10) are used as constraints. Only the part of g_2 not constrained by the matter distribution is varied.

the 3-geometry. That is, where

$$\delta_{g_2} \quad S(\tilde{g}, \tilde{\phi}) = 0. \tag{A14}$$

Similarly, to consider classical matter fields (wave packets), we can require the action to be stationary with respect to variation[13] of the matter field. That is,

$$\delta_{\phi_2} \quad S(\tilde{g}, \tilde{\phi}) = 0. \tag{A15}$$

The parameters being varied in (A9) and (A14) are one set of free parameters[8] that specifies the 4-geometry $\tilde{g}$. Any equivalent set of parameters[14] could be varied. The system (A9) and (A14) are equivalent to

$$\delta_{b_i} \quad S(\tilde{g}(b, \tilde{\phi}), \tilde{\phi}) = 0 \quad \text{for all } b_i \in b \tag{A16}$$

where $\tilde{g}(b,\phi)$ is a solution of the field equations for the matter distribution $\tilde{\phi}$. The b are a set of parameters[8] that specify the solution. The solution of (A16) (say $\bar{b}$) gives those solutions of the field equations that can form the core of the time history of a wave packet over 3-geometries. Thus only those particular solutions $\tilde{g}(\bar{b},\tilde{\phi})$ of the field equations can correspond to an approximate classical cosmology, and only then if the solution to (A16) is unique and if the wave packet is narrow enough.

Because the number of conditions in (A16) is equal to the number of free parameters b_i, (A16) will normally determine a unique 4-geometry as the core of the time history of the wave packet. When it does, we can call that 4-geometry a Machian cosmology. When those conditions do not lead to a unique solution, then Mach's principle does not hold for that matter distribution.

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[14] There are many alternate sets of parameters we could choose. We could choose, for example, the free-field initial conditions[8,15] as those parameters or the weyl tensor on a spacelike hypersurface.

[15] A classical initial condition is a 3-geometry plus lapse and shift functions(30), or a 3-geometry plus extrinsic curvature with the appropriate constraints.

We now need to consider the consequences when the total action $S(\tilde{g}(b,\tilde{\phi}),\tilde{\phi})$ for solutions to the field equations is either independent of or not strongly dependent on some of the parameters in the set b that specifies the 4-geometry $\tilde{g}$. The consequences are that the approximation that led to (A13) might not be valid and there would definitely be no wave packets over 3-geometries. There would be no "approximate classical" cosmology.

Appendix B

Lagrangian formulation[16]

The quantum amplitude of a space-time is proportional to

$$\exp(iS/\hbar) \tag{B1}$$

where $\hbar$ is Planck's constant divided by 2π.

S is the action given by

$$S = \int (-g(x))^{1/2} L(x)\, d^4x + \frac{1}{8\pi}\int K(h)^{1/2} d^3x + C \tag{B2}$$

where g is the determinant of the metric, and

$$L(x) = L_{geom}(x) + L_{matter}(x) \tag{B3}$$

is the Lagrangian[17]. In the surface term[(29,38)], K is the trace of the second fundamental form of the boundary, h is the induced metric on the boundary, and C is a term that depends only on the boundary metric h and not on the values of g at the interior points.

The Lagrangian for the geometry is

$$L_{geom}(x) = R(x)/16\pi, \tag{B4}$$

where

$$R(x) = R^{\mu\nu}(x)\, g_{\mu\nu}(x) \tag{B5}$$

is the 4-dimensional scalar curvature.

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[16] These calculations use the conventions of Misner, Thorne, and Wheeler [(31)].

[17] I do not include spin or fields other than gravitation here because they are not necessary for the main point of this paper.

The appendices to follow evaluate the total action for solutions to the field equations for specific types of matter distributions. In so doing, they freely use[18]

$$g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = -1 \tag{B6}$$

because it is valid for geodesics in space-times that are solutions to the field equations.

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[18] The signature of the metric is (- + + +). τ is the proper time. $d\tau^2 = - ds2$, where S is proper distance.

Appendix C

Black holes as matter distributions[19]

For a matter distribution of singularities whose masses are m_n and whose trajectories are given by x^μ_n, the energy momentum tensor is[(32)]

$$T^{\mu\nu}(x) = (-g(x))^{-1/2} \sum_n m_n \int d\tau_n \frac{dx^\mu_n}{d\tau_n} \frac{dx^\nu_n}{d\tau_n} \delta^4(x - x_n). \tag{C1}$$

Therefore, from Einstein's quations,

$$R^{\mu\nu} \equiv G^{\mu\nu} - \frac{1}{2} g^{\mu\nu} G = 8\pi \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right), \tag{C2}$$

we have

$$R^{\mu\nu}(x) = 8\pi (-g(x))^{-\frac{1}{2}} \sum_n m_n \left[\int d\tau_n \frac{dx^\mu_n}{d\tau_n} \frac{dx^\nu_n}{d\tau_n} \delta^4(x - x_n) - \frac{1}{2} g^{\mu\nu}(x) g_{\alpha\beta}(x) \int d\tau_n \frac{dx^\alpha_n}{d\tau_n} \frac{dx^\beta_n}{d\tau_n} \delta^4(x - x_n) \right]. \tag{C3}$$

Therefore, (B4), (B5), and (C3) give

$$L_{geom}(x) = -\frac{1}{2} g_{\mu\nu}(x) (-g(x))^{-\frac{1}{2}} \sum_n m_n \int d\tau_n \frac{dx^\mu_n}{d\tau_n} \frac{dx^\nu_n}{d\tau_n} \delta^4(x - x_n) \tag{C4}$$

as the Lagrangian for the geometry for solutions to the field equations.

Equations (B2) and (C4) give

$$S_{geom} = -\frac{1}{2} \sum_n m_n \int d\tau_n \frac{dx^\mu_n}{d\tau_n} \frac{dx^\nu_n}{d\tau_n} g_{\mu\nu}(x_n) \tag{C5}$$

as the action for the geometry.

The action for the matter is[(33)]

$$S_{matter} = \frac{1}{2} \sum_n m_n \int d\tau_n \frac{dx^\mu_n}{d\tau_n} \frac{dx^\nu_n}{d\tau_n} g_{\mu\nu}(x_n). \tag{C6}$$

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[19] When we have only one singularity, and we apply asymptotically flat boundary conditions, we get the Schwarzschild space-time. Although that space-time gives the same external field as a spherically symmetric distribution of mass such as a star, the action for the two is not the same. Thus, the calculations of this appendix do not apply to a collection of stars.

The sum of (C5) and (C6) is zero, which agrees with Hawking [29,38], that the only contribution to the action for this case comes from the surface terms.

The surface term for solutions to the vacuum field equations when asymptotically flat boundary conditions are applied is [38]

$$S = -\frac{M}{2}\int dt \tag{C7}$$

where M is the mass as measured from infinity.

Hawking[29] doubts the existence of solutions to the field equations for more than one black hole. However, if there were such solutions, then (C7) would give the action for those solutions that have asymptotically flat boundary conditions. Because it is possible to choose such boundary conditions in different ways relative to the matter distribution, (C7) shows that the action would be the same for all such solutions, even though the solutions are not equivalent. Thus, the total action would be independent of some of the parameters that specify the space-time.

Appendix D

Asymptotic space-times

We saw from appendix C that, for a matter distribution that consisted only of a collection of singularities, the total action for solutions to the field equations is independent of the initial and boundary conditions. If the number of singularities is finite, then these are asymptotic space-times. We will see here that the action for asymptotic space-times will always show some degree of independence of the geometry in the asymptotic regions, and therefore of the initial or boundary conditions or other parameters that specify the space-time.

In asymptotic space-times, the matter is confined to some region (say M) of space. The geometry outside of M (especially in the asymptotic regions) depends mostly on the boundary conditions, and only near M does it depend strongly on the matter within M. The geometry within M depends mostly on the matter within M (unless M is too sparse) and only near the boundary of M does it depend much on the boundary conditions.

The empty space outside of M makes no contribution to the action[9]. The action depends completely on the matter and geometry within M. Thus the geometry in the asymptotic regions influences the action only indirectly and only to the extent that the geometry inside M must adjust to fit with the geometry outside of M through the field equations.

Suppose there is a solution $\tilde{g}(\tilde{b},\tilde{\phi})$ to the field equations that satisfies (1). Then it can form the core of the time history of a wave packet over 3-geometries. Let $\tilde{S}$ be the action for that space-time. Then all other solutions $\tilde{g}(b,\tilde{\phi})$ to the field equations for the same matter distribution whose action S is such that

$$\left|S(\tilde{g}(b,\tilde{\phi}),\tilde{\phi})-S(\tilde{g}(\tilde{b},\tilde{\phi}),\tilde{\phi})\right| \lesssim \hbar \tag{D1}$$

will reinforce constructively with each other and therefore be contained in the time history of the wave packet. Because the action depends only weakly (and indirectly) on the geometry in the asymptotic regions, the space-times allowed by (D1) can include some whose geometries in the asymptotic regions are significantly different. Therefore the time history of the above wave packet includes significantly different geometries in the asymptotic regions.

Thus the wave packet will be too broad to approximate a classical space-time in the asymptotic regions, although it may be narrow enough outside of the asymptotic regions (especially within M) if the matter distribution within M is not too sparse.

Appendix E

Perfect fluids

For a perfect fluid, the energy-momentum tensor is[34]

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\, g^{\mu\nu} \tag{E1}$$

where ρ is the density and p is the pressure. From (E1) and the field equations (C2), we have

$$R^{\mu\nu} = 8\pi\left[(\rho+p)u^\mu u^\nu - \tfrac{1}{2} g^{\mu\nu}\left((\rho+p)\, g_{\alpha\beta} u^\alpha u^\beta + 2p\right)\right]. \tag{E2}$$

Therefore, (E2), (B4), and (B5) give the Lagrangian for the geometry for solutions to the field equations as

$$L_{geom} = -\tfrac{1}{2}(\rho+p)\, g_{\mu\nu} u^\mu u^\nu - 2p. \tag{E3}$$

We can take

$$L_{matter} = -\rho \tag{E4}$$

as the Lagrangian for the matter[35].

Adding (E3) and (E4) and using (B6) gives

$$L = L_{geom} + L_{matter} = \tfrac{3}{2}(\rho - p) \tag{E5}$$

for the total Lagrangian. From (B2) and (E5) we have

$$S = \frac{3}{2}\int (-g(x))^{\frac{1}{2}} \left(\rho(x) - p(x)\right) d^4x \quad + \text{surface terms}^{(29,38)} \tag{E6}$$

for the total action. Unlike the case where the only matter was a collection of singularities, the Lagrangian contribution to the action here is not identically zero unless the density and pressure are equal. The reason for the difference is that there, mass was a scalar and here density is a scalar.

The spatial limits of integration in (E6) are over the whole 3-space.

Appendix F

The symmetry property of Machian cosmologies

We often suppose that Machian cosmologies should have the same symmetries as the matter distribution. The first two of the following theorems establish such a symmetry property for the candidates for Machian cosmologies, solutions to (1).

Theorem 3 establishes existence by showing that cosmologies that have the above symmetry property are solutions of (1) and are thus candidates for Machian cosmologies.

Finally, theorem 4 combines theorems 2 and 3 to give the important result that the only solution of (1) for a homogeneous, isotropic matter distribution is the Robertson-Walker metric. Thus, the Robertson-Walker metric is the Machian cosmology for a homogeneous, isotropic matter distribution.

Theorem 1

If the form of the action $S(g,\phi)$ is homogeneous and isotropic (that is, there are no preferred directions or locations in our physical law) then for a homogeneous, isotropic matter distribution $\tilde{\phi}$, there are no solutions $\tilde{b}$ of (1) that give a 4-geometry $\tilde{g}(\tilde{b},\tilde{\phi})$ that has inertial frames that rotate relative to the matter distribution.

This theorem is significant for two reasons. First, one of the main results of this paper, is that equation (1) chooses those solutions of the field equations for a given matter distribution that are candidates for Machian cosmologies. Such Machian cosmologies should not have inertial frames that rotate relative to the matter distribution for a homogeneous, isotropic matter distribution. Second, Theorem 1 is the prototype for theorem 2, which is more general. The example of theorem 1 helps our understanding of theorem 2.

Proof:

For a homogeneous, isotropic matter distribution, there are many solutions $\tilde{g}(b,\tilde{\phi})$ to the field equations. Consider one such solution $\tilde{g}(a,\tilde{\phi})$ in which inertial frames rotate relative to the matter distribution about the x^1 axis. Consider the set of transformations T consisting of all rotations about an axis y other than the x^1 axis. Applying such a tranformation gives

$$T\,\tilde{g}(a,\tilde{\phi}) = \tilde{g}(b(a,T),\tilde{\phi}) \qquad \text{(F1)}$$

where $\tilde{g}(b(a,\tilde{\phi}),\tilde{\phi})$ is the same as $\tilde{g}(a,\tilde{\phi})$ except for the axis about which inertial frames rotate. T (for a fixed a) determines that axis. We can thus consider T to be a continuous parameter that specifies solutions to the field equations. All such transformations T leave the matter distribution unchanged and the form of the action unchanged. Therefore, the transformed matter distribution has the same set of solutions $\tilde{g}(b,\tilde{\phi})$ to the field equations as before. We now want to consider only those $\tilde{g}(b,\tilde{\phi})$ that are solutions of (1).

The transformation T is one of the parameters belonging to the set b, because varying T gives different solutions to the field equations. Equation (1) thus requires

$$\delta_T \quad S\left(\tilde{g}\left(b(a,T), \tilde{\phi}\right), \tilde{\phi}\right) = 0 \qquad \text{(F2)}$$

There is, however, no solution for (F2) because S is independent of T. Therefore $\tilde{g}(b(a,T),\tilde{\phi})$ can be a solutions of (1) only if T is not a parameter that specifies a solution to the field equations. That is true only if

$$\tilde{g}\left(b(a,t), \tilde{\phi}\right) = \tilde{g}\left(a, \tilde{\phi}\right) \quad \text{for all } T \qquad \text{(F3)}$$

That can only be true if $\tilde{g}(a,\tilde{\phi})$ does not have a preferred symmetry axis. That requires that $\tilde{g}(a,\tilde{\phi})$ does not have inertial frames that rotate relative to the matter distribution. That completes the proof.

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Theorem 2

All solutions $\tilde{b}$ of the system (1) give a 4-geometry $\tilde{g}(\tilde{b},\tilde{\phi})$ that has the same continuous symmetries as those shared by the matter distribution $\tilde{\phi}$ and the form of the action $S(g,\phi)$.

Proof:

Consider all solutions $\tilde{g}(b,\tilde{\phi})$ of the field equations for the given matter distribution. Consider one of those solutions $\tilde{g}(a,\tilde{\phi})$, that does not have one of the continuous symmetries shared by the matter distribution and the form of the action. Thus, there are a set $\hat{T}$ of transformations T such that $\hat{T}$ is a continuous group,

$$T\,\tilde{g}\,(a,\tilde{\phi}) = \tilde{g}\,(b\,(a,T),\tilde{\phi}), \tag{F4}$$

$$T\,\tilde{\phi} = \tilde{\phi}, \tag{F5}$$

and

$$T\;S\,(g,\phi) = S\,(g,\phi), \tag{F6}$$

where

$$\tilde{g}\,(a,\tilde{\phi}) \neq \tilde{g}\,(b\,(a,T),\tilde{\phi}) \tag{F7}$$

unless T = 1 (identity transformation).

Because T does not change the form of the action, it does not change the field equations. As it also does not change the matter distribution, the set of all solutions $\tilde{g}(b,\tilde{\phi})$ also remains the same. Applying the transformation T to the observer's frame of reference shows that $\tilde{g}(b(a,T),\tilde{\phi})$ is a solution of the field equations for all T in the group $\hat{T}$. Therefore we can consider T to be a continuous parameter that specifies solutions to the field equations.

We now want to consider only those $\tilde{g}(b,\tilde{\phi})$ that are solutions of (1). One of those parameters b_i that specifies the solutions to the field equations is the transformation T. Thus, (1) requires

$$\delta_T \; S(\tilde{g}(b(a,T),\tilde{\phi}),\tilde{\phi}) = 0 \quad . \tag{F8}$$

However, the action is independent of T as can be seen by applying the transformation T to the observer. Thus, the equation (F8) has no solution. Therefore, $\tilde{g}(b(a,T),\tilde{\phi})$ can be a solution of (1) only if T is not a parameter that specifies a solution to the field equations. That is true only if

$$\tilde{g}(b(a,T),\tilde{\phi}) = \tilde{g}(a,\tilde{\phi}) \quad \text{for all } T \text{ in } \hat{T} \quad . \tag{F9}$$

That means the geometry is invariant under the transformation T. As $\hat{T}$ represents all continuous groups of transformations for which the matter distribution and the form of the action are invariant, the 4-geometry $\tilde{g}(\tilde{b},\tilde{\phi})$ has all of the continuous symmetries shared by the matter distribution and the form of the action. That completes the proof.

Theorem 3

All solutions $\tilde{g}(b,\tilde{\Phi})$ of the field equations for the matter distribution $\tilde{\Phi}$ that have all the symmetries shared by the matter distribution and the form of the action are solutions of (1).

Proof: consider a parameterization b such that:

$\tilde{g}(b,\tilde{\Phi})$ has all the symmetries shared by the matter distribution and the form of the action when $b_i = 0$ for all b_i contained in b. Define

$$S(b,\tilde{\Phi}) \equiv S(\tilde{g}(b,\tilde{\Phi}),\tilde{\Phi}) \quad . \tag{F10}$$

Let

$$\tilde{g}(b,\tilde{\Phi}) = \tilde{g}(b_1, b_2, \ldots, b_i, \ldots, \tilde{\Phi}) = \tilde{g}(b_i, \overline{b_i}, \tilde{\Phi}) \, . \tag{F11}$$

Let

$$S(b,\tilde{\Phi}) = S(b_1, b_2, \ldots, b_i, \ldots, \tilde{\Phi}) = S(b_i, \overline{b_i}, \tilde{\Phi}), \tag{F12}$$

$$\text{where } b_i \notin \overline{b_i} \text{ and } \overline{b_i} \cup \{b_i\} = b \, .$$

It is sufficient to prove that

$$S(b_i, \overline{b_i}, \tilde{\Phi})\Big|_{b_j=0} = S(-b_i, \overline{b_i}, \tilde{\Phi})\Big|_{b_j=0} \tag{F13}$$

$$\text{for all } b_j \in \overline{b_i}$$

because, then, S is an even function of b_i and thus

$$\left.\frac{\partial S}{\partial b_i}\right|_{b_i=0} = 0 \quad \text{for all } b_i \in b \, , \tag{F14}$$

which shows that ($b_i = 0$ for all $b_i \in b$) satisfies (1).

We can prove (F13) by using the symmetry properties. Because $\tilde{\Phi}$ and S have common symmetries, there are a set of transformations $T_i \in T$ that leave the matter distribution and the form of the action unchanged; I will choose a parameterization b based on those transformations.

Choose a parameterization b such that for each $b_i \in b$ there is a transformation $T_i \in T$ and a parameterization $\hat{b}_i$ such that for all $b_{ij} \in \hat{b}_i$ we have

$$T_i \; \tilde{g}(b_i, b_{ij}, \tilde{\phi}) = \tilde{g}\left(-b_i, \sum_{k \neq i} \alpha_{ijk} b_k, T_i \tilde{\phi}\right) . \tag{F15}$$

Notice that the parameterization b_i, $\hat{b}_i$ does not have to be the same as b_i, $\bar{b}_i$. That is, for each i we may reparameterize the other b_j. To see that (F15) is correct, we simply notice that 1) the parameterization consisting of specifying all 10 independent components of the Weyl tensor on a spacelike hypersurface is general enough for all cases, and 2) the set of transformations that consist of interchanging spatial axes are general enough to represent any symmetries we may encounter. (They represent reflections and rotation with the appropriate choice of axes, for example.)

Then it is necessary only to display one parameterization of the Weyl tensor that satisfies (F15). Table II shows such a parameterization with the transformation T_i corresponding to each parameter b_i. As there are only 4 transformations (3 reflections and 1 rotation) in Table II, we need consider only 4 reparameterizations. Tables III, IV, V, and VI give those 4 parameterizations, and also give the result of applying the transformation to each parameter to demonstrate that each parameterization satisfies (F15).

The first 4 parameters in table II are odd eigenfunctions of interchanging spatial axes 2 and 3. Table III shows a parameterization where that transformation does not mix those 4 parameters with any other. Table IV does the same for the next 4 parameters and the transformation that interchanges axes 1 and 3. Table V does the same for the 9th parameter and the transformation that exchanges axes 1 and 2. Table VI does the same for the 10th parameter in Table II and the transformation, axis 3$\rightarrow$axis 2, axis 2$\rightarrow$axis 1, axis 1$\rightarrow$ axis 3. Using the known symmetries of the Weyl tensor easily establishes that the transformations are correct and that the 10 parameters are independent in each of the 5 parameterizations.

The point of having the property in (F15) is that I can then set all the parameters except one (b_i) to zero and then apply a transformation that gives another solution to the field equations, the same as the first, except for the sign of that one parameter b_i. Both the form of the action and the matter distribution are invariant under the transformation by postulate. Thus the transformation does not change the value of the action. The action is, however, formally a function of only the parameters and the matter distribution. Of these, only one parameter has been changed by the transformation, and that parameter has only changed sign. Therefore the action must be an even function of that parameter. Since this property is true for each parameter individually, we establish (F13), and thus complete the proof.

In detail I establish (F13) as follows:

$$S(b_i, \overline{b_i}, \tilde{\phi}) \equiv S(\tilde{g}(b_i, \overline{b_i}, \tilde{\phi}), \tilde{\phi}) \qquad \text{(F16)}$$

$$= S(\tilde{g}(b_i, b_{ij}, \tilde{\phi})$$

by definition of the parameterization in (F10), (F11), and (F12).

$$= S(T_i\, \tilde{g}(b_i, b_{ij}, \tilde{\phi}), T_i\, \tilde{\phi}) \qquad \text{(F17)}$$

because the form of the action S is invariant under the transformation T_i.

$$= S(\tilde{g}(-b_i, \sum_{k \neq i} \alpha_{ijk}\, b_k, \tilde{\phi}), \tilde{\phi}) \qquad \text{(F18)}$$

because the matter distribution is invariant under the transformation T_i and because of the property (F15) of the parameterization.

$$= S(-b_i, \sum_{k \neq i} \alpha_{ijk}\, b_k, \tilde{\phi}) \qquad \text{(F19)}$$

from the definition of the parameterization in (F10), (F11), and (F12).

Combining (F16) with (F19) with $b_{ij} = 0$ for all $b_{ij} \in \overline{b_i}$ establishes (F13) and finishes the proof.

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Theorem 4

The only solution $\tilde{b}$ of the system (1) for a homogeneous, isotropic matter distribution $\tilde{\phi}$ gives the Robertson-Walker metric, for $\tilde{g}(\tilde{b},\tilde{\phi})$.

Proof: From theorem 3, the Robertson-Walker metric is a solution of (1) because it has both symmetries (homogeneity and isotropy) shared by the matter distribution and the form of the action. That it is the unique solution of (1) follows from theorem (2) because it is the only solution of the field equations for the given matter distribution that has those symmetries.

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Table I

Classification of types of matter distributions according to how strongly the total action depends on the parameters that specify solutions to the field equations

	(a)	(b)	(c)
Does the action depend on the parameters that specify the space-time?	NO	Yes, weakly (geometry significantly different for $\|S-\tilde{S}\| < \hbar$)	Yes, strongly (geometry not significantly different for $\|S-\tilde{S}\| < \hbar$)
Physical interpretation	All classical space-times are equivalent with respect to the action	Some significantly different classical space-times are nearly equivalent with respect to the action	The various classical space-times are not equivalent with respect to the action
Some of the types of matter distributions included	Vacuum[9] Black holes (Appendix C)	Localized matter distributions (Appendix D)	Perfect fluids (Appendix E)
Examples of corresponding space-times	Minkowski space-time Schwarzschild metric	Asymptotic space-times Asymptotically flat space-times	Robertson-Walker metric
Condition for the formation of wave packets over 3-geometries	Can not form wave packets.	The total action is stationary with respect to variation of the parameters that specify the space-time.	
Conditions for such a wave packet to approximate a classical cosmology		1) Exactly one solution to the field equations fulfils the above condition, and 2) Only in regions of space-time where the geometry is not significantly different for $\|S-\tilde{S}\| < \hbar$ will the wave packet approximate a classical space-time.	Exactly one solution to the field equations fulfils the above condition.

S is the total action (matter plus geometry) for a solution to the field equations. $\tilde{S}$ is the total action for a solution to the field equations for which the action is stationary with respect to variation of the parameters that specify the space-time.

Table II

Ten independent parameters from the Weyl tensor $C_{\alpha\beta\gamma\delta}$ that are odd eigenfunctions of transformations based on interchanging spatial axes.

Parameter	Transformation
C_{1220}	axis 2 $\longleftrightarrow$ axis 3
$C_{1323} + C_{1223}$	axis 2 $\longleftrightarrow$ axis 3
$C_{2020} - C_{1212}$	axis 2 $\longleftrightarrow$ axis 3
C_{1023}	axis 2 $\longleftrightarrow$ axis 3
C_{1210}	axis 1 $\longleftrightarrow$ axis 3
$C_{1213} - C_{1323}$	axis 1 $\longleftrightarrow$ axis 3
$C_{1212} - C_{2323}$	axis 1 $\longleftrightarrow$ axis 3
C_{1302}	axis 1 $\longleftrightarrow$ axis 3
C_{1310}	axis 1 $\longleftrightarrow$ axis 2
$C_{1213} - C_{1323} - C_{1223}$	axis 3 $\longrightarrow$ axis 2
	axis 2 $\longrightarrow$ axis 1
	axis 1 $\longrightarrow$ axis 3

Table III

Results of interchanging axes 2 and 3 for one parameterization of the Weyl tensor

Parameter	Results of transformation (axis 2 $\longleftrightarrow$ axis 3)
C_{1220}	$-C_{1220}$
$C_{1323} + C_{1223}$	$-C_{1323} - C_{1223}$
$C_{2020} - C_{1212}$	$-C_{2020} + C_{1212}$
C_{1023}	$-C_{1023}$
$C_{1210} - C_{1310}$	$-C_{1210} + C_{1310}$
C_{2323}	C_{2323}
$C_{1230} - C_{1302}$	$C_{1230} - C_{1302}$
$C_{1210} + C_{1310}$	$C_{1210} + C_{1310}$
$C_{1323} - C_{1223}$	$C_{1323} - C_{1223}$
C_{1213}	C_{1213}

Table IV

Results of interchanging axes 1 and 3 for one parameterization of the Weyl tensor

Parameter	Results of transformation (axis 1 $\leftrightarrow$ axis 3)
C_{1210}	$-C_{1210}$
$C_{1213} - C_{1323}$	$-C_{1213} + C_{1323}$
$C_{1212} - C_{2323}$	$-C_{1212} + C_{2323}$
C_{1302}	$-C_{1302}$
$C_{1220} - C_{1310}$	$-C_{1220} + C_{1310}$
$C_{1220} + C_{1310}$	$C_{1220} + C_{1310}$
C_{1223}	C_{1223}
$C_{1323} + C_{1213}$	$C_{1323} + C_{1213}$
C_{2020}	C_{2020}
$C_{1023} - C_{1230}$	$C_{1023} - C_{1230}$

Table V

Results of interchanging axes 1 and 2 for one parameterization of the Weyl tensor

Parameter	Result of the transformation (axis 1$\leftrightarrow$axis 2)
C_{1310}	$-C_{1310}$
$C_{1213}+C_{1223}$	$-C_{1213}-C_{1223}$
$C_{2323}-C_{2020}$	$-C_{2323}+C_{2020}$
C_{1230}	$-C_{1230}$
$C_{1220}+C_{1210}$	$-C_{1220}-C_{1210}$
$C_{1220}-C_{1210}$	$C_{1220}-C_{1210}$
C_{1323}	C_{1323}
$C_{1223}-C_{1213}$	$C_{1223}-C_{1213}$
C_{1212}	C_{1212}
$C_{1302}-C_{1023}$	$C_{1302}-C_{1023}$

Table VI

Results of applying a transformation for one parameterization of the Weyl tensor.

Parameter	Result of transformation (axis 3 → axis 2, axis 2 → axis 1, axis 1 → axis 3)
$C_{1223} - C_{1213} + C_{1323}$	$-C_{1223} + C_{1213} - C_{1323}$
$C_{1223} + e^{-i\pi/3}C_{1213} - e^{i\pi/3}C_{1323}$	$e^{i\pi/3}(C_{1223}+e^{i\pi/3}C_{1213}-e^{i\pi/3}C_{1323})$
$C_{1223} + e^{i\pi/3}C_{1213} - e^{-i\pi/3}C_{1323}$	$e^{-i\pi/3}(C_{1223}+e^{i\pi/3}C_{1213}-e^{-i\pi/3}C_{1323})$
C_{1220}	$-C_{1310}$
C_{1310}	C_{1210}
C_{1210}	$-C_{1220}$
C_{2020}	C_{2323}
C_{2323}	$-C_{2020} - C_{2323}$
C_{1023}	C_{1230}
C_{1230}	$C_{1230} + C_{1023}$

31.1 Report of Referee 1

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SEPT 22

THE QUANTUM BASIS FOR MACH'S PRINCIPLE - Michael Jones.

While this paper contains some interesting ideas, to my way of thinking it contains mainly vague generalities and unsubstantiated suggestions that do not prove what the author sets out to prove. There may well be other referees who will disagree with me, and I would suggest sending the paper to others who may be more sympathetic to its style.

To give my comments substance, I will comment in more detail on some particular sections.

Equation (1) appears to be the key equation in the authors' development, and this is based on Appendix A. Here I am quite unclear as to the meaning of equation (A2), for the definition of the measures D(g) and D(ϕ) is left completely open; the domain of integration is unstated; the space-time topology is not given; in fact it is not clear if the stated integral exists, and if it does not the rest of the argument falls away. As examples, it probably does not exist in the Godel universe or Taub-Nut universe. Most crucially, it is unclear if one should add into equation (A2) a surface integral term as well as the volume integral shown - and Mach's principle is concerned precisely with the values of such surface integral terms. Then it is stated that (A13) is *an* integral of (A8); but Mach's principle is concerned with *all* solutions for the given conditions. In particular, after (A15) reference is made to "a set of parameters" that specify the 4-geometry. But a general 4-geometry is specified by a set of *functions*, that is, infinitely many parameters. Thus I simply cannot follow the bottom paragraphs of that page where it is said that the number of conditions in (A15) is equal to the number of free parameters. Maybe he means, under (A15), the number of parameters that determine the geometry at a point ? I don't know. One can make a number of guesses which may or may not correspond to what the author has in mind. On page 15, I have no idea what is meant by "not strongly dependent". We are given no criterion for this requirement.

Returning to section 3 on page 6, the meaning of equation (2) is quite ? obscure because the action can be normalised by any constant amount. Page 7 refers to table 1 which is apparently crucial to the rest of the argument. I do not follow where the information in table 1 has come from. What are the parameters whose variation has been examined ? What criterion has been used to decide that the action "depends weakly on the parameters"? On page 9, it is assumed that "our universe" is homogeneous and isotropic for all time. This may be quite untrue. If it is true, the question raised is *why* is it true ? - for it is well known that inertial frames and the rest frame defined by distant galaxies coincide in Robertson-Walker universes.

Appendix C1 is entitled "Black holes as matter distributions" but a δ-function in T is not the same thing as a black hole. In fact none of equations (C1) to (C6) appear to refer to black holes at all. (One *cannot* use a δ-function non-zero on a timelike world line to represent a black hole, for the singularity in a black whole is space-like.) Appendix F does not make clear in Theorem 1 whether the homogeneity and isotropy envisaged is 4-dimensional (i.e. a space-time) symmetry or 3-dimensional (i.e. referred to some preferred timelike vector field). In theorem 2, it is quite unclear what is meant by "all solutions ..... of the field equations for the given matter distribution" - how is the matter distribution described? in what way is it kept fixed while the space-time metric (and so all length and time measurements) is allowed to vary?

Overall, I find so many unanswered questions arise as I read the manuscript that I cannot recommend its acceptance. In a discussion of Mach's principle, *clarity* is essential above all else. This paper does not provide it.

31.2 My response to referee 1

The quantum basis for Mach's principle

Author's response to referee report #1

Referee: this paper . . . contains mainly vague generalities and unsubstantiated suggestions that do not prove what the author sets out to prove.

Response: I'll try to improve that.

Referee: I am quite unclear as to the meaning of equation (A2).

Response: If we knew everything about the meaning of equation (A2), we would have solved all of the problems of quantum gravity. I don't claim to have done that. What I have dome is to make some observations on the properties of (A2) in the classical limit. Of course, my observations depend on some kind of path integral formulation being valid. Many people would be surprised if a path integral formulation for quantum gravity turns out not to be valid. My general observations do not depend on the details of the path integral formulation, although the results of my calculations in the example depend on the Lagrangian I used.

Referee: The definition of the measures D(g) and D(ϕ) is left completely open.

Response: I left open the definition of the measure, because the measure has negligible effect in the classical limit. Kaku and Senjonović (Phys. Rev. D15, pp 1019-1025, 1977) claim to give the correct measure for quantum gravity on an asymptotically flat background, but I don't know if it applies to quantizing the cosmology.

Referee: The domain of integration is unstated.

Response: I said in my paper that the integral is taken over all field configurations with the given initial and final values. For the geometry, that means all 4-geometries that have the given initial and final 3-geometries. It should probably be limited to 4-geometries for which the action integral (B2) exists. It is probably too restrictive to include only 4-geometries that can be expressed as the time history of a 3-geometry. The correct domain of integration in quantum gravity is not yet known. However, it includes at least all of the solutions of Einstein's field equations. For the purpose of my paper, that is sufficient because I am considering the classical limit.

Referee: The space-time topology is not given.

Response: Once we have found a correct formulation of quantum gravity, we will probably find that all space-time topologies for which the action integral (B2) exists should be included in the path integral. The same will probably also be true for the signature.

Referee: It is not clear if the stated integral exists.

Response: If a valid path integral formulation for quantum gravity exists, then the integral (A2) exists.

Referee: If it does not the rest of the argument falls away.

Response: If the path integral (A2) does not exist (or something similar, perhaps involving path-ordered integrals), then that means a path integral formulation of quantum gravity is not valid. If it is not valid, then a lot of other work in addition to mine falls away. I feel that it is reasonable to assume that a valid path integral formulation of quantum gravity exists.

Referee: As examples, it probably does not exist in the Gödel universe or Taub-Nut universe.

Response: The Gödel universe would be one point in the domain of integration in (A2). I think the question is not whether the path integral (A2) exists for the Gödel universe, but whether the action integral (B2) exists for the Gödel universe. First, the existence of the action integral is a problem that must be solved in any successful path integral formulation of quantum gravity. My paper is not concerned with the problem of finding a correct formulation for quantum gravity, but rather with finding the correct classical limit assuming a correct formulation can be found. Second, there is the possibility that in a correct path integral formulation of quantum gravity, the action integral may not exist for some cosmologies. Such a phenomenon would, of course, require an interpretation. If the action integral does not exist for the Gödel universe, it might be a judgement against the Gödel universe rather than against the formulation of quantum gravity. I don't know if the action integral is defined for the Gödel universe in the formulation I used as an example, but because I had no cosmological constant in my Lagrangian, the Gödel universe will make no contribution in the classical limit for any of the matter distributions I considered. Similar comments apply to the Taub-Nut universe.

Referee: Most crucially, it is unclear if one should add into equation (A2) a surface integral term as well as the volume integral shown -

Response: Adding a surface integral term to a path integral

seems to negate the whole basis of the path integral formulation. Normally, we consider all possible paths (or, in general, ways) to get from the initial state to the final state. Each path makes a contribution (in the path integral) to the amplitude for getting from the initial state to the final state. Adding a surface term to the path integral (such as in (A2)) would correspond to a contribution to the amplitude for getting from the initial state to the final state that was in addition to the contribution from each path. (Of course, we could incorporate that contribution into the path integral formalism by calling that the contribution from the magic path, but I doubt the usefulness of that.)
I do, however, add a surface term to the action integral (B2).

Referee: Mach's principle is concerned precisely with the values of such surface integral terms.
Response: If there were any justification for adding a surface term to a path integral, then it might affect the results of my paper. Mach's principle might relate to surface terms in an action integral (such as (B2)), but I am not sure how much freedom there is in specifying those surface terms. I normally think of Mach's principle in relation to the surface terms in an integral formulation of classical general relativity such as that by Sciama, Waylen, and Gilman (Phys. Rev. 187, pp 1762-1766, 1969.).

In a path integral formulation of quantum gravity, the surface integral relevant to Mach's principle is the integral over the initial state in equation (A1) of my paper. Normally, the final state will depend on the initial state of both the matter fields and the geometry. However, as I argue in my paper, under some conditions (if the initial state was a long time ago, and if there is enough matter in the universe), the propagator in (A1) (or equivalently, the exponential factor in (A8)) will dominate over the initial state of the geometry in determining the final state. I also argue that those conditions are fulfilled for our universe.

Referee: It is stated that (A13) is an integral of (A8); but Mach's principle is concerned with all solutions for the given conditions.

Response: I think my wording was unfortunate. I will correct that. I meant that (A13) was the first term in the asymptotic evaluation of (A8). Because I was looking for a classical limit, I considered the case where there was only one stationary point in the integral in (A8). If there were more than one stationary point, and the stationary points were isolated, we would replace (A13) by a sum of similar terms, one term for each stationary point. If the stationary points are not isolated, there are standard methods for extending the calculation. It would have been more accurate if I had said that (A13) is the approximation to (A8) in the classical limit. I think that wording should avoid the confusion.

Referee: After (A15) reference is made to "a set of parameters" that specify the 4-geometry. But a general 4-geometry is specified by a set of functions, that is, infinitely many parameters. Thus I simply cannot follow the bottom paragraphs of that page where it is said that the number of conditions in (A15) is equal to the number of free parameters. Maybe he means, under (A15), the number of parameters that determine the geometry at a point? I don't know. One can make a number of guesses which may or may not correspond to what the author has in mind.

Response: Your first assumption was correct. The set of parameters are a set of functions, so that the effective number of parameters is infinite. When I said the number of conditions is equal to the number of parameters, I meant that in the usual way of comparing infinities. That is, by pairing parameters with conditions. If there is any free parameter associated with specifying the geometry, then we require that the action be stationary with respect to that parameter. That fixes the parameter, so that it is no longer free. We continue until there are no free parameters left associated with specifying the geometry. That the number of parameters is infinite offers no conceptual problem, and we don't have to actually carry out the operation in practice. Most of the exact solutions we actually know have a finite number of parameters, and I have applied the procedure to some simple cases as examples.

Referee: On page 15, I have no idea what is meant by "not strongly dependent". We are given no criterion for this requirement.

Response: This refers to the discussion in the third paragraph on page 12. I will try to make this connection clearer when

I modify the paper. The "strongly dependent" criterion is related to the approximation of the integral in (A8) by (A13). That approximation is valid only if the exponential factor in (A8) dominates the integration. That will be true only if the action depends strongly enough on the integration variables. That is the criterion for strongly dependent.

Referee: Returning to section 3 on page 6, the meaning of equation (2) is quite obscure because the action can be normalized by any constant amount.

Response: There are two ways to normalize the action by a constant amount.

a) Adding a constant - This adds a constant phase to all amplitudes. We can consider that to be a gauge transformation. It has no effect on calculated probabilities because all amplitudes would be normalized by the same phase, and all "interference patterns" would remain unchanged. It also would have no effect on equation (2), because the constant cancels out in the subtraction.

b) Multiplying by a constant - This would have no effect on the classical limit, because it would not affect the stationary conditions. However, it does affect the quantum amplitudes, because it changes the effective value of Planck's constant. It also changes the width of wave packets in the classical limit. Thus, while a normalization factor in the action is unimportant for a classical theory, it is significant for a quantum theory.

Referee: Page 7 refers to table 1 which apparently is crucial to the rest of the argument. I do not follow where the information in table 1 has come from.

Response: The information comes from footnote 12, and Appendices C, D, and E. These are referenced in the table at the appropriate places.

Referee: What are the parameters whose variation has been examined?

Response: For the empty spaces, the rotation rates of inertial frames. For the black holes, the rotation rates of the Minkowski frames to which the black hole solutions are asymptotic. For the asymptotic spaces, the parameters that specify the spacetime to which the asymptotic spacetimes are asymptotic. For the perfect fluid models, the relative rotation rate of the inertial frames and the matter distribution for a particular class of Bianchi cosmologies.

Referee: What criterion has been used to decide that the action "depends weakly on the parameters"?

Response: $|S-\tilde{S}| \leq \hbar$ for significantly different geometries.

Referee: On page 9, it is assumed that "our universe" is homogeneous and isotropic for all time. This may be quite untrue. If it is true, the question raised is <u>why</u> is it true? - for it is well known that inertial frames and the rest frame defined by distant galaxies coincide in Robertson-Walker universes.

Response: I didn't mean to imply that I was assuming our universe to be homogeneous and isotropic for all time. Our universe is not homogeneous, or course, as any look at the sky on a cloudless night will show. But to a very good approximation, the large scale structure of our universe seems to be homogeneous and isotropic, both for the matter distribution and for the geometry. I am not trying to answer why the matter distribution is so nearly homogeneous and isotropic. Rather, given that the matter distribution is so nearly homogeneous and isotropic, why is the geometry also nearly homogeneous and isotropic? The answer to this question is not obvious, because Einstein's equations do not require the geometry and the matter distribution to have the same symmetries. Consider, for example, the Bianchi type VI_h cosmologies (see reference 26 in my paper). Both the matter distribution and the geometry are spatially homogeneous because there exists a group of motions simply transitive on three-surfaces. The matter distribution is isotropic because the fluid flows normal to the surfaces of homogeneity. The geometry is not isotropic because in the rest frame of an observer moving with the fluid, a set of Fermi-propagated axes rotates with respect to a particular inertial triad. The relative rotation rate of the Fermi-propagated axes and a particular inertial triad depends only on time because these are spatially homogeneous models. The value of that rotation rate at some specified time can be considered to be a parameter that specifies these cosmologies. I argue in my paper that among those cosmologies only those for which the action is stationary with respect to variation of such a parameter contribute significantly in the classical limit.

--

A straightforward (but lengthy) calculation using the formulas in reference 26 shows that the action (from equation (3) in my paper) is an even function of the rotation rate parameter for these cosmologies. Thus, a rotation rate of zero is a solution of the stationary condition(equation (1)) for these cosmologies. That it is the only solution follows from theorem 1 in my paper, which shows that for all solutions of equation (1), the geometry will have the same spacetime symmetries as the matter distribution. Thus, among the Bianchi type VI_h cosmologies, only that for which the geometry is isotropic contributes significantly in the classical limit. That case corresponds to the Robertson-Walker metric.

Referee: Appendix C is entitled "Black holes as matter distributions" but a δ-function in T is not the same thing as a black hole. In fact none of equations (C1) to (C6) appear to refer to black holes at all. (One <u>cannot</u> use a δ-function non-zero on a timelike world line to represent a black hole, for the singularity in a black hole is space-like.)
Response: You are right, and I will remove equations (C1) through (C6) from my paper. The results of Appendix C remain unchanged, however, because my δ-function calculation agreed with Hawking's more rigorous result.

Referee: Appendix F does not make clear in Theorem 1 whether the homogeneity and isotropy envisaged is 4-dimensional (i.e. a spacetime) symmetry or 3-dimensional (i.e. referred to some preferred timelike vector field).

Response: The symmetries referred to in Theorem 1 are 3-dimensional. I will clarify that in the paper.

Referee: In theorem 2, it is quite unclear what is meant by "all solutions ..... of the field equations for the given matter distribution" - how is the matter distribution described? in what way is it kept fixed while the space-time metric (and so all length and time measurements) is allowed to vary?

Response: To separate the parameters that specify the matter distribution from those that specify the geometry is difficult, but not impossible. For example, the anisotropy in the Bianchi type VI_h models is clearly an anisotropy of the geometry rather than of the matter distribution because the fluid flow is normal to the surfaces of spatial homogeneity. I suspect that careful thinking (and a lot of work) would allow one to separate the parameters that relate to the matter distribution from those that relate to the geometry in all of the known solutions to the field equations. Considering the Weyl tensor can help in doing this, because the Weyl tensor is the part of the Riemann curvature tensor that is not constrained by the matter distribution. It is not important for us to actually carry out this separation, but only to know that there is a unique way to do it. The separation cannot be done completely at the classical level, but will require reference to atomic dimensions. This is reasonable, because we are talking about the classical limit of a quantum theory rather than a classical theory by itself.

Referee: Overall, I find so many unanswered questions arise as I read the manuscript that I cannot recommend its acceptance.

Response: I hope I have answered the questions. If you feel that my answers are satisfactory, I will incorporate the answers in my paper where necessary to clarify the paper.

Referee: In a discussion of Mach's principle, clarity is essential above all else. This paper does not provide it.

Response: I will try to correct that.

31.3 Report of Referee 2 and 3

M. Jones. The quantum basis for Mach's principle

The basic idea appears to concern the passage from a quantum universe, given say by a solution $\Psi\{g\}$ of the functional Schrödinger equations, to a classical interpretation in the case where $\Psi\{g\}$ is supposed to give an essentially complete description of the universe rather than of a subsystem subject to external measuring apparatus. This is, of course, a well-known problem in quantum gravity, but its bearing on Mach's Principle is a novel idea that could be well-worth discussion.

However, the development of the idea here seems to me to be wrong. Some particular points are:

(i) Equation A8: the result of the integration depends also on $\langle g_1\phi_1, t/\psi\rangle$. The analysis shows only that this will have to be sufficiently sharply peaked (i.e. just any old wave packet will not do) not that no wave packet construction is possible. Indeed, a wave-packet solution must be possible for some time duration.

(ii) Variation of the action w.r.t. endpoints gives the conjugate momentum ($\delta S/\delta g_1 = \pi_1$) so $\delta S/\delta g_1 = 0 \Rightarrow \pi_1 \Rightarrow 0$, i.e. the initial momentum is determined, so the uncertainty principle implies g_1 is undetermined, i.e. under this condition we do not have a wave packet!

(iii) $\delta S/\delta_{(\text{parameter})} = 0$ is precisely the condition that S does not depend strongly on the parameters (since its variation is small, i.e. zero to 1st order).

(iv) The specification of what are to constitute appropriate parameters is lacking. After all, one can introduce an arbitrary number of parameters with particular values in any given solution, and this can be done in such a way as to maximise the action for the chosen solution. I can see that the intention of the discussion here is clear, but that is only acceptable in an informal presentation. The formal "proofs" of theorems is then out of place.

I would make the following suggestion as to how this idea might be revised. Suppose $\Psi\{g\}$ is determined by some functional Schrödinger equation, say the Hamiltonian constraint in general relativity

$$\frac{\partial\Psi\{g\}}{\partial t} = H\{g, \frac{\partial}{\partial g}\}\Psi .$$

One can think of H as having the form $\sim \pi^2 + V\{g\}$. In that case quantum effects manifest themselves only when the WKB solution breaks down, and this depends on the form of $V\{g\}$, near particular solutions $\tilde{g}$. One could then investigate whether or not this would occur if $\tilde{g}$ were determined by matter in some way (e.g. by looking at particular examples).

Info. only. Paper rejected earlier.

3

The Quantum Basis for Mach's Principle

By Michael Jones

Referee Report

The author tries to arrive at the Mach principle through a classical limit to quantum geometrodynamics. Unfortunately, his technical treatment is totally inadequate to the subtleties of the subject. Let me mention just several examples: The quantum propagator in compact spaces should not refer to a time parameter. The gauge problem in the path integrals (the same geometry can be expressed through different metrics) is disregarded. To assume that Eqs. (A5), (A6) have unique solutions sidesteps the sandwich problem which clearly has a bearing on the author's formulation, but is nowhere discussed in the paper. The question how to separate the prarmeters which fix the matter field from those which specify the geometry is exceedingly tricky. Black holes are not solutions to the Einstein equations with the energy-momentum tensor having the δ-function singularity |Eq. (C1)|.

I cannot recommend the paper for publication.

31.4 Afterthoughts - 2008

I need to be more careful in future manuscripts.

31.5 Afterthoughts - 2009

Newton noticed that his laws of motion applied in a frame that does not rotate relative to the "fixed stars," and concluded the existence of an "absolute space" with the stars fixed in that absolute space (what we now refer to as an "inertial frame").

Mach denied the existence of absolute space, pointing out that only relative motions are observable. Instead of absolute space, he postulated that inertia is caused by an interaction with the rest of the universe. He suggested that what we now refer to as an inertial frame is caused by the rest of matter in the universe (now referred to as frame dragging).

Einstein based his General Relativity partly on Mach's ideas, and coined the term "Mach's principle" for the idea that inertial frames should be determined by the matter distribution. The term "Mach's principle" now means many things to many people, and there is no general agreement on what it means or on the validity of the various versions.

Although Einstein's General Relativity does include frame dragging, frame dragging is not complete. There are many solutions of Einstein's field equations in which there is inertia in the absence of matter or where there is rotation of inertial frames relative to the average matter distribution. This is because initial and boundary conditions also contribute to inertia.

Some researchers have proposed that boundary and initial conditions be chosen so that inertia and/or the metric be determined by the matter distribution. Some researchers have proposed that Mach's principle be used as a selection principle to discard solutions that have relative rotation of matter and inertial frames or have inertia without matter.

I have argued that if the metric (gravitation) is determined solely by sources (the matter), with no dependence on initial or boundary conditions, then gravitation would be very different from electromagnetic theory, in which the electromagnetic field has its own degrees of freedom, and can have arbitrary initial and boundary conditions. Thus, I have rejected Mach's principle in such a restricted form.

That leaves us still with the problem of explaining why we observe no relative rotation of inertial frames and the visible matter in the universe. I suggested that if we consider quantum gravity, that we could allow all of the solutions to the field equations, and that the non-physical solutions (those with relative rotation of inertial frames and matter) would cancel each other out if we look at the change in the action with variation of the initial and boundary conditions on the gravitational field.

It was at this point that I made a mistake in the present manuscript. I suggested that we vary the initial and boundary conditions on the gravitational field while holding the matter distribution constant. It was pointed out that there is no general procedure for doing that because it is not possible to specify the matter distribution independent of the geometry.

What I should do instead (and what I actually did in my calculations) is consider a family of solutions to the field equations (such as one of the Bianchi cosmologies) in which there is an initial condition that determines the amount of relative rotation of inertial frames and matter, and consider the change in action as that parameter is varied. In this way, we have no problems of the kind I mentioned above. Then, a saddlepoint approximation to the path integral for that parameter will give the solution for zero relative rotation.

Chapter 32

Measuring a quantum geometry[1]

32.1 Summary

Because any experim

32.2 Introduction

Suppose we have calculated a wave function

$$\psi(\phi, g, t) \tag{32.1}$$

over matter fields ϕ and 3-georaetries g as a function of time t for the universe. We wish to verify our calculations by direct measurement. We believe we understand the meaning of the wave function over the matter fields in terms of measurement[2] and therefore we will not be concerned with that aspect. Instead, we will consider the meaning of measuring the wave function over 3-geometries because it is considerably less obvious what that means, or how to go about it.

That (32.1) is a wave function for the entire universe, including the experimenter and his measuring equipment, is significant in the measuring process.

[1]Draft 15 June 1981. The summary seems a little short.

[2]Although we may disagree on the philosophical interpretation

There is the possibility that it is not correct to represent the state of the universe as a universal wave function as in (32.1). For the purposes of this discussion, I will ignore that possibility (except for an extension to density matrices). I will consider the consequences that (32.1) correctly represents the state of the universe.

I will begin by considering how we measure a classical geometry.

32.3 Experiments to measure the geometry

When we make a classical measurement of the geometry, we measure the 4-geometry, not the time variation of a 3-geometry. Although all of our measurements have been made in the solar system (and will probably continue to be made in the solar system for some time), it is convenient to divide our measurements into local or global measurements according to whether they give us information about the geometry inside or outside of the solar system.

Most of our measurements of the local geometry within the solar system consist of observing the trajectories of bodies or particles in gravitational fields in the absence of significant non-gravitational fields or in the presence of known non-gravitational fields. We usually use visible light to track the paths of large bodies such as metal spheres or planets. The final measurement may consist of the relative positions of dots on a photographic film. In all cases, the measurement consists of relative positions of some material body used as a tracer.

Part of our measurements concerns measuring time. We measure time with clocks, and it is usual to define a clock as any mechanism that returns to the same configuration often enough that counting repeats of that configuration gives us a measure (and a definition) of time. Some clocks are better than others (in that some physical laws look simpler in terms of time defined by good clocks), but useful clocks are as varied as the rotating earth, a water clock, a pendulum clock, and an atomic clock. In all cases, however, the measurement of time consists in counting the return of a physical system to the same (or nearly the same) configuration. To measure time, it is necessary (and sufficient) to measure the relative positions (or states) of matter[3].

When we track the trajectories of particles, we may use bubble chambers or scintillation counters. Although our purpose in such measurements is for other things, such tracks give us information (within some error) about the relation of the measurement apparatus to local inertial frames. Although we have more accurate ways to measure local inertial frames, we may still consider this type of measurement to be a "measurement of the geometry". As with the measurement of the trajectories of large bodies, all such measurements reduce to the measurement of the relative positions (or configurations, or states) of matter[4].

There are ways to measure the local geometry other than by tracking trajectories. The gravitational red shift of a spectral line in the Earth's gravitational field measures some aspect of the geometry. There are many experiments that give information about the local geometry. Every one of them involves measuring the relative position (or configuration, or state) of matter[5]. There are no exceptions.

The situation is much the same when we measure the geometry outside of our solar system. Astronomers measure the relative positions of spots on photographs. Measures of the cosmological red shift is in terms of the relative wavelengths of spectral lines. We measure electromagnetic radiation coming from outside the solar system at various frequencies, including visible, radio, microwave, and X-ray. We measure particles coming into the solar system. All of the information we have about the universe outside of the solar system is from matter[6] that comes into the solar system. From

[3] By "matter", I mean anything but geometry or gravitation.

[4] See previous footnote.

[5] See previous footnote.

[6] See previous footnote.

our measurements of the properties of that matter, we infer through often complicated calculations and models, which cosmological models for our universe are consistent with our measurements, and which are not. This is the process by which we measure the geometry of the universe outside of the solar system. In every case, the measurement itself can be reduced to that of measuring the relative position (or configuration, or state) of matter. Again, there are no exceptions.

When we turn from the process of measuring a classical geometry to that of measuring a quantum geometry, we find the same situation. What do we have available to us to measure a quantum geometry? Again, we have only matter. All measurements of a quantum geometry consist in measuring the relative states of matter. There are no direct measurements of geometry whether it is classical or quantum.[8]

I have so far argued that only matter is directly observable, that all measurements of the geometry are indirect, and are actually measurements on matter. Not all aspects of matter are directly observable. We will try to see which aspects of matter are observable.

Many measurements require time. For example, when we measure the distance to an object, we can send out a radar pulse, and see how long it takes it to return. We assume a clock to be essentially coincident with the radar. The measurement consists of two parts.

1. The reading on the clock is recorded when the pulse is sent out.

2. The reading on the clock is recorded when the returning pulse is received.

Reading the time on a clock consists of noticing which pieces of matter are coincident. On a simple analog clock, we read coincidence of the hour hand and part of the dial and the same for the minute and second hands. That there is a gap between the hands and the dial leads to a possible error in measurement. Such an error is inherent in any measuring device because it is impossible for two pieces of matter to be in the same place at the same time. On a digital clock, the coincidence is read internally. The digital readout is nothing more than a readout. The measurement is made internally and is in principle the recording of the near coincidence of two pieces of matter.

Sending the pulse corresponds to the coincidence of two pieces of matter (the electrons that make up the pulse of current in a particular part of some particular wire on their way to the transmitting antenna). Receiving the pulse corresponds to a similar coincidence of a pulse of electrons in the current pulse coming from the receiving antenna and a particular part of some particular wire.

For the purpose of analysis, we may idealise these pulse measurements as measuring the near coincidence of some particular electron (which we shall call the pulse) and some particular atom (which we shall call the gate) in a piece of wire. We may idealize the clock measurements as measuring the near coincidence of some chosen particle (which we shall call the hand) in the clock with some other particle (which we shall call the position on the dial).

Thus, we may redescribe the above radar measurement as the following idealization.

1. The particle (that gives the position on the dial) with which the hand of the clock is in nearest coincidence is recorded when the transmitted pulse is in near coincidence with the transmitting gate.

2. The particle (that gives the position on the dial) with which the hand of the clock is in nearest coincidence is recorded when the received pulse is in near coincidence with the receiving gate.

Exact simultaneity in the above two measurements requires that the pulse and the clock be coincident. To the extent that this is impossible, this leads to an inherent error in any such measurement.

[8]Stephen Hawking (private communication) agrees with me on this point.

The first description of the radar measurement was a shorthand description for the longer and more complicated one above. All measurements can be analysed in this way, and they will all be found to have the same common features.

1. Any measurement consists of a certain number of events (near coincidences in space-time). The above radar measurement consists of two events at the same point in space at different times.

2. Each event is a near coincidence of two or more particles. Each event in the above radar measurement consists of a near coincidence of four particles. Two of these four particles (the gate and the clock hand) were always assumed to be in coincidence, but that was never measured.

Careful reflection will show that all measurements can be described by the above model. We can call each event in such a measurement a primitive measurement. Thus, any measurement consists of some appropriate set of primitive measurements. From here on, I will use the term measurement to refer to a primitive measurement, because a primitive measurement forms the building block for any compound measurement, and we need only consider the properties of primitive measurements. Thus, a measurement always consists of a near coincidence of two or more particles.

Thus, we may consider that the only observable is the near coincidence of two or more particles.

32.4 Integrating out the geometrical parameters

We now return to the experimental verification of some particular calculation for the wave function for the universe (32.1), in particular, the wave function over 3-geometries. I have just argued in the previous section that there are no direct ways to measure the geometry. That all measurements of the geometry are indirect in that they involve measurements of the relative configuration (or state) of matter[9].

We still have the choice, however, to design specific experiments (which we may wish to standardize) that measure specific features of the geometry (even though indirectly, through measurements on matter), and define these to be "measurements of the geometry." I am just as free to analyze those same measurement processes in terms of measurements on matter. There will be no disagreement on the meaning of the measurements, only the procedure for interpreting the measurements.

I, thus, take the point of view that (32.1) does not represent a quantity that can be measured directly. Before it can be considered to be a measurable quantity, we must first remove (by some correct procedure, of course) the direct reference to the 3-geometry g in the wave function. I wish to repeat that there is no question about whether that procedure must be done, but only at what step it is done. I prefer doing it at the outset to make a point; however it would be equivalent to do that procedure as part of the description of the measuring process.

The next two sections will consider possible procedures to remove the 3-geometry g from the wave function.

32.5 Incoherent integration

Normally, when we have a wave function of several variables or parameters in a Hilbert space, and we measure only some of the variables, then we simply integrate over the unmeasured variables to

[9]See footnote 2.

give a probability density for measuring the remaining parameters. Thus

$$P(\phi, t) = \int \psi^*(\phi, g, t)\psi(\phi, g, t)D(g) \tag{32.2}$$

would give the probability density for the matter fields. $D(g)$ represents some measure over the space of 3-geometries.

The ϕ remaining in (32.2) include our measurement apparatus and all the rest of the matter in the universe. To complete the calculation corresponding to what is actually measured in the experiment, we would need to further integrate out the other matter variables that have nothing to do with the experiment. That process is reasonably straight forward, and does not concern us here.

What does concern us here is whether (32.2) is correct. I claim that it is not, for the following reasons.

1. ϕ and g do not represent independent parameters in a Hilbert space.

2. It is not even possible to specify ϕ without specifying the 3-geometry g.

32.6 Coherent integration

We need to convert the wave function in (32.1) into a wave function over observables. Section 32.3 showed that a 3-geometry g is not a direct observable. Also, a matter distribution ϕ is not expressible independent of the 3-geotnetry, so it is not a direct observable.

Because the matter distribution ϕ includes all of the matter in the universe, including the experimenter and his apparatus, whether or not a particular quantity is an observable will depend on whether the appropriate apparatus is included in ϕ. To avoid the philosophical discussion that such considerations might lead to, we will consider potential observables. That is, we will consider all quantities that might be observables. Some of these might not actually have the possibility of being observed, but at least we will include all quantities that we might be able to observe. Thus, our list of potential observables will include all observables, but it might include more.

I can't actually define a potential observable, but I can give enough examples that give the general idea. Basically, a potential observable is any invariant (coordinate-independent) quantity that we can form from the 3-geometry g and the matter field ϕ. Thus, if the mass distribution consist of a collection of particles, then the potential observables would include the distances between all pairs of particles. Not all such distances would be actually observable. For example, the distance between two arbitrary stars is not directly measurable by us. But the distance between two spots on a photographic plate made by an astronomer through a telescope aimed at the sky is an observable. The measurement of a spectrum can be reduced to the measurement of distance on a photograph, and such an[11]

Let us begin by forming all of the possible invariant functions from the matter distribution and the 3-geometry. Three of these invariant functions (which may not be functionally independent) characterize the 3-geometry, and we may take them to be [18, Weinberg, 1972, page 145] R, $R_{\mu\nu}R^{\mu\nu}$, and $|R|/|g|$. The rest characterize the matter distribution. For example, for a collection of point particles, the distances between all pairs of particles would be invariants that characterize the matter distribution.

[11] This part of the manuscript seems to have ended here. There may be a page missing.

32.7 Effect of density matrices on the quantum basis for Mach's principle

With a density matrix, we have various probabilities for various wave functions. My analysis applies to each of these wave functions. Thus, each wave function yields the same Machian cosmology. Thus, my calculation is extended.

Chapter 33

Analysis of the measurement process in remote sensing[1]

1. Summary
2. Introduction
3. The ruler illustrates the significant aspects of any measurement
4. In-situ measurements
5. Passive remote sensing
6. Active remote sensing
7. The independence of an ideal measurement on coordinate systems -What constitutes a good measurement?
8. References

33.1 Introduction

A remote sensing measurement consists of three parts.

1. The interpretation and modeling of the propagation to and from and the interaction with the object being sensed
2. The experimental apparatus that produces a measurement
3. The measurement process itself

Of the three, the first part is the most critical in any remote sensing measurement because it involves the most unknowns. Appropriately, most of the concern in a remote sensing measurement is with that part. The experimental apparatus is next in importance, and appropriately is usually given major consideration.

This report concerns the measurement process itself. While it is rarely the determining factor in the accuracy of a remote sensing measurement, it is only through understanding of its nature that it remain so.

[1]Draft 17 June 1981

While the accuracy of the measurement process depends on the design of the experimental apparatus, the nature of the measurement process is always the same. This report shows that the essential feature in all ideal measurements is the superposition of two objects at the same place and time. An ideal measurement is impossible because two particles cannot occupy the same position at the same time. A practical measurement involves bringing two particles into close proximity. The distance between the two particles during the measurement gives the limiting error in any classical measurement.

33.2 The ruler illustrates the significant aspects on any measurement

When I use a ruler to measure the length of a steel bar, I lay the ruler on the bar, align the end of the ruler with the end of the bar, and locate the position on the ruler adjacent to the other end of the bar. The measurement consists in bringing two points on the bar into correspondence with two points on the ruler. Any error in that correspondence results in an error in measurement.

Unless a point on the ruler occupies the same position at the same time as a point on the bar, there will be an ambiguity in associating a point on the ruler with a point on the bar. This can be easily seen with a thick ruler because the length measurement clearly depends on the angle with which we sight across the ruler. Even when the ruler is thin, the ambiguity exists, but is less. Only when a point of the ruler coincides with a point on the bar is the association of a point on ruler with a point on the bar independent of the geometry of space-time, strong electromagnetic, gravitational, or other fields, and the details with which that association is made. The impossibility of making a point of the ruler coincide with a point of the bar shows that such an ambiguity always exists in any real measurement.

A good (but not ideal) measurement results when a point of the ruler is very near a point of the bar. We may consider that the result of a good measurement is to associate one particle (say the nucleus of an atom) of the ruler with one particle of the bar. To the extent that such an association is only weakly dependent on the way that association is made is a measure of how good the measurement is.

If we were to attempt to make a particle on the ruler approach closer to a particle on the bar than is practical, the particle on the ruler would interact with the particles on the bar and alter the binding between adjacent particles on the bar, thereby changing the spacing in the crystalline structure, and changing the length we are trying to measure. While this shows a relationship with the quantum uncertainty principle, the effect we are considering is purely classical.

In the following sections I will give examples to illustrate that all measurements fit this general model. That is, that any ideal measurement involves making two particles coincide. A good measurement consists in bringing two particles close enough that the association of one with the other has small ambiguity.

Other aspects of error associated with the measurement, such as the changing relative lengths of the bar and ruler with temperature are part of the operation of the ruler as a measuring apparatus, and are not considered here.

More accurate length measurements are made with calipers. Their operation is more complicated than that of a ruler, but an analysis shows that the measurement process is the same as that of the ruler. Very accurate length measurements are made with lasers. That is a remote sensing measurement. We will see that the measurement process is still the same.

33.3 In-situ measurements

In remote sensing, the actual experimental apparatus that produces a measurement does so by making in-situ measurements. For example, a radar measurement consists in part of an in-situ measurement of a radio-wave pulse at a receiving antenna. Thus, an analysis of in-situ measurements is an analysis of the experimental apparatus that produces a remote sensing measurement.

We will consider several in-situ measuring devices, and show in each case that the measurement process in the end is the same as it was with the ruler.

33.4 Remarks - 2008

Apparently, I did not finish this manuscript.

Chapter 34

The nature of measurement[1]

abstract

Any experiment that makes a physical measurement can be analyzed so that the measurement proper consists of two physical associations of two or more pieces of matter or particles. A reading of the measurement consists of translating the physical association into a verbal association of the particles.

The extent to which the reading is unambiguous depends on how close the experiment brings the particles to make the physical association.

The measurement error resulting from such a reading ambiguity is intrinsic to any classical measurement, and is in addition to errors resulting from uncertainties in the model of the physical processes and from the quantum uncertainties.

34.1 Summary

Any experiment that makes a physical measurement consists of 5 parts.

1. A physical system upon which a measurement is to be made
2. A second physical system called the apparatus that interacts with the first system causing a measurement to be made
3. A model used to interpret the physical processes taking place in the combined physical system
4. A measurement
5. A reading of the measurement

I argue by example that such an experiment can always be analyzed so that the measurement always has the same simple form. That is, the measurement always consists of two physical associations of two or more pieces of matter or particles. A reading of the measurement consists of translating the physical association into a verbal association of the particles. The extent to which the reading is unambiguous depends on how close the experiment brings the particles to make the physical association. A completely unambiguous reading would require a physical association in which the particles are at the same place at the same time. Because it is impossible for two or more particles to occupy the same position at the same time, a reading will always have some ambiguity. This is an ultimate limit on the unambiguity of any classical measurement. It is in addition to

[1]draft 20 June 1981

any ambiguity resulting from uncertainties in the model of the physical process and the quantum uncertainty product.

To reduce the effect of this ambiguity the particles to be associated physically should be brought as close together as is reasonable without causing the other ambiguities in the experiment to increase. The reading should be insensitive to influences that are not directly measurable.

While the determining factor in the final accuracy of a remote sensing measurement is usually related to unknowns in the physical model of the process, it is advisable to estimate the error resulting from measurement reading ambiguity.

34.2 Measurement

1. Calculate the amplitude that two given non-interacting particles will have world lines that intersect at a given time.

2. Consider a measurement device that tries to associate a given particle (called the pointer or hand) with some other particle among a set of particles (called the scale or face). Consider the amplitude that the device will associate the pointer with some particular particle on the scale.

3. Consider a device (essentially an analog to digital converter) that prints out a digital measurement for some quantity being measured. Consider the amplitude that the device will print out some specified number for that measurement.

4. Consider a person reading a pointer of some measurement meter. Consider the amplitude that the person will read and write down some specified number for that reading. Some classical probability (and therefore density matrices) will probably enter into this calculation.

Classical probability (and therefore density matrices) may enter into any of the above calculations.

34.3 Eddington on pointer readings

As Eddington [52, 1928, pages 252, 253] correctly points out, any measurement on a physical system can be reduced to a collection of pointer readings or their equivalent.[2] He idealizes pointer readings to the intersection of world lines of two particles. (Of course, it is impossible for the world lines of two particles to actually intersect, because two particles cannot occupy the same place at the same time. This fact leads to an inherent ambiguity in the reading of any measurement, even at the classical level.)

The purpose of any experimental apparatus, then, is to cause appropriate pairs of particles to come close enough to each other that a nearly unambiguous reading can be made of the close physical association of the two particles. I thus define a measurement as the bringing of two particles into close proximity. I define the reading of the measurement to be the translation of the physical association of the two particles into a verbal association (particle A was close to particle B, the needle on the voltmeter was adjacent to 2.35 on the scale, the left end of the ruler was adjacent to bright spot number 1 on the photographic plate at the same time that the 2.35 cm mark on the ruler was adjacent to bright spot number 2 on the photographic plate, the second hand on the clock was adjacent to the 11 second mark on the clock face, etc.), whether it is done manually, or automatically. (The above remarks could have been written into an experimental log book by hand after a manual reading, or printed out on a computer terminal after an automatic scan.) Of

[2]Margenau's [110, 1959, page 165] criticism of Eddington's point of view indicates that he failed to understand the generality of Eddington's observation.

course the details of the measuring process may be more complicated or sophisticated, but they may always be analyzed as I have said. There is some ambiguity in the separation of the act of measurement from the act of reading the measurement. However, it is always possible to separate the two in the way I have said. We could, for example, consider the measurement to consist of sentences printed out on a computer printer, and the reading of the measurement to be simply the reading of the message by a person. The measurement still consists in bringing particles (the ink making up the letters and the paper) in near coincidence.

34.4 Notes from Dick Grubb about analog to digital converters

26 June 1981

All A/D converters make a comparison of the unknown voltage to a standard voltage, but in some of them the comparison is indirect. There are three main types.

The most accurate and fastest is called a tricycle successive approximation converter. We begin with a known standard, which is then broken down in powers of two with voltage dividers to make secondary standards. Sums of the secondary standards are then found which approximate the unknown voltage. The on or off value of each secondary standard in the binary succession then gives the binary value of the voltage. The successive approximation processes is the same as when we convert a decimal number to binary. A faster version has all possible combinations of secondary standards already made up.

A low speed, low precision A/D converter uses the unknown voltage to charge up a capacitor. The time it takes to discharge the capacitor through a known resistor is measured by counting cycles of a known frequency. There are offset errors.

An intermediate A/D converter is the joule slope converter. It also measures the time to discharge a capacitor through a known resistor, but this time, a comparison is made to find the known digitally determined voltage that the capacitor can be charged with and will discharge in the same time as with the unknown voltage. (I think I have that right, but it sounds very slow.) (gives 3-5 decimal digits.)

Chapter 35

Measurement from within a closed quantum system[1]

1. Summary
2. Introduction - Special problems occur when the measurement apparatus is part of the quantum system
3. Does the wave function for the total system predict the outcome of all possible measurements made from within the system?
4. Expressing the wave function in terms of primitive observables
5.

Quantum theory revolutionized measurement because it required all measurements to disturb the state of the system. Even so, it seems that it was only one step toward a real understanding of measurement. When we consider measurement of a quantum system, we normally consider a closed system that evolves according to Schroedinger's (or Dirac's, etc.) equation between measurements, but whose wave function jumps discontinuously at the time of measurement. While the philosophical description of this process leaves a little to be desired, the predictions of the theory as to the results of such measurements are unambiguous and agree with measurement.

Such systems need not be closed. For example, the standard theory still applies if we apply external potentials (that may be time dependent so long as these external potentials suffer no back reaction from the system. Thus, there is no possibility for such interactions to make any measurement on the system.

The normal way to consider measurement is to consider that we have two interacting systems: one called the apparatus, and the other called simply the system. The types of interactions between the two is usually idealized in various ways. However, suppose we want to consider measurements or observations that we make on the universe. We and our apparatus are part of the universe. The question is this. suppose we have calculated a wave function for the universe (including our apparatus and us). What is the appropriate way to calculate from that wave function the quantities that correspond to observables. In particular, when we integrate over the parameters in the wave function that are not observed, which parameters should be integrated coherently, and which should be integrated incoherently?

The simple answer is that those parameters that are not capable of observation should, be integrated coherently (as interfering alternatives), while those that are capable of being observed should be integrated incoherently (as exclusive alternatives).

[1] July 1981

The problem reduces then to deciding which parameters can be directly observed, and which cannot.

The standard procedure of setting up operators in a Hilbert space does not obviously apply here, because we and our apparatus are part of the system.

I prefer at this point to consider the problem from a physical rather than a mathematical viewpoint. I should point out here, that I am considering quantum gravity (or perhaps I should say quantum cosmology). Thus, I consider for my wave function for the universe, a wave function over 3-geometries and matter fields. That is the wave function I want to reduce down to a wave function over observables, by (coherently) integrating out all of the parameters that are not observable. Then it is straightforward to integrate out (incoherently) the parameters that are not observed..

When I say a physical viewpoint, I mean that I do not consider position, or momentum, etc. to be observables. I would rather consider (as Eddington did) something more direct such as pointer readings. It is tempting to try to reduce all observables to pointer readings or their equivalent. (Eddington idealized this to the intersection of world lines.) When we read time, we are making a logical association of a clock hand with a particular part of the clock face from the close physical association of the two. If the two actually coincided (the intersection of two world lines), then there would be no ambiguity in the logical association. Laying a ruler on a photograph to measure the distance between two stars can be analyzed in the same way. Modern measurement are usually done by converting some physical parameter to a voltage, which is then recorded by an analog-to-digital converter. Measurement within an analog-to-digital converter can be reduced to observing whether a current (possibly a pulse) does or does not flow through some particular part of a circuit. The measurement thus reduces to coincidence of one particle (a pulse) with another (a particular part of an electrical circuit).

If you think carefully, you will see that all measurements can be characterized by this idealization of observing the intersection of world lines. Thus, in integrating our wave function, we should integrate coherently over all parameters that do not correspond to the intersection of world lines of particles, we would integrate over all parameters describing the 3-geometry, for example, and "many more. We would end up with a wave function that is simply the amplitude for the intersection of world lines. This is particularly appealing, because the intersection of world lines is something that is independent of the geometry. Other properties of matter fields require the specification of the geometry as a background to make sense.

There is a difficulty, however. It is never possible for two world lines of particles to ever intersect because the two particles will interact. I don't know how to get around this objection.

I thought of one possibility. Instead of calculating the amplitude for the intersection of world lines, we calculate the amplitude for measurement apparatuses to print out various numbers after processing on a computer. It seems, though, that such a system is not universal enough to represent all possible kinds of measurements.

The intersection-of-world-lines idea was at least universal. As far as I can tell, all measurements can be idealized that way. In practice, however, the world lines never actually intersect. They simply get close enough that we can turn the close physical association into a logical association. However, as soon as we allow the world lines to be merely close, we have to define close, and then we have to bring in the metric, and that means to bring in the 3-geometry. Maybe we can simply bring in interaction rather than a metric distance. Suppose we say that two particles can be considered to be logically associated when their interaction is large. (That is, particle A (the pointer) is associated with particle B (a part of the dial) when particle A has a larger interaction with B than with any other particle.) But this doesn't look like pointer readings anymore. We wouldn't want to require that the hand on a clock has large interaction with a particular particle on the clock face. How do we get around this?

35.1 Comments and ideas from Dave Peterson, 21 July 1981

1. Not all intersecting world lines are observable, only when a real interaction occurs.
2. Do we need to specify a threshold for such interactions?

35.2 Response

Given a wave function for the whole universe over 3-geometries and matter fields at time t.

$\psi(g, \phi, t)$

Transform to a new basis.

$\psi(\alpha, \beta, t)$

where

α represents intersections of pairs of world lines of real particles, and β represents everything else. β also includes intersections of world lines in which at least one of the particles is a virtual particles. We can think of α as representing real interactions, and β as representing virtual interactions. α would include real photons and gravitons (as radiation fields). β would include virtual photons and gravitons (as terms in a perturbation expansion of an interaction). Thus, electromagnetic fields (other than radiation fields) and gravitational fields (other than radiation fields) would be included in β.

Basically, α can be directly observed, while β cannot. Thus, the βs are interfering alternatives (because they cannot be observed). Thus, the only part of the wave function that deals with observables is

$\psi_0(\alpha, t) = \int \psi(\alpha, \beta, t) d\beta$.

If only some of the αs are observed in a measurement, then

$P(\alpha_0) = \int \psi_0^*(\alpha, t)\psi_0(\alpha, t) d\alpha_n (n \neq 0)$.

gives the probability of observing α_0. If α_0 is an interaction, is there circular reasoning here, because we partly used interaction to define the difference between α and β?

Chapter 36

The relation between measurement in a quantum cosmology and a universal wave function[1]

1. Summary
2. Introduction
3. The universal wave function in a quantum cosmology
4. Measuring a quantum geometry
5. Calculating the amplitudes for observables
6. What are primitive observables?
7. Inelastic interactions
8. Non-Hermitian interactions
9. Extension to density matrices

36.1 Introduction

I should first explain what I mean by the title. By quantum cosmology, I mean quantum gravity or quantum geometry, but for the case where we consider the whole cosmology to have a quantum nature. Thus, I am not considering only local quantum fluctuations around a classical Minkowski background. By a universal wave function, I mean a wave function over 3-geometries and matter fields as a function of time that includes the whole cosmology including all experimental apparatus and all of the experimenters. By,"measurement in a quantum geometry", I mean that the experimenter is restricted to measurements that he can make from within the system. There are no external observers, nor external systems. By, "relation between measurement in a quantum cosmology and universal wave function", I mean, if we know the wave function, can we calculate the amplitudes for all possible measurements that we make? Answering the latter question is the purpose of this paper.

There are three difficulties with this undertaking.

[1] 25 July 1981

First. When there is no background geometry, there is some difficulty in uniquely specifying, understanding, and operating with a wave function,

Second. If there is no external observer so that all measurements must be made from within the system, some of the standard operations in quantum theory must be modified.

Third. As in ordinary quantum theory, there is an ambiguity about what is meant by a measurement. While that ambiguity can be tolerated in standard quantum theory because the predicted amplitudes for measurements is sufficiently independent of that ambiguity, the situation here is a little more difficult because of the difficulty in defining the Hilbert space. That is, in defining a complete and independent set of dynamic variables.

36.2 The universal wave function in a quantum cosmology

Hawking (1979)[111] uses the path-integral approach to calculate the amplitude

$$< g, \phi, t | g_1, \phi_1, t_1 > \tag{36.1}$$

to go from a state with a metric g_1, and matter fields ϕ_1 on a space-like hypersurface t_1 to a state with a metric g, and matter fields ϕ on a spacelike hypersurface t. We can consider t_1 and t to represent time. For a given initial state at time t_1, we can use (36.1) to give the amplitude

$$\psi(g, \phi, t) \tag{36.2}$$

for a state with metric g and matter fields ϕ at time t. We can consider (36.2) to be a wave function for the universe at time t.

Notice that the formulation in (36.1) and (36.2) seems to take care of the problem of not having a fixed background geometry on which to formulate quantum theory. That is, (36.2) gives the amplitude that the metric is g and that the matter fields specified on that metric are ϕ.

Because (36.2) includes all of the matter fields, it includes all of the experimental apparatus for measurements that can be made about the universe at time t. We may not yet know how to correctly calculate (36.2). We can assume that will eventually be solved. The problem considered here is given (36.2) and assuming it to be a complete description of the universe at time t, how do we calculate the amplitudes for measurements that are made at time t?

36.3 Measuring a quantum geometry

Does the wave function in (36.2) relate almost directly with measurable quantities as does a normal wave function? In particular, is the 3-geometry g directly measurable? Let us consider how we measure the geometry in practice.

The solar system demonstrates one way in which we measure geometry. (That is, a way in which we measure a gravitational field.) Most of our information about the geometry (or gravitational field) of the solar system comes from observing the positions of the planets. The geometry was inferred from the measurements through a theoretical model, first Newton's laws, and later Einstein's equations. (In practice, the geometry and the theoretical models were jointly constructed to satisfy the observed motions of the planets.)

A second way we measure geometry in the solar system is from the bending of star light as it passes the sun during an eclipse.

We also measure local gravitational fields on the earth. We do this in two ways, either by measuring the trajectories of macroscopic bodies or elementary particles, or by comparing gravitational forces with electrical forces with a spring scale or its equivalent. In the case of the trajectories of elementary particles, we are simply affirming that the geometry is locally MinKowski.

All of the above measurements of the geometry are indirect, in that they rely on measurements with matter. We thus suspect that the wave function in (2) is not directly verifiable, but rather some related wave function whose dynamical variables do not include the geometry.

We measure a quantum geometry in the same way in which we measure a classical geometry. The difference is that we may get different results. In either case, we make calculations (based on a classical or quantum model for gravitation) to predict the results of a measurement. If the predictions differ significantly, the measurement will decide the correct model.

There is one aspect of the geometry (or gravitational field) that I have not yet discussed, and which may not always fit the above pattern. It is gravitational radiation. Let us consider two cases. First, when many gravitons are present. The resulting gravitational field is macroscopic and classical. The interaction with matter fits the above pattern, and the present and planned methods to measure gravitational radiation can be analyzed in a manner similar to that above.

Second, when only one graviton is present. Our estimates of the interaction with a single graviton yield tremendously small values. However, a high frequency graviton would have as much energy as a photon, and if an inelastic interaction with a graviton existed that was strong enough, we might be able to detect gravitons directly. The situation seems similar to that for neutrinos, which we can now detect. Thus, it may in principle be possible to detect individual gravitons directly.

Except for the slight (in principle) possibility of detecting gravitons directly, all other measurements of the geometry are indirect.

36.4 Calculating the amplitudes for observables

We have seen that (with the possible exception of individual radiation gravitons) the geometry is not measurable directly, but only through certain aspects of the matter fields. This indicates that many of the dynamical variables (in particular, those for the metric) can somehow be gotten rid of without losing predictive ability for the outcome of measurement. Let us consider how we can do that.

Suppose we determine what the measurable dynamical variables are. Call them α. (Section 36.5 will propose such a set of dynamical variables.) Let us now transform the wave function in (36.2) to a new basis as follows

$$\psi(g, \phi, t) \to \psi(\alpha, \beta, t) \tag{36.3}$$

where α are the measurable dynamical variables, and β are the other dynamical variables necessary to complete the basis to make it equivalent to g, ϕ. By hypothesis, α include all of the dynamical variables that can be directly observed. Thus, none of the variables included in β can be directly observed. Because the β cannot be directly observed, they correspond to interfering alternatives [112, Feynman and Hibbs, 1965, page 13]. Thus, to calculate the amplitude for the dynamical observables, we must include the contributions for all values of the unobservable dynamical variables. Because the β correspond to interfering alternatives, the integration is coherent. Thus,

$$\psi(\alpha, t) = \int \psi(\alpha, \beta, t) D[\beta] \tag{36.4}$$

gives the amplitude for measuring the dynamical variables α, where $D[\beta]$ is the measure on the space of all β.

36.5 What are primitive observables?

In this section, I will propose a definition for primitive observables. I don't claim that my definition is the only one that would work, nor even the best one. However, it seems like a good one, and it does seem to satisfy the following requirements:

1. The definition is unambiguous.

2. In a complete basis, the observable dynamical variables correspond to interactions that are exclusive alternatives [112, Feynman and Hibbs, 1965, page 14], and they include all of the interactions that are exclusive alternatives.

3. In such a basis, all other dynamical variables correspond to interactions that are interfering alternatives [112, Feynman and Hibbs, 1965, page 13].

Let us start by considering in more detail the measurements of the positions of the planets. The positions of the planets were first observed with the naked eye, later with mechanical telescopes, then with optical telescopes, finally with photographic emulsions combined with optical telescopes. In all cases, the observation consists of an interaction of a photon with either, part of an eye retina or part of a photographic emulsion. In both cases, the measurement consists of an inelastic interaction of a photon with one or more particles.

The bending of starlight as it passes the sun during an eclipse is analyzed the same way, and leads to the same conclusion.

For trajectories of macroscopic bodies, we usually observe these with light. The analysis is the same as for the planets, and leads to the same conclusion.

For the trajectories of particles such as electrons, we use cloud chambers, bubble chambers, photographic emulsions, or particle counters. In the cloud chamber, the electron ionizes atoms along its path. We observe the track of ions because a supersaturated vapor condenses as drops on the ions. The interaction that allowed the observation is the ionization of the atom by the electron. Once that has occurred, the electron has left its mark, and we can observe the ion in relative leisure. The ionization consists of inelastic interaction of an electron with an atom. The measurement here fits the same pattern as that of photons with an eye retina or photographic emulsion, that of an inelastic interaction of one particle .with another. What we actually observe, is not the track of the electron, but rather a track of inelastic interactions.

In a bubble chamber, the electron also ionizes atoms along its path. The method we use to make the ions observable is slightly different, but not essential to this analysis. Again, the essential interaction is the ionization of the atom by the electron. The analysis is the same as for the cloud chamber, and leads to the same conclusion.

A photograph emulsion detects charged particles because the particle displaces electrons in grains of the emulsion. Again, the interaction is a relatively inelastic one between the particle we wish to observe and another. Analysis leads to the same conclusion as before.

A particle counter detects an energetic particle by the initiation of a cascade of electrons whose impulse can he amplified and recorded. Again, the interaction is a relatively inelastic interaction between the particle we wish to measure and some other particle in the counter. (Particles which are uncharged are detected by means of their interactions with charged particles when emitted or absorbed.)

Let us consider now the comparison of a gravitational field with an electromagnetic force by use of a spring scale or its equivalent. Unlike the interactions already considered, this is a classical measurement on a macroscopic system. We can lump all such classical measurements on macroscopic systems (such as instruments to measure electric or magnetic fields) together. As Eddington [52, 1928, page 252] pointed out, all such classical measurements can be reduced to pointer readings, which he then idealized as the intersection of the world lines of two particles.

If we analyze the manual reading of a pointer with the eye, the result is the same as for visual observation of planets, and leads to the same conclusion. A more accurate way to make a pointer reading is with some kind of automated optical system involving say photocells. Again, the analysis leads to the same result that the measurement corresponds to the inelastic interaction of a photon with some particle.

Of course a more accurate way to measure an electric or magnetic field is by observing the trajectory of a charged particle in the field. The analysis is the same as before for the observation of particles, and leads to the same conclusion.

In summary, all measurements can be divided into two sets.

1. In quantum measurements, the inelastic interaction of one particle with another
2. In classical measurements, pointer readings, which we can idealize as the intersection of two world lines.

Although both of the above could be generalized to the intersection of two world lines, it is more accurate to consider the classical measurement as being made up of very many of the quantum-type measurements. Section 36.6 will consider this point further.

36.6 Inelastic interactions

In the standard interpretation of quantum theory, a wave function can change in two ways:

1. In an isolated system, the wave function evolves according to Schroedinger's equation.
2. When the system interacts with an external classical system, the wave function can collapse to the eigenstate that was indicated by the measurement.

There are at least two well-known objections to this interpretation.

1. If we include the whole universe as the quantum system, then there is no external system to interact with, and Schroedinger's equation should apply, even during a measurement.
2. Most of us believe that there are no truly classical systems, that large systems that appear classical are really quantum systems that to a very good approximation can be treated classically.

There have been many attempts to find either an alternative theory or an alternative interpretation that would overcome the above objections. Here, I consider a slight modification to the standard interpretation that seems to overcome the above two objections. In this interpretation, [50, Green, 1965, chapter 2] the wave function in an isolated system can change in two ways;

1. The wave function evolves according to Schroedinger's equation if all interactions are elastic (interfering alternatives).
2. The wave function collapses when an inelastic interaction (exclusive alternatives) occurs,

The above interpretation overcomes the objections mentioned because it needs no external system, and because it substitutes a specific kind of interaction between quantum particles for the interaction with a classical system as the condition for the collapse of the wave function. The particular interactions that cause the wave function to collapse are the same as those considered in section 36.5 as those used to detect particles.

The factor that determines whether an interaction will cause the wave function to collapse is whether it is an interfering alternative or an exclusive alternative. The latter will cause the wave function to collapse. We can call the kind of interactions that cause the wave function to collapse inelastic interactions, because they cause changes in subsystems that make it easy to tell later that a change has occurred, and make it nearly impossible for the change to reverse itself. The ionization of an atom is an example.

Green (1965)[50] shows that this interpretation resolves the usual paradoxes of quantum theory.

In summary, inelastic interactions are the only observable dynamical variables in quantum theory, and thus these are the α variables to be used in section 36.4.

For the purposes of section 36.4, it is necessary only to identify inelastic interactions as the only observable dynamical variables in a quantum system, and to be able to enumerate what interactions are elastic and which are inelastic. It is not necessary to tell why some interactions are inelastic. However, section 36.7 will speculate on the latter point.

36.7 Non-Hermitian interactions

Suppose that the speculation that the inelastic interactions causes the wave function to collapse is correct. Is there any reasonable explanation for such behavior actually occurring, and is there a uniform way to describe that phenomenon in the same framework with the normal interactions?

Gottfried (1966, chapter IV)[51] analyzes the measurement process from the viewpoint of density matrices. He uses the Stern-Gerlach experiment to illustrate his point of view. In the Stern-Gerlach experiment, an inhomogeneous magnetic field separates a beam into separate beams according to spin components. Because spin components take on only discrete values, the separation is into discrete beams, that after a long enough time have negligible overlap.

Gottfried calculates the density matrix for the split beams in the Stern-Gerlach experiment. Because the sum of all the probabilities for alternatives is always one, the trace of the density matrix is always one. In addition, because we have in this case, a pure state, the trace of the square of the density matrix is also one. At this point, Gottfried makes an approximation to the density matrix that is valid because the overlap between the beams is negligible. The approximate density matrix depends no longer on the phase associated with the part of the wave function in each beam. As he points out, there is only one way to distinguish between the correct density matrix and the approximate one by experiment. If the beams could be brought back together with the correct relative phases, the correct density matrix would give the correct relative phases, whereas the approximate density matrix would not.

The approximate density matrix also has the trace of its square less than one, indicating that it is a mixed state rather than a pure state. The approximate density matrix gives exclusive alternatives rather than interfering alternatives for the beams. That is, the probability of a particle being in one beam or another is classical (exclusive alternatives) with the approximate density matrix, but quantum (interfering alternatives) with the correct density matrix.

In the development so far, we have considered the mixed-state density matrix to be an approximation for the pure-state density matrix. But what about the possibility that through some sort of interaction the pure-state density matrix actually was transformed into the mixed-state density matrix? Gottfried (1966, page 177)[51] shows that this is not possible if the system evolves through a Schroedinger equation with a Hermitian Hamiltonian, because then the trace of the square of the density matrix remains constant. However, a non-Hermitian Hamiltonian could change the trace of the square of the density matrix. Clearly, we don't believe that such an interaction occurs during the separation of the beams in the Stern-Gerlach experiment. If it happens, it would happen when the particles are detected to be separate, on a photographic emulsion or in a particle detector. Is it possible that all such inelastic interactions have non-Hermitian Hamiltonians that change quantum probabilities (interfering alternatives) into classical probabilities (exclusive alternatives)?

I don't know the answer to the last question, but I speculate that the answer is yes.

36.8 Extension to density matrices

The development in sections 36.2 and 36.4 assumed a universal wave function for the universe. This is equivalent to assuming that the universe is in a pure slate. To make the development more general, we should start with a density matrix description for the universe. Then, the development will apply to both pure states and mixed states.

To make that extension, we point out that the eigenvalues of the density matrix give the probability that the system is described by various wave functions (or pure states). We then make the development in section 36.4 for each of these pure states (or wave functions). At the end, we then apply the eigenvalues of the density matrix as probabilities to the wave functions represented in (36.4) to get the final density matrix.

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Chapter 37

The "exclusive alternative" interpretation of measurement in quantum systems[1]

1. Summary
2. Introduction - Is interaction with a classical system necessary for measurement of a quantum system?
3. Exclusive-alternative interactions versus interfering-alternative interactions
4. Schroedinger's cat
5. The Einstein-Rosen-Podolsky experiment
6. The Stern-Gerlach experiment
7. Particle detectors

See book by H. S. Green, Matrix Mechanics[50]

37.1 Summary

37.2 Introduction - Is interaction with a classical system necessary for measurement of a quantum system?

The orthodox view (the Copenhagen interpretation) on measurement of a quantum system is that the quantum system must interact with a classical system for a measurement to occur. Even leaving philosophical problems aside, there seem to be some internal problems with this viewpoint.

First, there seems to be no general agreement as to when the interaction with a classical system has taken place. The Schroedinger's cat paradox illustrates the degree to which the disagreement has gone. Although most people would consider a cat to be a classical system, the possibility of a quantum state consisting of a superposition of a dead cat and an alive one is seriously discussed even among those who subscribe to the orthodox view.

Second, there are situations in which we can say that a measurement has been made on a quantum system even though the system with which it interacted would not normally be considered

[1] 25 July 1981

to be classical. Everyone might not agree that the examples I will give illustrate my point, however, because of the ambiguity as to when a measurement has been made. My first example is a cloud chamber or bubble chamber. The quantum system we are measuring is an electron (say) as it travels through the chamber. I argue that the measurement occurs when the electron ionizes an atom, because that is the only interaction that involves the quantum system being measured (the electron). The subsequent condensation of a water droplet on the ion occurs later. Both the electron and the atom it ionizes are quantum systems. As a second example, consider neutron scattering from crystals [113, Feynman, 1962, page 3]. Suppose that the crystal initially has all the spins of the atoms aligned. If an ensemble of neutrons are scattered by the crystal, and do not change spin, the usual diffraction pattern will result. If, however, one neutron has its spin reversed by a local interaction with one of the atoms in the crystal, then that one atom will have its spin flipped. An ensemble of such neutrons will not produce a diffraction pattern. For each such scatter, the crystal can be interrogated later to reveal which atom had its spin flipped. That constitutes a position measurement for the neutron. The measurement of the quantum system (the neutron) occurred when the neutron was scattered from the atom in the crystal, not afterward during the interrogation of the crystal. During the interrogation of the crystal, the neutron was no longer being interacted with. Both the neutron and the crystal are quantum systems. (The large size of the crystal does not make it a classical system; the interaction was entirely of a quantum nature.)

We are thus left with the question: What does constitute a measurement on a quantum system? and: Is there a unique point at which such a measurement occurs?

We are left with two questions:

1. Is there a unique point at which a measurement on a quantum system occurs?

2. What constitutes a measurement on a quantum system?

In the following pages, I will present an interpretation that answers affirmatively to the first question, and gives a definite answer to the second. Although I admit that my interpretation is not necessary in quantum theory, the properties it has in terms of the answers to the above questions makes it a very useful interpretation.

37.3 Comments - 2008

This essay was never completed.

Chapter 38

Measurement in quantum cosmology[1]

1. Summary
2. Introduction - the difficulty in defining a quantum system without a background geometry
3. The wave function over 3-geometries and matter fields
4. The wave function for observables

Comment - 2008

Apparently I never went beyond the outline here.

[1] ca July 1981

Chapter 39

What is observable?[1]

39.1 Introduction

The question about what is an observable has been controversial for quite a long time. I don't intend to resolve that controversy today. In fact, any observation consists of a chain of events, and where in that chain the observation occurs is, I believe, somewhat arbitrary. I will express a point of view here that is not the only consistent point of view possible, but this point of view leads to some useful ideas concerning measurement.

I will argue here that the electromagnetic field is not directly measurable. Of course, as soon as I say that, the companies that manufacture equipment for measuring electric and magnetic fields will protest. As far as everyday practice is concerned, they are perfectly right. I am concerned with the interpretation of what their equipment measures. If we say that we have an instrument that will measure an electric field, we have to say what we mean by an electric field. We must choose a model of the electromagnetic field. If we choose a classical model (such as Maxwell theory) in which our measuring instrument does not disturb the field being measured, then we are led to no inconsistency if we maintain that our instrument measures the electric field. It is in the quantum regime where we must be more careful, and since we now know that the electromagnetic field is a quantum field, we must be more careful. To analyze what our instruments really do measure, I will consider measurement of electromagnetic radiation fields later. All electromagnetic fields other than radiation fields are measured by observing the effect of the field on some charged particle. Thus, what we really measure is an effect on a charged particle.

Let us now consider the measurement of the electromagnetic radiation field. I will first consider radio waves. The photon flux in a radio wave of measurable intensity is so great, that ordinary radio receivers with their antennas are not able to distinguish any quantum effects. The measurement of a radio wave is through interaction of the electric field with the conduction electrons in the metal of the antenna. As with measuring the non-radiation field, what we really measure is the effect on charged particles.

Let us consider the detection of photons in a photocell or photomultiplier tube. The detection begins with the photon interacting with a metal such that an electron is ejected by the photoelectric effect. What we really observe is what results from that electron. As in the previous cases, we really measure the effect on charged particles.

The detection of photons with photographic film is similar to that in a photocell, in that it begins with an ionization. Again, we are measuring the effect on a charged particle.

The detection of light in the human eye is probably the most difficult case to consider psychologically because here we really have the feeling of a direct sensation of the light without any

[1]Written sometime after 9 October 1981

intervening equipment. The eye is much more complicated, but it could be analyzed to show that it is the effect on charged particles that is observed. The important point is, however, that in the eye, we don't measure an electromagnetic field, we observe an interaction (which we may interpret as an interaction with light). Thus, although electromagnetic fields are not directly observable (or measurable), the interaction with them is observable.

In summary, when the electromagnetic field is treated classically, it is perfectly consistent to maintain that we can measure the electromagnetic field, but in considering the quantum aspects of the electromagnetic field, it is more appropriate to say that we observe interactions with the electromagnetic field.

Thus, the following amplitudes can be calculated and more or less directly compared with observation:

1. The amplitude that a photoelectric cell registers a count.

2. The amplitude that a piece of photographic film gets "exposed".

3. The amplitude that an interaction with light is registered on a particular part of the retina of my eye.

The significance of the above is that in calculating any of the above amplitudes, I need to integrate over the final state of the wave function for the electromagnetic field. In addition, because the same considerations apply to the preparation of the initial state, we again cannot say that the initial state consists of a wave function for the electromagnetic field, but rather that we had certain interactions occur. Thus, to get the amplitude for any physical process, we must integrate over the electromagnetic field wave function in both the initial and final state.

The real reason I wanted to go through these arguments, is to apply the same reasoning to the gravitational field (or the geometry). Thus, we do not measure the gravitational field (except in the classical approximation, where there is no difficulty in maintaining that point of view), we observe interactions with the gravitational field. Thus, quantum gravity should be able to give us the amplitude for the following kind of process. If we observe some set of interactions in the initial state, what is the amplitude for observing some particular interaction in the final state. To calculate that amplitude, we must integrate over the gravitational field coordinates (the 3-geometry) in both the initial and final state. Thus the amplitude (e.g., Hawking, 1979)[111]

$$< g, \phi, t | g_1, \phi_1, t_1 > \tag{39.1}$$

to go from a state with a metric g_1, and matter fields ϕ_1, on a spacelike hypersurface t_1, to a state with a metric g and matter fields ϕ on a spacelike hypersurface t is not directly observable. Such an amplitude will enter into the calculation of the amplitude for an observable, but that process will require an integration over the 3-geometries g, and g in the initial and final states.

The result is, that if we use a path-integral formulation to calculate (39.1), then when we combine that with the integration over the initial and final hypersurfaces, we will effectively have an integration over 4-geometries.

Chapter 40

Measurement in Quantum Cosmology[1]

abstract

How does a quantum-cosmology wave function (that is, a wave function over 3-geometries and matter fields) relate to measurement? Answering that question requires pointing out that we do not measure the geometry directly, but indirectly in terms of measurements on matter. Because the quantum-cosmology wave function must include the observer and his measuring apparatus, it in principle can yield the amplitude for all possible measurments. Further development depends on the fact that all measurements can be expressed in terms of primitive measurements that an interaction occurred or did not occur beteeen two particles. Finally, amplitudes for observables are obtained by integrating over the three-geometry because the geometry is not directly observable.

40.1 Introduction and summary

Recently, Hawking (1983)[114] presented a tentative quantum- cosmology wave function for our universe. Although there are questions concerning the accuracy of his particular calculation, of greater significance for the future, is how any such wave function should be interpreted. That is, how is such a wave function related to measurement? The purpose of this paper is to answer that question.

As is well known, there are three points of disagreement in the interpretation of standard quantum mechanics. First, in the standard Copenhagen interpretation, at least, the evolution equations (Dirac equation, Schroedinger equation, etc.) apply only to closed systems, and do not apply to the measurement process itself, which is normally interpreted as an interaction with a classical system. The controversy associated with trying to include the observer and the measuring process in a quantum description is well known. Second, wave functions and quantum amplitudes predict the behavior of ensembles, not individual systems. To compare theory with experiment, it is necessary to repeat an identical experiment many times. The difficulty of comparing theory with experiment when only one element of an ensemble is available for measurement is well known. Third, the initial wave function (or the initial density matrix, for a mixed state) must be known to make a quantum prediction. How to compare theory with measurement when the initial wave function is not known is a known difficulty.

In practice, all of these difficulties can be ignored by restricting experiments to those in which: (1) the quantum system is isolated except at the beginning and end, and the type of measure-

[1] File qcos.tex, DRAFT - 4 December 1983

ment used follows certain rules; (2) all measurements are made on ensembles; and (3) the initial wave function is known because the intitial state has been properly prepared. Experiments that do not fulfill these restrictions are usually considered in the same light as ill-posed problems in mathematics.

In quantum cosmology, none of these difficulties can be ignored (e.g., Wheeler, 1964)[115, 116, 117]. (1) A wave function for the universe must include the observer and his apparatus, so that we cannot consider our quantum system isolated. (2) We have only one universe available for observation; we do not have an ensemble. (3) We do not know the initial wave function for the universe.

I shall not deal with these difficulties here because I have nothing new to add.

Rather, I consider here a fourth problem, one that does not appear until one considers quantum cosmology and quantum gravity. In particular, what does a wave function over three-geometries mean in terms of measurement? That is, if we have calculated a wave function over three-geometries for our universe, what measurements can we do to compare with the calculations?

To answer this question, I break the problem up into the following steps. I first review how measurements of the geometry are made, and point out that such measurements are indirect in that they involve the behavior of matter (section 40.2).

Second, I argue that an ideal classical measurement corresponds to the intersection of the world lines of two particles (section 40.3). The significance of this point is that the intersection of two world lines can be specified independently of the geometry. Such an ideal classical measurement never exists, but a practical classical measurement occurs when the world lines of two particles are so close that they can be associated. This can be considered to be an idealization of Eddington's idea that all measurements can be reduced to pointer readings.

Next, I point out that all quantum measurements consist in registering the occurence of events, such as registering a photon in a photocell (section 40.4). All such primitive quantum measurements can be described without reference to the geometry.

With these considerations in mind, I point out that any measurement of the geometry consists in a combination of primitive quantum measurements (section 40.5). It is possible to predict a probability amplitude for the outcome of such measurements from a knowledge of the wave function over 3- geometries and matter fields. The matter fields include the observer and his measuring apparatus. In making that calculation, it is necessary to integrate over the final 3-geometry, because the geometry is not directly observable.

In practice, we are not given a wave function for the cosmology. We have the possibility of inferring such a wave function from quantum measurements that we make in the present (section 40.6).

If we take a cosmological wave function and integrate over 3-geometries, we shall be left with a wave function over matter fields (section 40.7). How does such a wave function differ from the usual wave function in quantum mechanics? As far as I can tell, where is no way to tell by measurement. This seems to imply that there is no way to tell in principle between quantum fluctuations of the geometry and quantum fluctuations in the matter fields. However, different models for the quantum cosmology will normally lead to different wave functions for the matter fields.

Although some of the details in the present analysis may need further clarification and expansion in the future, the concepts presented here should increase our understanding of what measurement means in quantum cosmology.

40.2 Measuring the geometry is indirect.

When we measure the geometry, we do so by observing the behavior of matter in that geometry. For example, we determine local geometry by observing the trajectories of bodies in the earth's gravitational field or in the solar system, or the trajectories of particles in bubble chambers or similar apparatus. We determine global geometry by astronomical observations of stars and galaxies. In all cases, our measurements are on matter. We cannot observe the geometry directly.

40.3 Classical measurements can be considered to correspond to the association of particles (ideal pointer readings).

Let us consider how we measure the geometry from photographs of star fields. We infer the trajectories of planets from the changing patterns of the star fields on film. We can consider a photograph to consist of a combination of primitive measurements, which consist of the association of a bright spot (the star or planet) with a particular part of the photographic plate or paper. The ability to define a classical measurement as the association of two pieces of matter is general.

Eddington's (1928)[52, p. 247] idea that all measurements can be reduced to pointer readings is similar. A pointer reading consist in the association of one piece of matter (the pointer) with another (a particular part of the scale).

That all classical measurements can be thought of as the association of two pieces of matter or of two particles (if we idealize) is useful for the present purposes becauses the association of two particles can be described independently of the geometry.

In any realistic pointer reading, parallax will lead to an uncertainty in the reading. The process of associating two particles from their proximity also leads to an uncertainty. Only if the world lines of the two particles intersect, will the association of the two particles be unambiguous, and only then can the two particles be associated independently of the geometry. We can consider such an intersection of world lines to be an ideal classical measurement.

The impossibility of the intersection of the world lines of two particles shows the limitations of that line of reasoning. However, if the world lines of two particles are close enough, there is the possibility that the two particles may interact. That interaction could be considered to associate the two particles. That possibility will be considered in the next section.

40.4 Quantum measurements consist in registering the occurrence of events.

A photocell or photomultiplier tube can be considered to be a prototype (archtype?) for all quantum measuring devices. It registers the interaction of a photon with some atom in the photocell. All quantum measuring devices register the interaction of one particle with another. Those devices that behave similarly to a photocell are clear.

Other devices require some explanation to show that they also fit this general description. A track in a bubble chamber consists of many bubbles. Each bubble marks the place where a charged particle ionized a hydrogen atom. A bubble chamber thus registers many interactions (ionizations in this case) of one particle (the moving charged particle) with another particle (a hydrogen atom).

Using film as a measuring device (for example in astronomy) can be analyzed similarly. Many photons may fall on a piece of film. Some of those photons will expose parts of the film by first ionizing atoms in the film. We can consider that each exposed grain in the film registers the ionization of an atom by a photon. Thus, recording on film can also be considered to register the occurrence of events, each event being the interaction of one particle with another.

All quantum measuring devices (in fact, all measuring devices when analyzed on the quantum level) can be characterized as registering the interaction of one particle with another. Sometimes the analysis to show that may be complicated. Sometimes the characterization in terms of registering interactions may not be unique (as when a bubble chamber track is photographed), but such a characterization can always be made.

The significance of this point is that a quantum measurement (as registered by the interaction of two particles) can always be decribed without reference to the geometry. The above analysis applies also to many body interactions. The important point is that all primitive quantum measurements consist in registering the occurrence of interactions, and the occurrence of an interaction can always be described without reference to the geometry. Thus, a deficiency in the corresponding analysis for the classical case is removed in the quantum analysis. In the next section we shall see how this helps relate a cosmological wave function to measurement.

40.5 Relation of a quantum cosmology wave function to quantum measurements - Integration over final 3-geometries

Let us assume that we have a wave function

$$y = y(g, f) \tag{40.1}$$

for the 3-geometry g and matter fields f. The matter fields f include the observer and all of his measuring apparatus including his clocks. We want to relate y to measurements that might be made, for example, to determine the geometry.

As pointed out in section 40.1, a measurement of the geometry is indirect in that it must be made in terms of matter. Thus, g can not be measured directly, but must be inferred from measurements on f. As pointed out in section 40.4, all measurements can be reduced to registering interactions. From a knowledge of y, we can, in principle, calculate the amplitude for those interactions.

That we calculate the amplitude for an interaction, but a measurement registers (or does not register) that interaction brings us back to the difficulty mentioned in section 40.1 that we do not have an ensemble of universes available. Section 40.6 will consider this difficulty slightly. The present section will consern itself only with calculating the amplitudes for those interactions.

The exact calculation of a particular quantum amplitude in detail would be tedious, and would not give any more insight into the ideas involved. Instead, I indicate only schematically how such a calculation would go.

Suppose we are going to make a series of measurements to, say, infer something about the geometry. Let us assume we have analyzed the experiment carefully, and have identified a number of interactions that we register as having occurred or having not to have occurred. (In practice, this may be tedious . It would require identifying the grains in a piece of photographic film, for example.) Let us call these interactions a_i. Let us call the amplitudes for the occurrences of the interactions a_i (as determined from y) A_i. We need to find out how to calculate A_i from y.

Let us consider what y(g,f) means. It is the amplitude that the 3-geometry is g, and that on that 3-geometry we have matter fields f. In general, f cannot be specified without reference to g. To indicate that explicitly, we write

$$y = y(g, f_g), \tag{40.2}$$

where the subscript g indicates that f is specified in terms of g.

Although specification of f_g on a given 3-geometry g may be difficult because time is not a parameter, the difficulties should not be insurmountable. In practice, time is measured by observing repetitions in the configuration of matter, such as in the planets (rotation and revolution of the earth, for example), atoms, and various man-made clocks. Time measurements can be analysed in the same way as other measurements, and can thus be reduced to a combination of the occurrence of events, each event being an interaction between two or more particles. Thus time measurement is inherent in f_g.

There is a great deal of flexibility in the way that f_g can be specified. This flexibility is analogous to the choice of basis in standard quantum theory.

One of the ways of specifying f_g has a great advantage for the present purposes, as we shall see later. f_g can be specified in terms of the possible interactions between pairs of particles. As far as measurements are concerned, this is a completely general representation of f_g, because as pointed out in section 40.4, all possible measurements can be shown to consist of combinations of such primitive measurements.

Thus, we can always choose a representation in which we can write

$$y = y(g, f_o), \tag{40.3}$$

where f_o is a representation of the matter fields in terms of primitive observables, that is, in terms of interactions between pairs[4] of particles. Equation (40.3) is a completely general representation of y in terms of observables f_o. "Observable" here has a more physical definition than the usual mathematical one in terms of the eigenalues of Hermitian operators, although a mathematical definition could probably be found. The present definition of "observable" is narrower than the usual one. For example, position and momentum are not primitive observables; they are derived quantities.

The advantage of the representation (40.3) is that f_o can be specified independently of the geometry g, as pointed out in section 40.4. This allows us to write[5]

$$y(f_o) = \int y(g, f_o) Dg, \tag{40.4}$$

where Dg is a measure for g.[6]

Because the 3-geometry g can not be directly observed (section 40.2), the wave function $y(f_o)$ is a wave function in terms of all possible observables. (Remember that f_o includes the observer and all of his measuring apparatus.)

Because f_o includes all possible observables, it includes the set a_i for the original experiment that was designed to measure some aspect of the geometry. Thus, the set $\{a_i\}$ is a subset of f_o. We can thus write

$$f_o = a_i U \bar{a}, \tag{40.5}$$

where $\bar{a}$ is the complement of $\{a_i\}$ with respect to f_o. Thus, we can write

$$y(f_o) = y(a_i, \bar{a}). \tag{40.6}$$

Finally, we have[7]

[4]This construction could easily be extended to 3-body and higher interactions, if necessary. Actually, it could be extended to the amplitudes of any events.

[5]In an earlier attempt, I had written (40.4) with f_o replaced by f_g. I thank Stephen Hawking (1980, private communication) and Julian Barbour (1980,private communication) for pointing out that that formulation was not valid.

[6]Stephen Hawking (1980, private communication), Gary Gibbons (1980, private communication), and A. Ashtekar (1981, private communication) agreed that one must always integrate over the 3-geometry to calculate the amplitude of an observable because the geometry is not directly observable. Ashtekar felt that the idea was already known, but he could not think of a reference.

[7]Equation (40.7) assumes that the elements of $\bar{a}$ are interferring alternatives. In general, $\bar{a}$ will include both interfering alternatives and exclusive alternatives. Normally, a specific calculation will require a thorough examination to identify the observables as interferring or exclusive alternatives. Then (40.7) must be generalized accordingly.

$$A_i = y(a_i) = \int y(a_i, \bar{a}) D\bar{a} \tag{40.7}$$

for the calculated amplitudes associated with the measurement a_i.[8]

40.6 Quantum measurements of the geometry

In practice, we can measure the geometry quite well. The measurements are all indirect, in that they involve measurements on the matter (section 40.2), and they can be analysed classically (section 40.3) or from a quantum viewpoint (section 40.4).

How can we use such measurements to infer a cosmological wave function, and what are the consequences that we have only one universe and not an ensemble?

I shall deal with the second question first. In section 40.5, we saw how to calculate the amplitude A_i for an observable a_i. In normal practice, many of the a_i can be treated as an ensemble because they are in a similar situation (such as neighboring grains in a photographic emulsion, or neighboring atoms in a bubble chamber). When such ensembles are not naturally present, we create them by repeating an experiment with a time delay.

We so far have no experimental evidence of the existence of quantum gravity. I suspect that if we discover a situation in which measurements cannot be formed in such ensembles, that that will be related to experimental evidence for quantum gravity.

Now to the question of how we can infer a cosmological wave function from measurements. We assume a model for y(g,f), make calculations of A_i corresponding to a_i that can be measured by experiment, and make a comparison for those a_i that can be formed into ensembles. Of course, the calculations as outlined in section 40.5, would be tedious, but shortcuts could be found.

40.7 Quantum gravity versus quantum mechanics

The function $y(f_o)$ defined in (40.4) looks like a normal wave function or quantum amplitude as calculated in standard quantum mechanics, because it makes no reference to the geometry. How does it differ from a normal wave function or quantum amplitude?

In principle, there is no difference. $y(f_o)$ is the amplitude for observing the interactions f_o. However, $y(f_o)$ is not referenced to any particular geometry (a background geometry, for example). (Remember, that f_o can be specified independent of the geometry.) $y(f_o)$ does contain indirect information about the geometry (and its quantum fluctions) as explained in sections 40.2 and 40.4. However, quantum fluctuations implied by $y(f_o)$ differ in no qualitative way from normal quantum fluctuations. That is, it seems impossible in principle to distinguish between quantum fluctuations in the geometry and normal quantum fluctuations. However, $y(f_o)$ does depend on $y(g, f_o)$ through (40.4), so that quantum fluctuations in the geometry will contribute to observable quantum fluctuations.

40.8 Acknowledgements

In the summer of 1967, Douglas Gough showed me the paper, "On the origin of Inertia", by Dennis Sciama (1953)[12]. The concepts in that paper so intrigued me that I started along a line of reasoning that eventually led me to the present work.

[8]All of the calculations here have assumed that the universe is a pure state that can be described by a wave function y. If the universe is in a mixed state, described by a density matrix, then all of the calculations presented here can be extended to include that generalization in a straightforward way.

Partly because my main field of research is only distantly related to this one, and partly because my main research activities take priority over those leading to the present work, my progress has been slow. During those 15 or more years, I have been influenced by many people, both directly, and through their publications.

Because of the long time period, I find it difficult to reconstruct exactly how I developed my present viewpoint, especially because I took so many wrong paths before I reached the present viewpoint. I find it especially difficult now to remember to whom I owe various ideas, especially because I found myself independently rediscovering ideas that were already known.

Still, I shall try to mention here those people that have influenced my development of the present work. Because that development has been along a very long path, it may not be obvious to the reader the relevance of some of the influences. In roughly cronological order, here are the people I can remember to have influenced the development of the work.

First, Leopold Felsen first introduced me to the concept of a path integral (although not by that name) during a series of lectures in the summer of 1967 in Boulder, Colorado. Later, my understanding of path integrals increased through reading the book by Feynman and Hibbs (1965)[21].

Trying to understand Mach's principle played a strong role in the development of my present viewpoint. It is well known that there are many viewpoints as to what Mach's principle really is. In that regard, I have been influenced by Sciama (1953)[11], Wheeler (1962,1963,1964c)[118, 119, 115], Sciama, Waylen, and Gilman (1969)[16], Raine (1975[109], and private communications), Barbour (1974[108], and private communications), but mostly by Mach, himself (1911, 1960)[120, 102, 1, 121, 122, 15].

My path eventually led to quantum cosmology, where I was mostly influenced by Wheeler (1964a,b, 1968)[116, 117, 19], Misner, Thorne, and Wheeler (1973)[20], and Hawking (1978, 1979)[106, 111, 123].

I thank Don Page (1980, private communication) for introducing me to the concepts of mixed states and density matrices.

I also thank Stephen Hawking (1980, private communication), Julian Barbour (1980, private communication), and Don Page (1980, private communication) for pointing out errors in my previous thinking.

Most of all, I thank my wife for putting up with my scientific indulgences.

Chapter 41

Relation Between Measurement and the Wave Function in Quantum Cosmology[1]

The following comments were handwritten on the first page of the manuscript.

- 4th draft?
- 1985 or later
- unsharp measurements? (cloud chamber)
- quantum logic?
- projection?
- put math in appendix.
- explain in English in text.
- put measurement example in appendix.

Abstract

The relationship between a wave function in quantum cosmology and measurement is considered. It is shown that the geometry is not directly measured, but is inferred from measurements on matter. A representation of the matter fields in terms of directly measurable interactions allows a wave function to be calculated that specifies the amplitudes for those interactions without direct reference to the geometry.

41.1 Introduction

A wave function $\psi[h, \phi]$ in quantum cosmology gives the amplitude that the 3-geometry on some spacelike hypersurface is h, and that on that 3-geometry, the matter fields are ϕ (e.g., Hartle and Hawking, 1983[124]). Careful consideration shows, however, that geometry is not directly measurable. We are thus led to the question of what a wave function in quantum cosmology means

[1]Previous title: Measurement in Quantum Cosmology. 1985 or later

(that is, how a wave function in quantum cosmology or quantum gravity is related to measurements). It is not at all clear a priori how the amplitudes for various 3-geometries relates to measurements. It is the purpose here to answer this question.

Section 41.2 begins by discussing how we measure the geometry classically, and points out that all measurements of the geometry are indirect in that they involve measurements on matter. Measuring the geometry in the quantum case must also be an indirect measurement in terms of matter.

If we need to make measurements on the matter to measure the geometry, but the wave function for the matter depends on the geometry, then it is difficult to define measurements in quantum cosmology. Section 41.3 considers this difficulty and argues that quantum measurements can be defined independent of the geometry in terms of the amplitudes for interactions.

Section 41.4 considers how to calculate directly measurable quantities in terms of a quantum cosmology wave function.

Many of the ideas developed here for quantum cosmology also apply to quantum gravity (in which quantum fluctuations of the geometry on only a local scale are considered).

Although only pure states are considered here, the ideas can be extended to mixed states in a straightforward way.

41.2 Measuring the geometry

A quantum cosmology is described by an amplitude for each 3-geometry. To find out what such an amplitude means, we have to first consider how we measure the geometry.

When we measure the geometry, we do so by observing the behavior of matter in that geometry.

For example, we determine local geometry by observing the trajectories of bodies in the Earth's gravitational field or in the solar system, or by observing the trajectories of particles in bubble chambers or similar apparatus. We determine global geometry by astronomical observations of the positions of stars and galaxies or the spectra from stars and galaxies.

A complete measurement of the geometry requires also a measurement of time. We always measure time by counting the number of times some system returns to the same configuration (for example, revolutions of the Earth around the sun, rotations of the Earth on it's axis, swings of a pendulum, or oscillations of atoms in an atomic clock). In every case where we measure time, we are making measurements on matter.

Thus, whenever we measure the geometry, we must make measurements on matter. We cannot observe the geometry directly.

In quantum cosmology also, measurements of the geometry are in terms of measurements on matter.

41.3 Defining measurement independent of the geometry

One of the usual ways of expressing a wave function in quantum cosmology is as $\psi[h,\phi]$ (e.g.. Hawking, 1978[106], 1979[111]; Hartle and Hawking, 1983[124]; Halliwell and Hawking, 1985[125]), where ψ is a functional of the 3-geometry h and the matter fields ϕ. ψ gives the probability amplitude that the 3-geometry is h and that on that 3-geometry the matter fields are given by ϕ. The time is not directly a parameter, but must be derived from ψ.

As pointed out in Section 41.2, however, the geometry is not directly measurable. We always infer the geometry (including time) by measurements on matter. We thus might be tempted to integrate ψ over all possible 3-geometries to leave a wave function in terms of the matter distribution alone. This is not possible, however, because it is not in general possible to specify the matter distribution independent of the geometry.[2]

Clearly that approach does not work. We need to consider more specifically which aspects of the matter distribution we observe during a measurement, and whether we can specify such aspects independent of the geometry.

With regard to classical measurements, a little reflection leads to a definition of an ideal measurement as the intersection of the world lines of two particles. This idea seems to be an idealization of Eddington's (1928)[52] idea of reducing all measurements to pointer readings. The intersection of two world lines can be specified independent of the geometry, so that criterion is satisfied. However, the intersection of two world lines is an idealization that cannot occur in practice.

As an example, consider the photograph of a star field in astronomy. In measuring the distance between two stars in the photograph, we may use a ruler. We then associate a point on the ruler with the bright spot on the photograph corresponding to each of the two stars. The association will not be perfect because of parallax. However, as the distance between the film and the scale of the ruler approach zero, the effect of parallax approaches zero, and the measurement becomes better. A perfect measurement would be possible only if the world line of some part of the bright spot on the film corresponding to the star intersected the world line of a point on the scale of the ruler.

Other, possibly more sophisticated, methods to measure the positions of stars on a photograph can be analyzed in the same way. In fact, any kind of classical measurement can be analyzed in this way. Although in practice most classical measurements are limited by other considerations, the above development shows why we can consider ideal classical measurements to consist of the intersections of the world lines of particles.

The error associated with assuming that the world lines of two particles can coincide is probably of the same order of magnitude as the error associated with the classical approximation anyway. Thus, we may think of the assumption that the world lines of two particles can coincide as another aspect of the classical approximation.

A similar analysis of quantum measurements is much more satisfactory. An ideal quantum measurement consists in observing whether a particular event or interaction does or does not occur. Such events or interactions can be specified independently of the geometry.

We can analyze a photograph of a star field according to quantum measurements. The photographic emulsion consists of many grains. Each grain is either exposed or not exposed, and we can observe which case occurs for each grain. If we want to observe the positions of the grains very accurately, we might use lasers and photomultipliers. The position measurements would then consist of very many observations of whether photons are or are not observed in the photomultipliers.

In practice any experiment can be analyzed in terms of observing whether some events or interactions do or do not occur. Measurements of time can also be analyzed this way.

To compare predictions with experiment we must have some way to calculate from the wave function $\psi[h, \phi]$ the amplitudes for all of the interactions associated with the chosen experiment.

As an example, we can consider an experiment that verifies within some accuracy that particles follow paths that deviate from uniform motion in a straight line only on account of the rotation of the Earth relative to the stars and the gravitational field of the Earth. We can consider this to be an experiment that measures an aspect of the local geometry.

I shall not analyze how the rotation of the Earth relative to the stars or how the gravitational field of the Earth are determined independently, but those experiments can be analyzed in a similar way. Also, any peripheral measurements necessary to ensure that the experiment is properly set up (such as measuring the relative positions of counters, calibrating instruments, and checking that the effects of ambient electric or magnetic fields are negligible) can be analyzed similarly.

[2] I thank Julian Barbour and Stephen Hawking for pointing out the importance of this to me.

For the experiment, we shall produce a charged particle, such as a proton, accelerate it to some speed, send it through a series of detectors such as spark chambers, and record the time of passage of the particle through whatever detectors with which it interacts. In this way, we can measure an aspect of the local geometry.

I want to show that this experiment can be analyzed in terms of whether specific events do or do not occur (such as the detection of a particle by a counter).

If we were to use a bubble chamber to measure the trajectory of the particle, we would get nice measurements of the path, but no time information. If we were to capture the particle in a detector, we would get a good measurement of the position and time, but no detailed information about the trajectory. If we use a spark chamber, we can get reasonable information about the trajectory including time of flight.

We can consider that the spark chamber divides up space into little boxes. If a charged particle enters one of these boxes, then we get a pulse of current flowing through the pair of wires associated with that box.

Of course each detection of the particle corresponds to an interaction of the particle with the detector. Each of these interactions will deviate the particle slightly from the path it would have taken in the absence of the detector. Thus, the measured path is only an estimate of the path the particle would have taken in the absence of the detector, and thus this local measurement of the geometry has some error associated with it. Normally the size of this error can be estimated.

The position associated with each of these boxes in space is estimated from knowing the mechanical construction of the spark chamber.

The time measurements can be analyzed at several levels. Normally for time measurements, we would have an atomic clock that stabilizes an oscillator. We would then divide the frequency of that oscillator down to get a frequency that is useful for our experiment. We would then get clock pulses that we would use to give timing information throughout our experiment. The precision of any time measurement would be one clock pulse.

In practice, so many particles would be involved at each stage of the clock that we could consider the time measurements to be classical. We would, in effect, have divided time into intervals, and by using coincidence circuits with the spark chamber pulses and the timing pulses, we would have divided space-time into boxes. Each detection would then be an event in which we could assign a space-time value (accurate to within the size of the space-time box).

At this level of analysis, we could consider the spark chamber and the clock as simply part of the environment, and needing no further analysis. Our measurements would consist entirely of recording space-time boxes in which detections occur. We could repeat the experiment under nearly identical circumstances, and obtain a distribution of measurements for the trajectory. (A trajectory is described by the set of detections for one passage of the particle.) We could then compare the predicted probabilities of trajectories with the measured distribution of trajectories.

We could analyze the spark chamber and the clock also on quantum level, and obtain similar results, but the analysis would be tedious because there would be so many interactions involved. If one wanted to analyze the clock on a quantum level, it would be better to design a clock in which measurable quantum effects were present.

Consider all of the spacetime boxes through which this spacelike hyper-surface passes. Suppose there are K_n of these. Suppose J_n of these correspond to the trajectory a_n which is embedded in this spacelike hypersurface. Let us number the K_n spacetime boxes in this hypersurface so that the first J_n correspond to the trajectory. Then the trajectory a_n in this spacelike hypersurface corresponds to the detection of a particle in the first J_n spacetime boxes and no particle detected in the boxes numbered J_n+1 to K_n. Of course these spacetime boxes are limited to the intersection of the spacelike hypersurface and the spark chamber for the duration of the experiment.

Clearly, the probability amplitude A_n for the trajectory a_n is the amplitude that there is a detection in the first J_n spacetime boxes and no detection in the boxes $J_n + 1$ through K_n. More generally, A_n is the amplitude that a specified outcome occurs for the detection of a particle in the K_n spacetime boxes.

The probability of the trajectory a_n is $|A_n|^2$, and this can be compared with the fractional occurrence of that trajectory when the experiment is repeated. We shall consider in Section 41.4 how A_n is actually computed from knowing the wave function. For now, we write schematically

$$A_n = A_n[\psi[h_{ij}, \phi]] \; . \tag{41.1}$$

At this point, the main consideration is that the a can be specified independently of the geometry.

Although this analysis may seem somewhat tedious, it helps clarify in general the relationship of a wave function in quantum cosmology and measurement.

41.4 Predicting measurements in quantum cosmology

If we know the wave function $\psi[h, \phi]$ for the universe on a specified space-like hypersurface, how do we calculate the probability amplitude

$$A_n = A_n[\psi[h, \phi]] \tag{41.2}$$

for the occurrence of a specified outcome c_{nj}. (with amplitudes C_{nj}, for $j = 1$ to K_n) for the detection of a particle in each of K_n specified detectors?

The wave function $\psi[h, \phi]$ is the amplitude that the 3-geometry is h, and that on that 3-geometry we have matter fields ϕ. In general, ϕ cannot be specified without reference to h. To indicate that explicitly, we can write

$$\psi = \psi[h, \phi_h] \; , \tag{41.3}$$

where the subscript h indicates that ϕ is specified in terms of h.

There is considerable flexibility in the way that ϕ_h can be specified. This flexibility is analogous to the choice of basis in standard quantum theory.

One way to specify ϕ_h, is in terms of all possible interactions b_m for $m = 1$ to M that are compatible to be observed without interrupting each other. Such a set of interactions is not unique, because the total number of interactions that can be observed is much larger than the number of interactions that are compatible (in that an experimenter could in principle choose to measure all of them). This is simply Bohr's complementarity principle, that the experimental apparatus necessary to measure one interaction may be incompatible with measuring some other interaction.

Such a basis is clearly a complete representation for ϕ, because it includes all possible compatible measurements. In addition, because such interactions can be specified independent of the geometry, such a basis for ϕ can be specified independent of h. Thus, for such a representation, we can write

$$\psi = \psi[h, \{b_m\}] \; , \tag{41.4}$$

where the set $\{b_m\}$ is a representation of ϕ that is independent of the 3-geometry h.

Because the geometry is not directly observable, we want to Integrate over the 3-geometry h so that we are left with a wave function of observables only. because the basis $\{b_m\}$ can be specified independent of the geometry, we are allowed to do this. Thus

$$\psi(\{b_m\}) = \int \delta h \psi[h_{ij}, \{b_m\}] \tag{41.5}$$

where δh is a measure for h.

The wave function $\psi(\{b_m\})$ gives the amplitude that a specified outcome for the interactions b_m (for $m = 1$ to M).

Because we have freedom to choose the set $\{b_m\}$ to be any complete set of compatible interactions, we can choose the set $\{b_m\}$ to include the set $\{c_{nj}\}$ providing the c_{nj} are compatible. In particular, we can choose

$$b_m = c_{nm} \text{ for } m = 1 \text{ to } K_n \ . \tag{41.6}$$

If, in designing the experimental apparatus, we exclude the possibility of observing b_m for $m = K_n + 1$ to M, then we can integrate over the interactions not observed to give

$$A_n\psi(\{c_{nm}\}) = \int_{m=k_n+1\to M} \delta b_m \psi(\{b_m\}) \ , \tag{41.7}$$

where δb_m is a measure for b_m.

41.5 Summary

Considerations on how the geometry is measured show that the geometry is not directly measurable, but instead is inferred from measurements on matter. Because of this, all measuring apparatus must be included in the wave function.

The matter distribution can be represented in terms of measurable interactions. This representation has the advantage that it can be specified independent of the geometry. Because the matter distribution can be specified independent of the geometry, it is permissible to integrate out the geometry (which is not directly observable) to obtain a wave function that gives the amplitude for directly measurable interactions.

41.6 Acknowledgments

I would like to thank Julian Barbour, Gary Bornzin, Douglas Gough, Stephen Hawking, Don Page, and David Peterson for helping me clarify my ideas during discussions over the past several years.

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Chapter 42

The principles in Mach's principle[1]

42.1 Principles

Can we use Mach's principle in any way as a guiding principle in forming physical law? That is, is there really a principle there?

Einstein claims to have based his General Theory of Relativity on Mach's principle along with the principle of equivalence. However, as is well known, there are solutions of Einstein's equations that do not satisfy Mach's principle. In particular, there are multiple solutions for the same distribution of matter.

This would seem to contradict Mach's idea that matter should determine inertial properties.

Sciama, Waylen, Gilman - Mach's principle as a boundary condition for General Relativity.

Mach - Only relative motion is important, not motion relative to any arbitrary reference frame.

——— Absolute space does not exist.

——— Flat empty space does not exist.

Raine - Matter determines curvature, curvature determines the metric.

———- Asymptotically flat spaces are non-Machian, conformally flat spaces are Machian

Jones – Considering the physics of a sparse universe will throw some light on physical laws.

42.2 Rotation of inertial frames

Mach recognized that the lack of rotation of inertial frames relative to the visible star field is unlikely to be a coincidence. In looking for a causal relation, he postulated that matter was the source of all inertia. His conjecture has since become known as Mach's principle and was one of the guiding principles for Einstein in developing his General Theory of Relativity.

Unfortunately, as is now well known, although in General Relativity, matter does generate inertia, there is also source-free inertia. In particular, there are matter-free solutions with inertia, for example, the empty Minkowski space.

It has been suggested that Mach's principle be used as a boundary condition to allow only those solutions that satisfy Mach's principle. This has had limited success. [Expand: Sciama, Waylen, and Gilman, Raine]

[1] Joensu, Finland, 1985

Mach[2] may have assumed too much. Although his conjecture was sufficient to explain the observations, it is not necessary. All we really need to do is explain why our inertial frame seems not to rotate relative to the visible universe.

In more modern language, we can express the question as why our universe seems to be the Robertson-Walker metric rather than other similar but anisotropic metrics. We know of many homogeneous, anisotropic solutions of Einstein's equations, for example.

42.3 Photons?

The experimental evidence supporting the intrinsic quantization of the EM radiation field is from a few classes of observation:

1. Photoelectric effect
2. Black-body radiation
3. Spontaneous emission
4. Compton scattering
5. Photon antibunching
6. EM fluctuations

[2]or was it someone else? I need to check on this.

Chapter 43

Relation between measurement and a wave function for the universe[1]

43.1 Introduction

Hartle and Hawking (1983)[124] and others have made some estimates of what a wave function for the universe might be in some simple cases.[2] The idea of a wave function for the universe is, of course, controversial. It is not clear what a wave function for the universe would mean or how one would use it, especially if such wave functions are available in the future with more detail.

We would like to compare such wave functions with experiments or observations (to test the model that produces it).[3] There are three major difficulties that keep such a wave function from being interpreted in the usual way.

First, it seems that the experimenter and all of his apparatus will be included in the wave function rather than having a separate "system" (described by a wave function) and an observer (or experimenter) that is allowed to interact with the system in the standard way to yield an acceptable measurement.

Second, the experimenter has not had a chance to prepare the initial state of the system (the universe, in this case) so that a good quantum mechanical experiment can be gotten.

Third, we have only one universe, not an ensemble. Because the wave function yields probabilities, then usually it is necessary to repeat an experiment under identical conditions and compare fractional occurrences with predicted probabilities.

One side result of an attempt to extend quantum mechanics in this way is that we may learn more about quantum mechanics in its usual application.

The idea here is to try to apply a wave function for the universe according to a common definition to see if it might be possible to use it to make calculations that could be compared to observations. It may not work, but it makes sense to try, rather than dismiss the idea of a wave function for the universe out of hand.

There are further difficulties with regard to wave functions for the universe that are outside the scope of this paper. First, is the enormous task of calculating such wave functions, such as the start made by Hartle and Hawking (1983)[124] and Tipler (1986). Second, is the task of including

[1] Joensu, Finland, 1985, or slightly later

[2] The purpose of this paper is to try to interpret what such a wave function would mean with regard to comparing it with measurements. For that purpose, I take the definition of a wave function $\psi(S, h, \phi)$ to be the amplitude that on a hypersurface S, the 3-geometry is h, and on that 3-geometry there are matter fields ϕ[124].

[3] There are two major aspects to the difficulty of interpreting such a wave function. The first aspect regards how we might verify experimentally a prediction about the quantum nature of the geometry. For this, we must consider how we measure the geometry. The second aspect is that . . .

measurement instruments in such a wave function, roughly along the lines indicated here. Third, is the task of calculating from such wave functions the amplitudes for processes that can be measured.

43.2 Measuring the geometry

43.3 Include the experimenter

Yes/no experiments.

43.4 Mixed ensembles

43.5 Initial wave function

Mixed states

Action peaked about some state

43.6 Relativity of the quantum nature

43.7 Violation of uncertainty?

43.8 Discussion

Chapter 44

Does quantum theory apply to the whole universe?[1]

abstract

The difficulties caused by not being able to perform an ideal quantum experiment (in which an initial state is prepared, a closed system is allowed to evolve, aspects of the final state are measured, and these measurements are repeated on identically prepared systems) on the universe as a system is considered. That we do not know the initial wave function for the universe and that we do not have an ensemble of universes to observe are shown to cause no difficulties in principle. The role the observer's apparatus plays when included in the wave function is clarified. Some of the considerations here may clarify some difficulties in ordinary quantum theory also.

44.1 Introduction

In an ideal quantum experiment, a system is prepared in some initial state, the system is allowed to evolve as a closed system (governed by the Schrödinger equation, Dirac equation, or Klein-Gordon equation, as appropriate), and finally measurements are made on the final state. Knowledge of the initial state allows one to calculate the probability amplitude and therefore the probability for the outcomes of various measurements in the final state. To make a comparison of prediction with measurements, it is necessary to repeat the experiment on an ensemble of systems that have been prepared in an identical way. The predicted probability of a particular outcome is then compared with the fraction of times that outcome occurred.

When we try to extend quantum theory to the whole universe (quantum cosmology), we immediately find difficulties, as is well known (e.g.. Wheeler, 1977)[126].

First, we do not know the initial state of the universe. Second, because any possible observer we may consider and his measuring apparatus are part of the system under consideration, the concept of the evolution of a closed system seems no longer to apply. Third, we do not have an ensemble of universes available for measurement, only the one we live in.

We note that these same difficulties can occur in normal quantum mechanics if we are not careful to perform an "ideal" (as described above) quantum experiment. Discussions of such non-ideal experiments often involve interpretations of quantum theory, philosophy, and sometimes psychology.

In trying to apply quantum theory to the universe, however, we are forced into using non-ideal quantum experiments. By analyzing how we actually make measurements in such cases, we not only have the possibility of explaining the relationship between quantum cosmology wave functions

[1] ca 1985

and measurements, but also of shedding some light on the significance of non-ideal experiments in ordinary quantum mechanics.

Section 44.2 considers the initial state of the universe, and argues that a priori knowledge of the initial state of the universe is not necessary to make practical calculations in quantum cosmology. Section 44.3 considers the interaction of the observer with the rest of the universe, and Section 44.4 considers how ensembles fit into quantum cosmology.

Many of the ideas developed here for quantum cosmology also apply to quantum gravity (in which quantum fluctuations of the geometry on only a local scale are considered).

Although only pure states are considered here, the ideas can be extended to mixed states in a straightforward way.

44.2 Initial state of the universe

The introduction pointed out that it is not possible to perform an ideal quantum experiment in quantum cosmology because, among other reasons, we do not know the initial state of the universe. Although this is true, there are reasons why its importance is diminished.

Suppose that our knowledge of the initial state is imperfect. We can represent the uncertainty in our knowledge by considering the initial state to be a mixed state. In doing that, we assign appropriate probabilities (these are classical probabilities) to various possible wave functions for the initial state, and represent that as a density matrix in the usual way. We can then calculate the evolution of the system to predict amplitudes for various events or interactions. Because such an experiment will generally yield less information than an ideal experiment, an ideal experiment is nearly always preferable, but a non-ideal experiment still yields information.

Suppose we know nothing about the initial state. Then we can represent the initial state by a density matrix in which all possible wave functions for the initial state have equal (classical) probability. We can still calculate the evolution of that state to predict amplitudes for various measurements. That may or may not be useful, depending on the evolution equations.

If we have a model for the quantum cosmology (for example, Hartle and Hawking, 1983[124]; Halliwell and Hawking, 1985[125]), then we can assume an initial state and derive a wave function for the quantum cosmology as a function of time. Once we have a wave function, we can predict amplitudes for various events or interactions. In the case of such a model, we can put limits on what the initial state must have been in order that predictions agree with measurements.

44.3 Interaction with the observer

The introduction pointed out that it is not possible to perform an ideal experiment in quantum cosmology because if the system is the whole universe, then the observer is part of the system and we cannot treat the system as an Isolated, closed, system. If that occurs, then we cannot apply the evolution equations to the system.

In dealing with this problem, it is useful to distinguish between the observer as a person and his measuring apparatus. The difficulties concerning the observer as a person are no more difficult for quantum cosmology than for ordinary quantum mechanics if we choose to leave the observer (but not his apparatus) out of the wave function for the universe. This may not be necessary, but it seems sufficient to reduce that difficulty to the same level as in ordinary quantum mechanics. Leaving one person out of the wave function cannot have a large effect. Considering the role of the observer leads to various interpetations of quantum theory and philosophical questions about when a measurement really occurs as illustrated by the examples with Schrödinger's cat and Wigner's friend.

It is important that we leave the observer's apparatus in the wave function, however. Then we have the possibility of calculating from the wave function the amplitudes for various events or interactions that may occur in the observer's apparatus. Jones (1985)[127] considers how to calculate the amplitude for events or interactions when the measuring apparatus is included in the wave function. If we can do that, then perhaps the role of the observer as a person become irrelevant with regard to the physics.

44.4 Ensembles in quantum cosmology

The introduction pointed out that it is not possible to perform an ideal quantum experiment in quantum cosmology because we have only one universe rather than an ensemble. Thus, we might be able to predict the amplitudes for several events or interactions, but a single measurement to observe whether that interaction did or did not occur cannot be used to make any meaningful comparison between prediction and experiment. In quantum theory, an ensemble is necessary to compare a predicted probability of occurrence of an event with a measured frequency of occurrence of that event.

In practice, however, gravitation and cosmology seem to behave classically. In fact, we know of no direct indication that the gravitational field (geometry) is quantized, and we do not expect to see any direct evidence in the near future that gravitation is quantized. We expect that gravitation is quantized, however, because otherwise it would be possible in principle to violate the uncertainty principle by observing very accurately the gravitational field of an electron, say.

Such ensembles must occur naturally then, and, of course, they do. In any gravitational interaction there are so many gravitons that we automatically have an ensemble.

We also have to consider ensembles for ordinary quantum interactions. For those, quantum behavior is readily observed, so that ensembles are not automatic. Still, there are several ways to form ensembles, just as in standard quantum mechanics. By a careful choice of an experiment a set of interactions can be chosen whose circumstances are so similar that their amplitudes to occur are the same. This set of interactions, then, forms a suitable ensemble. The point is, that we do not need an ensemble of universes, but merely an ensemble of interactions to observe.

44.5 Summary

The role of quantum cosmology is considered in light of its inability to be treated as an ideal quantum experiment (in which the initial state is prepared, the system is allowed to evolve without interruption, aspects of the final state are measured, and these measurements are repeated on identically prepared systems). It seems that complete knowledge of the Initial state is not necessary in a quantum experiment, although the experiment is usually improved by more knowledge of the initial state, especially for a simple experiment. Our lack of a priori knowledge of the initial state of the universe does not seem in principle to make the study of quantum cosmology useless.

Removing the observer (but not his apparatus) from the quantum cosmology wave function reduces philosophical difficulties concerning the question of "when has a measurement been made?" to the same order as those in ordinary quantum mechanics. By leaving the apparatus in the wave function, amplitudes for measureable quantities can be calculated without philosophical difficulties.

That we have only one universe rather than an ensemble to observe is shown not to cause difficulties. It is necessary only to have an ensemble of interactions that are so similar that they have the same amplitudes for occurring.

Acknowledgments

I would like to thank Julian Barbour, Gary Bornzin, Douglas Gough, Stephen Hawking, Don Page, and David Peterson for helping me clarify my ideas during discussions over the past several years.

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Chapter 45

A semiclassical explanation for the ground state of electromagnetic energy[1]

45.1 Introduction

Is the electromagnetic radiation field intrinsically quantized into packets of energy $\hbar\omega$ because of the nature of the electromagnetic field, or is it simply produced in such units because of it's interaction with quantized matter?

Einstein[128] first suggested that light might be quantized to explain the photoelectric effect. Much later, Lamb and Scully[129] showed that the photoelectric effect could be explained semiclassically (that is, by a classical electromagnetic field interacting with quantized matter).

Loudon[130] and Milonni[131] have reviewed the evidence supporting an intrinsic quantization of the electromagnetic radiation field, and point out that other experiments (in addition to the photoelectric effect) that are often considered evidence for the intrinsic quantization of the electromagnetic field can be explained semiclassically.

Often, the Planck radiation spectrum is considered evidence for the intrinsic quantization of the electromagnetic field. However, Boyer[132] has shown the Planck spectrum can be derived from a classical electromagnetic zero-point radiation, if it is of the correct magnitude. This implies that spontaneous emission by atoms can also be explained semiclassically as stimulated emission by the same classical electromagnetic zero-point radiation.

One apparent lack in Boyer's explanation is a good argument for choosing the magnitude of the classical electromagnetic zero-point radiation. Although Boyer argues that the zero-point radiation should be proportional to frequency to satisfy Lorentz invariance, he chooses the proportionality factor of one-half of Planck's constant to be consistent with the Planck spectrum.

Here, we remedy this lack by suggesting an independent argument for why we would expect classical electromagnetic zero-point radiation to be exactly one-half of a quantum. Although we do not propose that all measurements can be explained by a classical electromagnetic radiation field, our arguments here strengthen the viewpoint that the Planck spectrum and spontaneous emission belong in the category of of experiments that can be explained classically.

Whatever the answer to this question, our understanding of physics is enhanced by classifying experiments according to whether the results of the experiment require a quantized electromagnetic field for its explanation.

We note in passing, the kinds of experiments that cannot be explained by a classical theory of

[1] DRAFT 5 June 1986

radiation. Such experiments are those that measure aspects of the electromagnetic field which do not obey Maxwell's equations. (Equivalently, experiments that show that the electric and magnetic field can not be simultaneously measured with exact precision.) Apparently, the Lamb shift[112, p. 256] falls in this category.

Thus, those experiments that depend on classical statistical fluctuations of the electromagnetic field can be explained by a semi-classical treatment, whereas those experiments that depend on quantum fluctuations of the electromagnetic field (fluctuations that are not constrained to obey Maxwell's equations) require a quantum treatment of radiation for their explanation.

45.2 The ground state for classical radiation

Boyer[132] shows that the Planck radiation spectrum can be explained using classical electrodynamics with classical electromagnetic zero-point radiation. He also shows that

Let us consider experiments that involve interactions between matter and a classical electromagnetic radiation field. In particular, we want to consider interactions that involve atomic transitions in atoms, and the associated absorption or emission of radiation.

The ground state of classical electromagnetic radiation in the presence of matter must be less than one quantum for each mode. Otherwise, it would be possible for an atom to absorb radiation from the ground state, a situation inconsistent with the definition of ground state.

In any experiment involving matter and classical electromagnetic radiation, amount of radiation present within each radiation mode

If we consider statistical fluctuations of the ground state of electromagnetic radiation, then there would be no reason for there to be any restrictions on such fluctuations, except that the ground state for each mode could never exceed one quantum. Thus, we would expect within any ensemble that for each radiation mode the ground state energy would be uniformly distributed between zero and one quantum.

questions

1. Can a background classical EM fluctuating field cause spontaneous emission?

2. Can we get such a field of half a quantum classically?

3. Can we tell whether the background is classical[2] by measurements using matter?

[2] A "Classical" EM field means that the field obeys Maxwell's equations.

Chapter 46

Effective rest mass of a photon in a plasma[1]

abstract

When electromagnetic waves (such as radio waves) propagate through a cold plasma (such as the ionosphere), the dispersion relation is the same as that for free, massive particles (such as electrons). Thus, in this mathematical analogy, solutions to Maxwell's equations have the same form as the solutions to the Dirac equation or to the Klein-Gordon equation. The effective rest mass of a photon in the ionosphere equals hf_N/c^2, where h is Planck's constant, f_N is the plasma frequency, and c is the free-space speed of light.[2] Maybe the rest mass of particles (such as electrons) also results from interactions of the particle with the medium, just as for photons in a plasma.

46.1 Dispersion relation for radio waves in the ionosphere

It is well-known (e.g., Budden, 1961[134], 1985, Ratcliffe, 1962[135]) that the refractive index n for radio waves propagating in a cold plasma (such as the ionosphere) is given by

$$n^2 = 1 - f_N^2/f^2 = 1 - \omega_N^2/\omega^2 \ , \tag{46.1}$$

where f is the radio-wave frequency and f_N is the plasma frequency of the medium. (ω and ω_N are the wave frequency and plasma frequency in radians per second.) The plasma frequency is related to the electron density N of the plasma by

$$\omega_N^2 = 4\pi r_o c^2 N = 4\pi e^2 N/m \ , \tag{46.2}$$

where c is the free-space speed of light, e is the charge of the electron, m is the rest-mass of the electron, and

$$r_o = e^2/(mc^2) \tag{46.3}$$

is the classical electron radius.

The phase velocity of the radio wave is

$$v = c/n = \omega/k \tag{46.4}$$

Combining (46.1) and (46.4) gives

$$\omega^2 - c^2k^2 = \omega_N^2 \ , \tag{46.5}$$

the dispersion relation for radio waves propagating in a cold plasma.

[1] draft - 16 Oct 1986.

[2] I recently (2022) found mention of the effective rest mass of a photon in a plasma in a 1992 paper by Kulsrud and Loeb [133]. I am not surprised. The idea is very obvious.

46.2 Dispersion relation for a free electron

We can compare (46.5) with the dispersion relation for a free electron. We have the usual formula

$$E^2 - c^2p^2 = m^2c^4 \tag{46.6}$$

The energy E of the electron is related to the frequency ω of the electron wave by

$$E = \hbar\omega \ , \tag{46.7}$$

and the momentum p of the electron is related to the wavenumber k of the electron wave by

$$p = \hbar k \ , \tag{46.8}$$

where

$$\hbar = h/(2\pi) \tag{46.9}$$

is the reduced Planck's constant. Substituting (46.7) and (46.8) into (46.6) gives

$$\omega^2 - c^2k^2 = m^2c^4/\hbar^2 \tag{46.10}$$

as the dispersion relation for a free electron wave.

46.3 Effective rest mass of the photon

We notice that (46.5) and (46.10) give the same dispersion relation for radio waves in the ionosphere and and free electron waves if we associate

$$\omega_N^2 = m^2c^4/\hbar^2 \tag{46.11}$$

Or,

$$m = \hbar\omega_N/c^2 = hf_N/c^2 \tag{46.12}$$

Thus, the photons associated with radio waves in the ionosphere have an effective rest mass given by (46.12).

46.4 Interpretation

I should point out that there is no trick here. Radio waves propagating in a cold plasma really do have the same dispersion relation as free particles, such as electrons. That is, they have the same relation between frequency and wavenumber. Therefore, they will also have the same phase velocity and the same group velocity. Also, both types of waves are isotropic. That is, the phase velocity is independent of direction of propagation.

Just as the minimum energy that an electron may have is the rest energy, a radio wave becomes evanescent in a plasma that has a plasma frequency greater than the radio wave frequency. If radio waves are sent straight up into the ionosphere, then they will actually come to rest at a height where the plasma frequency equals the wave frequency, and will then reverse direction and return to the ground, just as a ball thrown into the air will stop, reverse direction, and return to the ground.

The agreement may be pure coincidence. However, the knowledge that the effective rest mass of the photon results from interaction of the electromagnetic waves with the electrons in the plasma suggests that the rest mass of other particles (such as electron) also results from some kind of (as yet unknown) interaction.

Following this line of reasoning may lead to some insight into the origin of mass. Notice that there are some similarities with this way of thinking and Mach's principle that the inertial properties of matter (and possibly rest mass) are due to some kind of interation of matter with the stars.

The particular form for the dispersion relation depends on the kind of interaction involved. If we had included the effect of the earth's magnetic field on the propagation of radio waves in the ionosphere, the resulting dispersion relation would have been much more complicated than that given by (46.5), and would have been anisotropic. Similarly, if we had considered an electron propagating in an electric or magnetic field rather than a free electron, the resulting dispersion relation would have been much different than that given by (46.10), and would have also been anisotropic.

46.5 Note added in February 2013 - Higgs mechanism

I realize now that I had independently considered an origin of mass for particles that others, including Peter Higgs, had already thought of. Is it possible that the actual mechanism for giving an effective rest mass to particles (the Higgs mechanism) is the same as the mechanism by which photons get mass in a plasma? I will consider that possibility here.

Unfortunately, in my original derivation here, I failed in my notation to make a distinction between mass of the electron and mass of a photon. I shall correct that here, using m_e for the mass of the electron and m_γ for the effective mass of the photon.

Substituting (46.2) into (46.12) gives

$$m_\gamma^2 = \frac{\hbar^2}{c^4}\frac{4\pi e^2}{m_e}N \ . \tag{46.13}$$

If we want to compare this with the Higgs field creating effective rest mass of particles, then we should consider the electron density N as analogous to the strength of the Higgs field, and the ratio of the square of the charge on the electron and the mass of the electron as analogous to the strength with which the Higgs field interacts with the particle in question. Considering $4\pi e^2/m_e$ as the strength with which a photon interacts with a plasma is very reasonable. This concept seems very easy to generalize.

Here, I consider the above mechanism for creating effective mass only for the W and Z bosons, not for any other particles. The arguments for needing a mechanism for creating mass for Fermions does not seem very strong.

We rewrite (46.13) for the Z boson as

$$m_Z^2 = \frac{\hbar^2}{c^4}(\text{weak interaction strength of Z boson with Higgs field})(\text{strength of Higgs field}) \ . \tag{46.14}$$

Similarly for the W bosons,

$$m_W^2 = \frac{\hbar^2}{c^4}(\text{weak interaction strength of W boson with Higgs field})(\text{strength of Higgs field}) \ . \tag{46.15}$$

In the above, I have considered only the weak interaction, even though the W bosons have electric charge. The main argument for doing that is that the Higgs particle (or field) has no electric charge. However, we notice that a plasma is also uncharged although the electrons and ions that make up the plasma have electric charge. Therefore, it might be that for the W boson, there might be an electric contribution to the effective mass.

Suppose the Higgs mechanism is the same as that for photons in a plasma. That is, consider a Higgs field to be a weak plasma. That is, a composite of two parts that have weak charge and have been separated in the same way that a plasma is a partially ionized gas. In that case, the Higgs field could be considered to have a resonant frequency in the same way that a plasma has a resonant frequency, the plasma frequency.

However, now we run into a problem, because in the plasma mechanism, the plasma gives mass to the photon by giving it a rest frequency equal to the plasma frequency. To be able to give two different particles different masses, the Higgs field would need to have two rest frequencies.

Let's go back to how a plasma gives an effective mass to a photon. Photons interact with a plasma in two different ways. First, the attraction of electrons and ions in a plasma by the coulomb force is by exchange of photons. Second, a passing electromagnetic wave accelerates those electrons by the electric field associated with the electromagnetic wave.

If there is something similar going on with the Higgs field, then we must first separate the two parts of the Higgs field. Then, there must be an attraction between those two parts through the weak interaction so that we can have a resonant frequency. To get two resonant frequencies, we could have two forces operating, an electromagnetic interaction in addition to a weak interaction. With two resonant frequencies, we have the possibility of getting different masses for the W and Z bosons. Thus, we rewrite (46.15) as

$$m_W^2 = \frac{\hbar^2}{c^4}(\text{weak \& EM interaction strength of W boson with Higgs field})(\text{strength of Higgs field}) \,. \tag{46.16}$$

Notice that we cannot extend this mechanism to give additional masses (unless we include another interaction, such as the strong interaction) to more particles. Thus, we could not use this mechanism to give mass to the Fermions, as well. However, as I pointed out earlier, the arguments against Fermions having intrinsic mass is weak.

The usual explanation of the Higgs mechanism is a mathematical explanation in terms of SU(2) and U(1) rather than a physical mechanism.

Let's look closer at this. How many kinds of weak charges are there? To help answer that, we notice that the strong force, governed by SU(3), has three color charges plus three anti-color charges, and the electromagnetic interaction, governed by U(1), has one charge plus an anti-charge. I therefore postulate that the weak interaction, governed by SU(2), has two weak charges plus two weak anti-charges. In fact, the weak interaction has two kinds of charges, namely weak isospin and weak hypercharge, although they are related through their electric charge.

Thus, I further postulate, in analogy with a plasma, that the Higgs field is a weak-interaction plasma composed of weak isospin charges separated from weak anti-isospin charges plus weak hypercharge separated from weak anti-hypercharge.

Now, we need to find out which kinds of weak charge with which the W and Z bosons interact. W_1 and W_2 are eigenstates of weak isospin, but W_+ and W_- are not; they act as raising and lowering operators instead. However, W_1 and W_2 are not eigenfunctions of electric charge. The Z boson has zero electric charge, zero weak isospin, and zero weak hypercharge. So, probably W interacts with weak isospin and Z interacts with weak hypercharge. Yes, I checked on this. The quote is "Weak hypercharge is the generator of the U(1) component of the electroweak gauge group."

There is still the question of what we mean by a Higgs particle. To try to answer that, we could try to find out what we mean by a plasma particle. (Plasmon?) I don't know. There are further considerations, however. The usual calculation of the dispersion relation for electromagnetic waves in a plasma is classical. It treats the plasma as a medium that has an intrinsic frequency, the plasma frequency, for electrons to oscillate. As an electromagnetic waves comes by, the electric field of the wave drives the electrons to oscillate at the wave frequency. Energy is transferred from the wave to the electrons. As the wave passes by, it takes the energy back, but suffers a phase shift, whose final result is the usual dispersion relation. However, as we make the wave weaker and weaker, we will start to get effects from the quantum nature of the wave. Energy is exchanged between the wave and the electrons only in single quanta. When the number of quanta becomes small, then we may be able to discover what we mean by a plasma particle.

Let's look again at the question of whether the Higgs field can or needs to give mass to Fermions. The usual argument for needing to have the intrinsic mass of Fermions to be zero has to do with left- and right-handedness. It has been deduced from parity measurements with the weak interaction that W bosons interact only with left-handed particles or right-handed antiparticles. If these particles have mass, then it is possible to find a frame in which a left-handed particle is right handed. Therefore, these particles cannot have mass. However, this may be a weak argument because it may be important only in the frame of the interaction what the handedness is. (I am starting to be a little unsure of this conclusion now.)

However, I have now discovered that the parity measurements have led to labeling the weak isospin of particles according to the handedness of the particles. That is, the weak isospin of right-handed Fermions and left-handed anti-Fermions has been defined to be zero. In this case, if the Fermions have mass, then the weak isospin of a particle will depend on the frame of reference. That does seem to be a problem. However, defining the intrinsic mass of the particles to be zero and requiring the Higgs field to give their observed mass does not seem to solve the problem. Once the particles have mass, whether it is intrinsic or effective, we still have the possibility of changing handedness of a particle by a change in reference frame.

I have the feeling here that the theorists have begun to dig themselves into a hole from which they cannot get out. Instead of continuing to dig deeper, it might be better to find something completely new.

I admit that my reason for looking into this was to see if the correct Higgs mechanism for giving mass was the same as the mechanism of a plasma giving effective mass to photons. If so, then we need an analogous "plasma frequency" for each mass. Since there are only two kinds of weak charges, namely weak isospin and weak hypercharge, that mechanism can give only two values for an effective mass, namely for the W and Z bosons.

46.6 Note added in 2008 - Comments by David Peterson

I gave a copy of this essay to David Peterson in October 1986, and he suggested some appropriate references on 23 October 1986.

These are: Schechter (1986)[136], Weinberg (1986)[137], Shupe (1985)[138], Batakis (1983)[139], Moriyasu (1983)[140, p. 96], Higgs (1964)[141, 142], Higgs (1966)[143], Veltman (1986)[144], a 1983 article in the CERN Courier[145], Kaul (1983)[146], Wignall (1985)[147], Horak (1970)[148], Jackson (1962)[149, pp. 222, 226, 336], Gingras (1980)[150], and Anderson (1963)[151].

Chapter 47

What is wrong with the geodesic equation?[1]

The geodesic equation does not act like the kind of force equation I would expect from looking at the Lorentz force equation. That is, it is not the product of a current with some derivative of some kind of potential. Supposedly, $T^{\mu\nu}$ represents the source for gravitational fields, and $g_{\mu\nu}$ represents the potential. So, the force law should look like $T^{\mu\nu}$ times some derivative of $g_{\mu\nu}$. This seems to be true only in the rest frame of the body. In other frames, the force law does not take that form. In one representation, it is $\nabla \cdot T^{\mu\nu}$. That looks more like a derivative of the current rather than a current times a derivative of a potential.

So, maybe I should start in the frame of the body and try to transform to an arbitrary frame. Γ represents a derivative of $g_{\mu\nu}$, but it is not a tensor. Can I define a tensor that is the same as Γ in the frame of the body? That just might work. In the geodesic equation, the term for the covariant derivative that is a product of Γ times $T^{\mu\nu}$ does have the desired form for a force law. It is only the $\ddot{x}$ term that does not follow the proper form.

On the other hand, what is wrong with an $\ddot{x}$ term? In one form of writing the geodesic equation, we have $T^{\mu\nu}_{;\nu} = 0 = T^{\mu\nu}_{,\nu} + T^{\epsilon\nu}\Gamma^{\mu}{}_{\epsilon\nu} + T^{\mu\epsilon}\Gamma^{\nu}{}_{\epsilon\nu} = T^{\mu\nu}_{,\nu} + T_{\alpha\beta}\Gamma^{\mu\alpha\beta} + T^{\mu\epsilon}\Gamma^{\nu}{}_{\epsilon\nu} = 0$. Maybe in this equation, $T_{\alpha\beta}$ is a current, but $T^{\mu\nu}$ is a potential.

It seems that $T^{\mu\nu}$ in the $T^{\mu\nu}_{,\nu}$ term must be a potential instead of a current, because it is potentials that we take the derivative of, not currents in the force equation. But we need something more concrete than that.

Representing gravitation by geometry forces a choice of units in which inertial mass and passive gravitational mass are equal. Thus, within the geometrical framework it is not possible to to discuss the distinction between inertial and passive gravitational mass. I don't think allowing the gravitational constant G to vary will help, because there is still active gravitational mass to consider.

For a spin 2 gravitational system, we have more that just mass; we have tensor currents. In Einstein's system, $T^{\mu\nu}$ represent the tensor currents. To replace inertial mass, we must have a tensor potential. The metric $g_{\mu\nu}$ seems to do part of the job, but not the whole job. The $T^{\mu\nu}$ in the $T^{\mu\nu}_{,\nu}$ term in the geodesic equation seems to do the rest of the job. Thus, this $T^{\mu\nu}$ acts more like a potential than like a current.

How can I argue conclusively that this $T^{\mu\nu}$ must be a potential rather than a current?

How to separate geometry from gravitation: Transform to some arbitrary coordinate system which satisfies no pre-conceived conditions. For example, $G = T$ is not necessarily satisfied for that coordinate system. Also, the geodesic equation is not necessarily satisfied for that coordinate

[1] 8 December 1989, Corrected 1 May 2015

system, and the metric $g_{\mu\nu}$ does not necessarily give distances in that coordinate system. But, there will be a geometry in which all of these things are satisfied, so we must express all of those things in terms of this arbitrary coordinate system.

For the geodesic equation locally, I could choose a coordinate system that coincides with the geometry except that we have $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$. I hope that will work.

Chapter 48

Inertia of a charged particle in a charged universe from electromagnetic induction[1]

abstract

Try to construct a pure electromagnetic cosmology. Achieve at least obtaining inertia from electromagnetic induction in a charged universe.

48.1 Introduction

Of the 4 known fundamental forces of nature, the gravitational interaction seems special because it forms a background arena on which the other forces interact. That is, solutions of the Einstein field equations give a geometry as such an arena. In fact, there are cosmological solutions to the field equations that involve only gravitation (if we ignore details in the stress-energy tensor) and no other interaction. We know of no cosmologies that involve only one interaction except for the gravitational interaction.

This seems to set gravitation apart from the other forces. Is that because gravitation is the only metric theory? Does it require a metric theory of an interaction to form a cosmology? If so, does that require that in any formulation of the laws of nature at least one of the interactions must be a metric theory?

To shed some light on these issues, I have tried to form a cosmology using only a purely electromagnetic interaction. I was not completely successful, but I did manage to get inertia from an electromagnetic induction interaction in a charged universe. It seems to require a metric theory of interaction to form a cosmology.

Sciama (1953)[11] used an electromagnetic analogy to derive inertia from an induction interaction with a universe that has a matter density to show a possible mechanism for Mach's principle (Mach, 1911, 1904)[120, 102, 1, 121, 122, 15]. Davidson (1957)[14] pointed out that General Relativity already included an induction interaction of the type needed, and Sciama, Waylen, and Gilman (1969)[16] derived an integral formulation of Einstein's field equations to show a possible mechanism for Mach's principle.

We can apply Sciama's (1953)[11] calculation directly to the motion of charges in a charged universe rather than as an analogy. In this way, we can derive inertia of charged particles in such a universe from electromagnetic induction.

[1] 8 December 1989, revised, 29 August 2022

Because the inertia in these examples arises from an interaction with all of the matter in the universe, and therefore involves an integration over the whole universe, there must be a way to keep the calculations finite. Sciama (1953)[11] assumed an expanding universe, and cut off the calculations at the distance where bodies would move with the speed of light if the expansion were extended that far as a simple way to get an approximately correct result.

48.2 Electromagnetic inertia

We follow the calculation of Sciama (1953)[11], except that we are dealing here with pure electromagnetic interactions. We assume a background flat Minkowski space-time, upon which we make electromagnetic calculations. We shall assume standard Maxwell electromagnetic theory applies.

We assume otherwise that the universe is roughly as we observe it to be except that the bodies in the universe have a small positive electric charge so that on the average there are N excess positive electric charges (equal in magnitude to the charge on the electron) per cubic centimeter. We expect that N will be a very small number. Because the electromagnetic interaction is so much stronger than gravitational, we should not need a large charge density to achieve a sizable induced inertial mass on charged bodies.

We assume that any charged bodies contain also their intrinsic (non-electromagnetic inertia), but we are looking for solutions in which the charge density of the universe is large enough that the inertia from electromagnetic induction is much larger than the intrinsic inertia.

Let us further assume, for simplicity, that each particle or body is identical (we shall allow test charges, however), with charge e (positive) and intrinsic mass m. Let us now proceed to follow Sciama's (1953)[11] derivation of an electromagnetically induced inertial mass for each of these charged particles from its interaction with all of the other charges (with a few changes, however).

We shall choose one of these charged particles as the origin of our coordinate system. The scalar electromagnetic potential due to all of the other charges in the universe will be

$$\phi = \int \frac{Ne}{r} d^3r = \int \frac{4\pi r^2 Nedr}{r} = 4\pi Ne \int rdr \ . \tag{48.1}$$

The domain of integration, is over all charges in the universe, but we shall not be explicit about that yet.

Equation (48.1) assumes that the range of the electromagnetic interaction is infinite, corresponding to a zero rest mass for the photon. However, a photon propagating in a charged medium (such as radio waves in the ionosphere) has a dispersion relation that is the same as that of a massive particle.[2] Thus, if the photon has an effective rest mass, then the correct potential is not (48.1), but a Yukawa potential.

$$\phi = \int e^{-\frac{r}{r_0}} \frac{Ne}{r} d^3r = \int_0^\infty e^{-\frac{r}{r_0}} \frac{4\pi r^2 Nedr}{r} = 4\pi Ne \int_0^\infty e^{-\frac{r}{r_0}} rdr \ , \tag{48.2}$$

where r_0 is the reduced effective Compton wavelength of the photon.

Suppose now that our test particle (which is the origin of our coordinate system) is moving to the right with velocity v. Then the rest of the universe is moving to the left with velocity v. That is, the average velocity of the universe is $-v$. We can perform a Lorentz transformation on (48.2) to calculate the scalar and vector potential in this moving coordinate system. This gives

$$\phi = \int \gamma e^{-\frac{r}{r_0}} \frac{Ne}{r} d^3r = \int_0^\infty \gamma e^{-\frac{r}{r_0}} \frac{4\pi r^2 Nedr}{r} = 4\pi Ne\gamma \int_0^\infty e^{-\frac{r}{r_0}} rdr = 4\pi Ne\gamma r_0^2 \tag{48.3}$$

[2]This is an error. A charged medium is not the same as a plasma, which is neutral because the positive ions balance the charge of the electrons. A photon does not acquire an effective rest mass in a charged medium as it does in a plasma. The part of the following equations that depend on a rest mass and a Compton wavelength for the photon are wrong. Instead, the cut-off for the integration will come at the Hubble radius.

and

$$\mathbf{A} = -\int \frac{\mathbf{v}}{c}\gamma e^{-\frac{r}{r_0}}\frac{Ne}{r}d^3r = -\int_0^\infty \frac{\mathbf{v}}{c}\gamma e^{-\frac{r}{r_0}}\frac{4\pi r^2 Nedr}{r} = -\frac{\mathbf{v}}{c}4\pi Ne\gamma\int_0^\infty e^{-\frac{r}{r_0}}rdr = -\frac{\mathbf{v}}{c}\phi \ , \quad (48.4)$$

where

$$\gamma = (1 - v^2/c^2)^{-\frac{1}{2}} \quad (48.5)$$

is the usual relativistic factor. For the calculations that follow, however, we shall neglect relativistic effects, and take γ equal to unity.

We can now calculate the electric and magnetic field in the frame of the particle, and then calculate the force on the particle. However, the magnetic force on the particle in its own reference frame is zero, because the velocity of the particle is zero in its own reference frame (We assume zero magnetic moment for these particles). Therefore, it is sufficient to calculate the electric field only.

The electric field is

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\nabla\phi + \frac{\phi}{c^2}\frac{\partial\mathbf{v}}{\partial t} = \frac{\phi}{c^2}\frac{\partial\mathbf{v}}{\partial t} = \frac{4\pi Ner_0^2}{c^2}\frac{\partial\mathbf{v}}{\partial t} \ , \quad (48.6)$$

where the gradient of the scalar potential is zero, because we are considering a uniform universe. Therefore, the force on the particle is

$$\mathbf{F} = q\mathbf{E} = \frac{4\pi Neqr_0^2}{c^2}\frac{\partial\mathbf{v}}{\partial t} = 0 \quad (48.7)$$

thus

$$\frac{\partial\mathbf{v}}{\partial t} = 0 \ , \quad (48.8)$$

which gives Newton's law of inertia, with some kind of electromagnetically induced inertial mass.

To find out the value for that electromagnetic mass, we introduce another test charge Q, close enough to the charge at the origin of our coordinate system that it's contribution must be considered in addition to the uniform charge of the universe. The scalar and vector potentials are now (neglecting the relativistic factor γ)

$$\phi = 4\pi Ner_0^2 + \frac{Q}{r} \quad (48.9)$$

and

$$\mathbf{A} = -\frac{\mathbf{v}}{c}\phi = -\frac{\mathbf{v}}{c}(4\pi Ner_0^2 + \frac{Q}{r}) \approx -4\pi Ner_0^2\frac{\mathbf{v}}{c} \quad (48.10)$$

Therefore, the electric field is now

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\frac{Q}{r^2} + \frac{4\pi Ner_0^2}{c^2}\frac{d\mathbf{v}}{dt} \quad (48.11)$$

so that the total force is

$$\mathbf{F} = q\mathbf{E} = -\frac{Qq}{r^2} + \frac{4\pi Neqr_0^2}{c^2}\frac{d\mathbf{v}}{dt} = 0 \quad (48.12)$$

Comparing (48.12) with the usual formula for the motion of a charged particle q with mass m

$$m\frac{d\mathbf{v}}{dt} = \frac{Qq}{r^2} \quad (48.13)$$

shows that the particle in (48.12) has an effective electromagnetically induced inertial mass

$$m_{EM} = \frac{4\pi Neqr_0^2}{c^2} = \frac{4\pi Ne^2r_0^2}{c^2} \ , \quad (48.14)$$

where the second equality applies if we take all particles to have charge e.

As noted in the footnote above, a photon does not have an effective rest mass or Compton wavelength in a charged medium as it does in a plasma. Therefore, the correct cut-off in the above integrations is at the Hubble radius, not the Compton wavelength. Therefore, (48.14) should be replaced by

$$m_{EM} = \frac{2\pi NeqR^2}{c^2} = \frac{2\pi Ne^2R^2}{c^2} \ , \tag{48.15}$$

where R is the Hubble radius.

48.3 Interpretation

As in Sciama's (1953)[11] calculations, in a positively charged universe, local positive charges will accelerate toward one another. The same will be true for negative charges in a negatively charged universe. In general, the acceleration of a particular charged particle is independent of it's charge (as long as it is charged). Thus, in a positively charged universe, any charged particle will accelerate toward a near-by positively charged particle, and accelerate away from a near-by negatively charged particle. Thus, two negatively charged particles in a positively charged universe will accelerate away from each other.

The case of a pair of charged particles of opposite charge is more interesting. In a positively charged universe, the negatively charged particle will accelerate toward the positively charged particle, and the positively charged particle will accelerate away from the negatively charged particle. What actually happens is that the two particles accelerate in the same direction with the positive particle in the lead. (In a negatively charged universe, the negative particle would be in the lead.)

At first, this seems like strange behavior. In particular, it seems to contradict conservation of momentum. However, in the electromagnetic case, current is the analog of momentum. The two charged particles carry opposite momentum (current) because they have opposite charges, even though they are traveling in the same direction.

48.4 Including gravitation

There will be gravitational interactions among the photons and the charged particles because of their associated energy and momentum. A consistent account of this effect can be made by writing the stress-energy tensor for the electromagnetic field and the charged particles, and solving Einstein's equations. Qualitatively, this procedure should lead to results similar to those of the previous section, so will not be pursued in further detail here.

We shall consider one feature, however. Sciama, Waylen, and Gilman (1969)[16] developed an integral formulation of Einstein's field equations. They showed that each component of the metric tensor at one point in space-time could be represented as an integration over all of space-time of the stress-energy tensor with a propagator that coupled the 16 components of the metric tensor. The propagator obeyed a coupled Klein-Gordon type equation with a tensor mass term given by components of the Riemann tensor.

Clearly, this propagator represents gravitons as propagators of the gravitational field. However, these gravitons have an effective (induced) mass because they are propagating in a medium (the universe). Thus, the effective range of the gravitational interaction is no longer infinite, but equal to the effective Compton wavelength of these gravitons. Because the effective mass of the gravitons comes from the components of the Riemann tensor, the effective square of the Compton wavelength will be on the order of the inverse of the local curvature. That is, for a Friedmann-LeMaître-Robertson-Walker space time, the Compton wavelength will be roughly equal to the Hubble radius. Therefore, the Hubble radius can be used as a cut off in any integration of induced inertial mass.

48.5 Discussion

Let us consider how well we can answer some of the questions brought up in the introduction. Is a metric theory required to form a cosmology? The answer is no. As pointed out by Weinberg (1972, Section 6.9)[13], it is not necessary to treat gravitation as a geometry (the geometric analogy, as he refers to it). Although gravitation can be treated as a metric theory, it is not necessary to interpret it as a geometry.

Is gravitation set apart from the other interactions because it is a metric theory? No. Is gravitation set apart from the other interactions in that it is the only interaction that can form a cosmology? Yes, because any other interaction will pro-duce energy and momentum, which are sources of gravitation. Thus, gravitation is the only interaction which can exist without the others.

In the case of the cosmology constructed here with a charged universe, the electromagnetic interaction was the dominant interaction. Although gravitation could be considered small, it could not be neglected because neglecting gravitation led to a situation that was very sensitive to small changes.

We need to consider what constitutes a cosmology in the Machian sense. In the example presented here, we showed how electromagnetic induction could induce inertial mass in charged particles. In addition, it produced inertial frames for charged particles that do not accelerate with respect to the average charge distribution. In addition, this charged universe induced an effective rest mass for photons (which do not have charge). It is not clear whether this charged universe also generated inertial frames for the photons. I do not think so. The equations we assumed for electromagnetism assumed that inertial frames already existed. Possibly, the charge distribution altered the inertial frames for the photons.

Suppose we knew how to implement Mach's principle in a consistent way. Then, a mass distribution would determine inertial frames for energy and momentum, and a charge distribution would determine inertial frames for charged particles. These two frames would not have to coincide. That is, they could have relative accelerations. Then, a charged, massive particle would experience a force proportional to its acceleration with respect to each of these frames. Such a situation would place gravitation and electromagnetism on a more equal footing.

Even massless particles (such as the photon) experience inertia, in that they propagate in straight lines at constant speed in inertial frames (in the absence of other forces). Thus, inertial mass is not the only measure of inertia.

The question of what happens in a sparsely populated universe, where inertial induction (either for charges or masses) is small may lead to some insight into the answer to these questions.

I realize that I am rambling here. I'll fix it up after I get my ideas organized better.

Chapter 49

Quantum Selection of a Classical Cosmology[1]

abstract

A mechanism is proposed to meet two challenges implicitly left for us by Ernst Mach:

1. Formulate physical law completely in terms of relative configurations.

2. Give a reasonable explanation for the observation that inertial frames seem to have no rotation relative to the bulk of observable matter in the universe.

Although the first challenge is met if the action is expressed in terms of kinematically observable quantities, it is not possible to meet the second challenge within a framework of classical gravitation unless one gives up independent degrees of freedom for the gravitational field. It is possible to meet the second challenge within a framework of quantum cosmology, however, without giving up independent degrees of freedom for the gravitational field. Using a saddlepoint approximation (which is valid for our universe), initial conditions for the classical field equations are determined from the requirement that the action be stationary with respect to variation of the 3-geometry on an initial spacelike hypersurface. Observable classical 4-geometries are restricted to those whose actions differ from the action at the saddlepoint by an amount less than Planck's constant. Such 4-geometries have inertial frames that have very small rotation relative to the matter distribution.

49.1 Introduction

Ernst Mach (1872, 1933)[120, 102, 1, 121, 122, 15] criticized Newton's formulation of mechanics and gravitation because it involved quantities (such as inertial frames) that are not kinematically observable. That inertial forces (having no apparent origin) exist in frames that rotate or accelerate relative to certain "inertial frames" is no more than a definition of inertial frame since there is no other way to determine such frames. That inertial frames seem not to rotate or accelerate relative to the "fixed stars" has no operational use in mechanics unless there is a specific connection between inertial forces and the rotation and acceleration of the stars.

Thus, Mach implicitly presented a challenge for us that has two parts:

1. Formulate physical law in terms of relative configurations.

[1] 18 April 1994. Presented at (and submitted to the Proceedings of) the conference on Mach's Principle: "From Newton's Bucket to Quantum Gravity", held at Tübingen, Germany, July 26-30, 1993, File: q-cos2.tex

2. Give a reasonable explanation for the observation that inertial frames seem to have no rotation relative to the bulk of observable matter in the universe.

In spite of many tries (too numerous to mention here), no generally accepted solution to Mach's challenge that "our law of inertia is wrongly expressed" has been found. Further, in spite of Mach's ideas having guided Einstein in his formulation of General Relativity and been given the name "Mach's Principle" by Einstein (1918)[152], there have been doubts about whether General relativity satisfies either of Mach's two challenges.

Barbour (1993)[153] has probably made the most thorough investigation into the first of Mach's two challenges. He argues that General Relativity is "perfectly Machian" in that Einstein's equations can be expressed in terms of relative configurations of particles and fields. He puts forth convincing evidence that Einstein's field equations are already expressed in terms of kinematically observable quantities. More generally, his work shows how one can specify fields (and the geometry in particular) in terms of kinematically observable quantities. Barbour has certainly made the most progress in solving the first of Mach's two challenges, and very likely has succeeded.

The problem with accepting General Relativity as "perfectly Machian," however, is that there are solutions of Einstein's field equations in which there is relative rotation of matter and inertial frames. The problem arises because of the arbitrariness of initial and boundary conditions. Thus, it would be difficult to explain the very existence of our own universe because of the improbable initial conditions that would yield the apparently small or zero relative rotation of matter and inertial frames. Thus, we are still left with Mach's second challenge.

There have been attempts to solve Mach's second challenge by somehow restricting the allowable cosmological solutions. These attempts have usually fallen into two categories. In the first category have been attempts to find a formulation in which solutions would automatically satisfy Mach's second challenge. In the second category have been attempts to specify initial and boundary conditions for Einstein's field equations.

In the first category, Lynden-Bell (1992)[5] presents a formulation that would have probably satisfied Mach at first glance, because it expresses dynamics in terms of the relative configuration of the particles and gives no relative rotation of matter and inertial frames. A related approach uses integral formulations of General Relativity (Al'tshuler 1967[154]; Lynden-Bell 1967[155]; Sciama, Waylen, and Gilman 1969[16]; Gilman 1970[156]; Raine 1975[109]). However, because formulations such as these contain implicit initial and boundary conditions on the gravitational field, we see that the first category differs from the second only in whether the initial and boundary conditions are implicit or explicit.

In the second category, initial and boundary conditions enter more explicitly. Thus, Wheeler (1964)[115] proposed using Mach's principle as a criterion to select solutions of Einstein's field equations. Making such initial and boundary conditions part of the laws of physics, however, would remove independent degrees of freedom from the gravitational field.

It can be argued (Earman 1993[157]; Kuchař 1993[158]; Raine 1993[159]; and others: this conference) that the gravitational field should be considered as fundamental as matter. For example, should we consider the energy in gravitational waves a part of the matter? Pair creation in a strong gravitational field seems like an example where the gravitational field is primary rather than secondary to matter. The difficulty of deciding whether matter or the gravitational field is primary suggests that we keep independent degrees of freedom for the gravitational field, and allow arbitrary initial conditions. To meet Mach's second challenge, therefore, requires that we find a physical mechanism for selecting the initial or boundary conditions.

The failure to find such a mechanism in classical gravitation that does not remove independent degrees of freedom from the gravitational field suggests that we look at quantum cosmology. In that case, all initial conditions on the 3-geometry are allowed through a wave function over 3-geometries specified on an initial spacelike hypersurface. Then we look for a way in which our observable nearly classical cosmology might be recovered in some approximation.

49.2 An example from ordinary wave mechanics

To help explain the ideas that follow, we first consider elementary wave mechanics. If we have an initial single-particle state specified by an initial wave function $< x_1, t_1|\psi >$ at time t_1 then the wave function $< x_2, t_2|\psi >$ at time t_2 is (e.g., Feynman and Hibbs 1965, p. 57)[21, p. 57]

$$< x_2, t_2|\psi >= \int_{-\infty}^{\infty} < x_2, t_2|x_1, t_1 >< x_1, t_1|\psi > dx_1, \tag{49.1}$$

where $< x_2, t_2|x_1, t_1 >$ is the propagator for the particle to go from (x_1, t_1) to (x_2, t_2). We consider the case where the semiclassical approximation for the propagator is valid. That is, (e.g., Feynman and Hibbs 1965, p. 60)[21, p. 60]

$$< x_2, t_2|x_1, t_1 >\approx f(t_1, t_2) e^{\frac{i}{\hbar} I_{cl}[x_2,t_2;x_1,t_1]}, \tag{49.2}$$

where $I_{cl}[x_2, t_2; x_1, t_1]$ is the action calculated along the classical path from (x_1, t_1) to (x_2, t_2). Thus, (49.1) becomes

$$< x_2, t_2|\psi >\approx f(t_1, t_2) \int_{-\infty}^{\infty} e^{\frac{i}{\hbar} I_{cl}[x_2,t_2;x_1,t_1]} < x_1, t_1|\psi > dx_1. \tag{49.3}$$

Notice that because of the initial wave function we have an infinite number of classical paths contributing to each value of the final wave function.

There are two cases to consider. In the first, I_{cl} is not a sharply peaked function of x_1. In that case, there will be contributions to the wave function at t_2 from classical paths that differ greatly from each other.

In the second case, which we now consider, I_{cl} is sharply peaked about some value of x_1, say x_{sp}. That is, we have

$$\frac{\partial}{\partial x_1} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}} = 0. \tag{49.4}$$

Thus, x_{sp} is a saddlepoint of the integral (49.3), and significant contributions to the integral are limited to values of x_1 such that

$$|x_1 - x_{sp}|^2 < \left| \frac{2\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}}} \right|. \tag{49.5}$$

If $< x_1, t_1|\psi >$ is nearly constant over that range, then we can take it outside of the integral. A saddlepoint evaluation of the integral then gives

$$\begin{aligned} < x_2, t_2|\psi > \quad &\approx \quad f(t_1, t_2) < x_{sp}, t_1|\psi > \\ & \left[\frac{2\pi i\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}}} \right]^{1/2} e^{\frac{i}{\hbar} I_{cl}[x_2,t_2;x_{sp},t_1]}. \end{aligned} \tag{49.6}$$

We notice from (49.4) that the momentum at t_1 at the saddlepoint is zero. That is,

$$p_1|_{x_1=x_{sp}} = 0. \tag{49.7}$$

However, for the paths that contribute significantly to the integral in (49.3), there is a range of values of the momentum, namely

$$|p_1^2| < 2\hbar \left| \frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}} \right|, \tag{49.8}$$

consistent with (49.5) and the uncertainty relation.

As a check, using a special case, we consider the free-particle propagator (e.g., Feynman and Hibbs 1965, p. 42)[21, p. 42]

$$< x_2, t_2|x_1, t_1 >= \left[\frac{m}{2\pi i\hbar(t_2 - t_1)}\right]^{1/2} exp\left[\frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)}\right], \tag{49.9}$$

and we choose

$$< x_1, t_1|\psi >=< A, t_1|\psi > exp[-B(x_1 - A)^2] \tag{49.10}$$

to represent a broad initial wave function. For this case, the integral in (49.1) or (49.3) can be evaluated exactly to give

$$< x_2, t_2|\psi >=< A, t_1|\psi > \left[1 + \frac{2B\hbar(t_2 - t_1)}{im}\right]^{-1/2} exp\left[\frac{-B(x_2 - A)^2}{1 - \frac{2B\hbar(t_2-t_1)}{im}}\right]. \tag{49.11}$$

The condition that the initial wave function is slowly varying is now

$$|B| \ll \left|\frac{m}{2\hbar(t_2 - t_1)}\right|, \tag{49.12}$$

so that (49.11) is approximately

$$< x_2, t_2|\psi >=< x_2, t_1|\psi > \tag{49.13}$$

in agreement with (49.6), since

$$I[x_2, t_2; x_1, t_1] = I_{cl}[x_2, t_2; x_1, t_1] = \frac{m}{2}\frac{(x_2 - x_1)^2}{t_2 - t_1}, \tag{49.14}$$

and

$$x_{sp} = x_2. \tag{49.15}$$

Notice that it is the sharply peaked action that determines which classical paths in (49.3) dominate the integral in this case, not the maximum of the initial wave function.

The calculations can clearly be generalized to two or three dimensions. The main point is that whenever the initial wave function is broad, many paths may contribute to the wave function in the final state, even when a semiclassical approximation is valid for the propagator. In that case, many classical paths may contribute significantly to the wave function in the final state.

When the classical action is sharply peaked as a function of the coordinates of the initial state, however, only a narrow range of classical paths contribute significantly to the wave function in the final state. This is thus a mechanism for selecting classical paths in wave mechanics. As we shall argue in the next sections, this principle has broader application.

49.3 Quantum cosmology

In the case of quantum cosmology, we have a formula analogous to (49.1) to give the wave function over 3-geometries g_2 and matter fields ϕ_2 on a 3-dimensional hypersurface S_2.

$$< g_2, \phi_2, S_2|\psi >= \int < g_2, \phi_2, S_2|g_1, \phi_1, S_1 >< g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1), \tag{49.16}$$

where $< g_1, \phi_1, S_1|\psi >$ is the wave function over 3-geometries g_1 and matter fields ϕ_1 on a 3-dimensional hypersurface S_1, and $< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >$ is the amplitude to go from a state with

3-geometry g_1 and matter fields ϕ_1 on a surface S_1 to a state with 3-geometry g_2 and matter fields ϕ_2 on a surface S_2 (Hawking 1979)[123]. $D(g_1)$ and $D(\phi_1)$ are the measures on the 3-geometry and matter fields. The integration is over all initial 3-geometries g_1 and matter fields ϕ_1 for which the integral is defined.

We assume that the initial wave function is broad. This is the most reasonable assumption without any knowledge of why it might be peaked.

49.4 Semiclassical approximation

As in section 2, we want to consider the case where the semiclassical approximation for the propagator is valid. That is, (e.g., Gerlach 1969)[160]

$$< g_2, \phi_2, S_2 | g_1, \phi_1, S_1 > \approx f(g_2, \phi_2, S_2; g_1, \phi_1, S_1) e^{\frac{i}{\hbar} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]}, \tag{49.17}$$

where the function outside of the exponential is a slowly varying function. Substituting (49.17) into (49.16) gives

$$\begin{aligned} < g_2, \phi_2, S_2 | \psi > \ &= \int f(g_2, \phi_2, S_2; g_1, \phi_1, S_1) e^{\frac{i}{\hbar} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]} \\ &\quad < g_1, \phi_1, S_1 | \psi > D(g_1) D(\phi_1). \end{aligned} \tag{49.18}$$

Each value of the integrand in (49.18) corresponds to one classical 4-geometry. As in (49.3), because the initial wave function can be broad, there will be an infinite number of classical 4-geometries that contribute to each value of the final wave function. Here, however, we do not have only one single integration, but an infinite number of integrations, because the integration is carried out over all possible 3-geometries and all matter fields on the initial surface.

In the simple example in Section 2, there were two cases to consider for the single integration being carried out. In the first case, the classical action was not a sharply peaked function. In the second case, the classical action was a sharply peaked function so that a saddlepoint approximation could be applied to the integration. Following that strategy, we would need to consider those two cases for each of the infinite number of integrations in (49.18).

Here, however, we consider two cases. In the first, I_{cl} is not a sharply peaked function of the 3-geometry g_1 for at least one of the infinite number of integrations in (49.18). In that case, there will be contributions to the wave function on S_2 from classical 4-geometries that differ significantly from each other. We consider this case in a later section.

In the second case, I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (49.18). We consider this case in the following section.

49.5 Saddlepoint approximation for the integral over initial states

We consider the case here where I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (49.18). In that case, we can formally make the saddlepoint approximation for each of the integrations in (49.18). In analogy with (49.4), we have the saddlepoint condition

$$\frac{\partial}{\partial g_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] \bigg|_{g_1 = g_{sp}} = 0 \tag{49.19}$$

and

$$\frac{\partial}{\partial \phi_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] \bigg|_{\phi_1 = \phi_{sp}} = 0 \tag{49.20}$$

We consider the case where there is only one solution to the saddlepoint conditions (49.19) and (49.20). In that case, (49.19) selects a single classical 4-geometry. However, there will be a range of classical 4-geometries in the neighborhood that contribute significantly to the integral in (49.18). These are determined by

$$|I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] - I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]| < \hbar. \tag{49.21}$$

We can formally write the saddlepoint approximation to the integration in (49.18) as

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > &= f(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) < g_{sp}, \phi_{sp}, S_1|\psi > \\ & f_1(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) e^{\frac{i}{\hbar} I_{cl}[g_2,\phi_2,S_2;g_{sp},\phi_{sp},S_1]} \end{aligned} \tag{49.22}$$

where the exponential is assumed to dominate the behavior in (49.22). The main result, however, is that classical 4-geometries that contribute significantly to (49.22) lie within a narrow range specified by (49.21).

Equation (49.19) requires that the momentum canonical to the initial 3-geometry for the classical 4-geometry at the saddlepoint be zero. That is

$$\pi^{ij}\Big|_{g_1 = g_{sp}} = 0. \tag{49.23}$$

(The extrinsic curvature on S_1 will therefore also be zero at the saddlepoint.) However, there will be a range of initial canonical momenta and a range of initial 3-geometries corresponding to the range of classical 4-geometries that satisfy (49.21), so that the uncertainty relations between initial 3-geometries and their canonical momenta are satisfied.

Whether there is a narrow or broad range of classical 4-geometries that satisfy (49.21) depends on the second derivative of the action with respect to the initial 3-geometry.

49.6 Spatially homogeneous spacetimes

The integration in (49.18) is an integration over functions g_1 and ϕ_1 defined on S_1. In that sense, it is similar to a path integral. For example, there are six independent functions that define g_1. As in the integration for a path integral, there are approximations that can be made to reduce the number of integrations that must be performed. For example, in the single-scattering approximation, one considers a subset of paths that are straight lines broken only at the scattering point. That reduces the path integral to three integrations in the case of 3-dimensional space.

Here, we want to consider matter distributions similar to that observed, at least for the large scale. Thus, we want to restrict the integration in (49.18) to classical spatially homogeneous 4-geometries that have a homogeneous matter distribution in calculating the classical action in the exponential. The integration in (49.18) would then be over the 3-geometries that form the boundary of those 4-geometries on S_1.

As an example, we shall use Einstein's General Relativity for the classical 4-geometries, but the same calculations could be done for other classical gravitational theories, in case it turns out that General Relativity is not the correct theory of gravity. Thus, we want to consider the integration in (49.18) in which the classical 4-geometries used to calculate the action in the exponential are restricted to Bianchi cosmologies.

The appropriate calculation would be to consider the most general Bianchi model, with all of the parameters that describe that model, and carry out the integration over all of those parameters. We notice that the Bianchi parameters (which are time independent) define the initial three geometry, and therefore are valid integration variables in (49.18). On the other hand, if it is suspected that the saddlepoint for the integration will correspond to the Friedmann-Robertson-Walker (FRW) model, then one can restrict consideration to only those Bianchi models that include the FRW model as a special case, and consider integration in (49.18) for only one Bianchi parameter at a time, holding the others fixed at the FRW value. Here, we do that for only one of the Bianchi models for illustration.

In choosing which Bianchi model to use, we would like one that has a parameter that can be varied continuously to give the FRW model. In addition, we would like to choose a parameter that represents rotation of inertial frames relative to the matter distribution. In that way, we could directly test the ability of quantum selection to implement Mach's ideas about inertia.

So far, I have not been able to find a completely satisfactory example. The Bianchi VI_h model seems to partially satisfy these criteria, since it has a parameter that represents an angular velocity of inertial frames relative to matter, and setting that parameter to zero seems to give the FRW metric. However, there seem to be some difficulties with the Bianchi VI_h model being able to change continuously into the FRW model, and also a possible problem with the topology. Until I find a better example, however, I shall use this one.

We can take the action to be

$$I = \int (-g^{(4)})^{1/2}(L_{geom} + L_{matter})d^4x + \frac{1}{8\pi}\int (g^{(3)})^{1/2} K d^3x, \tag{49.24}$$

where Hawking (1979)[123] shows the importance of the surface term. Hawking (1979)[123] also points out a potential problem in that the action can be changed by conformal transformations, but suggests a solution.

$$K = g^{(3)ij}K_{ij} \tag{49.25}$$

is the trace of the extrinsic curvature. Although the extrinsic curvature is zero on S_1 at the saddlepoint, it will be nonzero in a region around the saddlepoint. The extrinsic curvature is given by

$$K_{ij} = -\frac{1}{2}\frac{\partial g^{(3)}_{ij}}{\partial t}, \tag{49.26}$$

where $g^{(3)}_{ij}$ is the 3-metric. In this example, we take the Lagrangian for the geometry as

$$L_{geom} = \frac{R}{16\pi}, \tag{49.27}$$

where R is the scalar curvature, but we realize that a different Lagrangian might eventually be shown to be more appropriate in a correct theory of quantum gravity.

For a perfect fluid, the energy momentum tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{49.28}$$

where p is the pressure, ρ is the density, and u is the 4-velocity. For solutions to Einstein's field equations for a perfect fluid, (49.27) becomes

$$L_{geom} = \frac{1}{2}\rho - \frac{3}{2}p, \tag{49.29}$$

and we can take the Lagrangian for the matter as (Schutz and Sorkin 1977)[161]

$$\begin{aligned} L_{matter} &= \rho \\ or &\quad \text{(Schutz 1976)[162]} \\ L_{matter} &= p \end{aligned} \tag{49.30}$$

Thus, for the former choice, the classical action for perfect fluids is

$$I_{cl} = \frac{3}{2}\int(-g^{(4)})^{1/2}(\rho - p)d^4x - \frac{1}{16\pi}\int(g^{(3)})^{1/2}g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t}d^3x. \tag{49.31}$$

The latter choice gives a third of (49.31). A correct theory of quantum gravity will choose which (if either) of these two choices is correct, but for this illustration, a factor of three in the action will make little difference. We can take

$$p = (\gamma - 1)\rho \tag{49.32}$$

for the equation of state, where $1 \leq \gamma < 2$. Then (49.31) becomes

$$I_{cl} = \frac{3}{2}\int(-g^{(4)})^{1/2}(2-\gamma)\rho d^4x - \frac{1}{16\pi}\int(g^{(3)})^{1/2}g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t}d^3x. \tag{49.33}$$

Equation (49.33) diverges for an open universe. The significance of that might be that only closed universes make sense. On the other hand, it might be that the calculation of the action for the correct theory of quantum gravity will give a finite value for the action, even for an open universe, but here, we shall restrict our calculation to the case of a closed universe.

We use the solution for the Bianchi VI_h model from Ellis and MacCallum (1969)[163] with $h = -1/9$. After some algebra, we have

$$I_{cl} = \frac{3\pi^2}{4a_0}\int_{T^*}^{t}\frac{Y(t)Z(t)}{X(t)}dt, \tag{49.34}$$

where the spatial part of the 4-volume integration has already been carried out, a_0 is a parameter of the model, T^* is the Planck time, and X(t), Y(t), and Z(t) are functions of the model that must be determined by differential equations given by Ellis and MacCallum (1969)[163]. As expected, the surface term in (49.33) has canceled.

This cosmological model is relevant here because it has a relative rotation of inertial frames with respect to the matter. Specifically,

$$\Omega(t) = \frac{b}{Y^2(t)Z(t)} \tag{49.35}$$

is the angular velocity in the rest frame of an observer moving with the fluid, of a set of Fermi-propagated axes with respect to a particular inertial triad. The parameter b is an arbitrary constant of the model, and is zero if and only if there is no rotation of inertial frames relative to matter. Thus, we are interested to know the dependence of the classical action on b.

If we define

$$r^3(t) = X(t)Y(t)Z(t)\left(\frac{-3k}{3a_0^2 + q_0^2}\right)^{3/2} \tag{49.36}$$

and

$$1 + \alpha(t) = \frac{Y(t)^{2/3}Z(t)^{2/3}}{X(t)^{4/3}} \tag{49.37}$$

then the classical action in (49.34) becomes

$$I_{cl} = \frac{3\pi^2}{4}\left(\frac{3 + q_0^2/a_0^2}{-3k}\right)^{1/2}\int_{L^*}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{49.38}$$

where L^* is the Planck distance and $k = +1$ for a closed universe. We choose the lower limit to be the boundary where quantum effects would be important. We expect that the value of the action would not depend significantly on the value of the lower limit as long as it is small, and this turns out to be the case.

For the $h = -1/9$ case, we have

$$q_0 = -3a_0. \tag{49.39}$$

if and only if $b \neq 0$. However, when integrating over b, the behavior for small b dominates over the exactly $b = 0$ point. Therefore, we shall use (49.39) in any case. Substituting (49.39) into (49.38) gives

$$I_{cl} = \frac{3\pi^2}{2}\left(\frac{-1}{k}\right)^{1/2}\int_{L^*}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{49.40}$$

The form of the equation of state in (49.32) allows one of the differential equations for the model to be integrated in closed form to give

$$8\pi\rho = 3r_m^{3\gamma-2}r^{-3\gamma} \tag{49.41}$$

where r_m is a constant of integration that depends on the amount of matter in the universe and the speed of expansion relative to the gravitational attraction. Equation (49.41) shows that r_m is a measure of the amount of matter in the universe for a given value of r. Therefore, we might expect Machian effects (inertial induction) to increase for larger values of r_m.

Using (49.41), we have

$$\dot{r}^2 = \left(\frac{r_m}{r}\right)^{3\gamma-2} - k - k\alpha - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{r^2}{12}\left(\frac{\dot{\alpha}}{1+\alpha}\right)^2. \tag{49.42}$$

For the isotropic case, only the first two terms on the right hand side of (49.42) are nonzero. r_m is the value of r where those two terms are equal. For a closed universe for the isotropic case, r_m is the maximum value of r.

Equation (49.42) can be written

$$\dot{r} = \sqrt{\left(\frac{r}{r_m}\right)^{2-3\gamma} - k - k\alpha - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{V^2}{3r^4}}, \tag{49.43}$$

and the remaining differential equations to solve are

$$\dot{V} = 3k(1+\alpha)r - \frac{k}{(2a_0)^6}\frac{2b^2}{(1+\alpha)^2r^3} \tag{49.44}$$

and

$$\frac{\dot{\alpha}}{1+\alpha} = \frac{2V}{r^3}, \tag{49.45}$$

where (49.45) is the definition of $V(t)$, and $r(t)$ represents the size of the universe. The angular rotation (49.35) of an inertial frame relative to local matter is given by

$$\Omega(t) = \left(\frac{-k}{4a_0^2}\right)^{3/2}\frac{b}{[1+\alpha(t)]r^3(t)} \tag{49.46}$$

It is not possible to solve the differential equations exactly in closed form, but we can find approximate solutions assuming that α is less than one and neglecting all but the first term under the radical in (49.43). This gives

$$\alpha = \frac{12k}{(3\gamma+2)(3\gamma-2)}\left(\frac{r}{r_m}\right)^{3\gamma-2} - \frac{k}{72a_0^6r_m^4}\frac{b^2}{(\gamma-2)^2}\left(\frac{r}{r_m}\right)^{3\gamma-6}. \tag{49.47}$$

This solution is valid for small r if b is small enough.

We then perform the integration in (49.40). The result is

$$\begin{aligned}
\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m1}}{L^*}\right)^2\left\{\frac{2}{3\gamma_1+2}\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{L^*}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]\right.\\
&+\frac{2}{3\gamma_2+2}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}-\left(\frac{r_1}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}\right]\left(\frac{r_{m2}}{r_{m1}}\right)^2\\
&+\frac{24k\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}-\left(\frac{L^*}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}\right]}{(3\gamma_1+2)(3\gamma_1-2)(9\gamma_1-2)}\\
&+\frac{24k\left[\left(\frac{r}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-1}-\left(\frac{r_1}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-1}\right]}{(3\gamma_2+2)(3\gamma_2-2)(9\gamma_2-2)}\left(\frac{r_{m2}}{r_{m1}}\right)^2\\
&-\frac{kb^2\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}-\left(\frac{L^*}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}\right]}{36a_0^6r_{m1}^4(\gamma_1-2)^2(9\gamma_1-10)}\\
&\left.-\frac{kb^2\left[\left(\frac{r}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-5}-\left(\frac{r_1}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-5}\right]}{36a_0^6r_{m2}^4(\gamma_2-2)^2(9\gamma_2-10)}\left(\frac{r_{m2}}{r_{m1}}\right)^2\right\}, \qquad (49.48)
\end{aligned}$$

where I have assumed that γ changes from an early-universe value of γ_1 to its late-universe value of γ_2 at $r = r_1$. To satisfy continuity of ρ at $r = r_1$, we must also have r_m change from r_{m1} to r_{m2} at $r = r_1$, where

$$\frac{r_{m2}}{r_{m1}} = \left(\frac{r_1}{r_{m1}}\right)^{\frac{3\gamma_2-3\gamma_1}{3\gamma_2-2}}. \qquad (49.49)$$

Substituting (49.49) into (49.48) gives

$$\begin{aligned}
\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m1}}{L^*}\right)^2\left\{\frac{2}{3\gamma_1+2}\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{L^*}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]\right.\\
&+\frac{2}{3\gamma_2+2}\left[\left(\frac{r}{r_{m1}}\right)^{\frac{3}{2}\gamma_2+1}-\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_2+1}\right]\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1-\frac{3}{2}\gamma_2}\\
&+\frac{24k\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}-\left(\frac{L^*}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}\right]}{(3\gamma_1+2)(3\gamma_1-2)(9\gamma_1-2)}\\
&+\frac{24k\left[\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_2-1}-\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_2-1}\right]}{(3\gamma_2+2)(3\gamma_2-2)(9\gamma_2-2)}\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-\frac{9}{2}\gamma_2}\\
&-\frac{kb^2\left[1-\left(\frac{L^*}{r_1}\right)^{\frac{9}{2}\gamma_1-5}\right]\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}\left(\frac{r_1}{r}\right)^{\frac{9}{2}\gamma_1-5}}{36a_0^6r_{m1}^4(\gamma_1-2)^2(9\gamma_1-10)}\\
&\left.-\frac{kb^2\left[1-\left(\frac{r}{r_1}\right)^{5-\frac{9}{2}\gamma_2}\right]}{36a_0^6r_{m1}^4(\gamma_2-2)^2(9\gamma_2-10)}\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}\left(\frac{r_1}{r}\right)^{\frac{9}{2}\gamma_1-\frac{9}{2}\gamma_2}\right\}, \qquad (49.50)
\end{aligned}$$

To get a rough estimate, we take

$$\gamma_1 = 4/3 \qquad (49.51)$$

to represent a relativistic early universe,

$$\gamma_2 = 1 \tag{49.52}$$

to represent a matter-dominated late universe, and

$$r_1 = \frac{r}{100} \tag{49.53}$$

as an estimate that the universe changed from radiation-dominated to matter-dominated when the universe was about one-hundredth of its present size (Weinberg 1972, Section 15.3, p. 481)[13, Section 15.3, p. 481]. Substituting (49.51), (49.52), and (49.53) into (49.48) gives

$$\begin{aligned}
\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m1}}{L^*}\right)^2 \left\{\frac{1}{3}\left[10^{-6}\left(\frac{r}{r_{m1}}\right)^3 - \left(\frac{L^*}{r_{m1}}\right)^3\right]\right. \\
&+\frac{2}{5}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}} - 10^{-5}\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}\right]\left(\frac{r_{m1}}{r_1}\right)^2 \\
&+\frac{k}{5}\left[10^{-10}\left(\frac{r}{r_{m1}}\right)^5 - \left(\frac{L^*}{r_{m1}}\right)^5\right] \\
&+\frac{24k}{35}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{7}{2}} - 10^{-7}\left(\frac{r}{r_{m2}}\right)^{\frac{7}{2}}\right]\left(\frac{r_{m1}}{r_1}\right)^2 \\
&\left.-\frac{kb^2}{4a_0^6 r_{m1}^4}\left[\frac{1}{800}\left(\frac{r}{r_{m1}}\right) - \frac{1}{8}\left(\frac{L^*}{r_{m1}}\right) + \left(\frac{r_1}{r_{m1}}\right)^2\left(\frac{r_{m2}}{r}\right)^{\frac{1}{2}}\right]\right\}.
\end{aligned} \tag{49.54}$$

Neglecting some small terms, letting $k = +1$ for a closed universe, and using (49.49) gives

$$\begin{aligned}
\frac{I_{cl}}{\hbar} &= \frac{3i\pi^2}{2}\left(\frac{r_{m1}}{L^*}\right)^2 \left\{\frac{10^{-6}}{3}\left(\frac{r}{r_{m1}}\right)^3\right. \\
&+\frac{2}{5}\left(\frac{r_{m1}}{r_1}\right)^2\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}} + \frac{10^{-10}}{5}\left(\frac{r}{r_{m1}}\right)^5 \\
&+\frac{24}{35}\left(\frac{r_{m1}}{r_1}\right)^2\left(\frac{r}{r_{m2}}\right)^{\frac{7}{2}} \\
&\left.-\frac{9b^2}{16000a_0^6 r_{m1}^4}\frac{r}{r_{m1}}\right\}.
\end{aligned} \tag{49.55}$$

Because the parameter b is an initial value for the cosmology, it is one of the variables of integration in (49.18). In making the saddlepoint approximation for that integration, we need to locate the saddlepoint (that is, the value of b that makes the action in (49.50) stationary. We see that the action is stationary with respect to variation of b at the isotropic case of $b = 0$, as expected. The range of values of b that contribute significantly to the integral in (49.18) is given by (49.21). That is

$$\left|\frac{I_{cl}(b)}{\hbar} - \frac{I_{cl}(b=0)}{\hbar}\right| < 1. \tag{49.56}$$

Thus, substituting (49.50) into (49.56) gives

$$\frac{3\pi^2}{2}\left(\frac{r_{m1}}{L^*}\right)^2 \frac{b^2}{36a_0^6 r_{m1}^4}\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_1 - 5}$$

$$\left\{\frac{\left[1 - \left(\frac{L^*}{r_1}\right)^{\frac{9}{2}\gamma_1 - 5}\right]}{(\gamma_1 - 2)^2(9\gamma_1 - 10)}\right.$$

$$
\left. + \frac{\left[1 - \left(\frac{r}{r_1}\right)^{5 - \frac{9}{2}\gamma_2}\right]}{(\gamma_2 - 2)^2(9\gamma_2 - 10)} \left(\frac{r_1}{r}\right)^{\frac{9}{2}\gamma_1 - \frac{9}{2}\gamma_2} \right\} < 1. \tag{49.57}
$$

The restriction on the size of b in (49.57) guarantees that α in (49.47) is less than one for r less than r_m and also that the terms neglected in (49.43) are small enough.

Substituting (49.51), (49.52), and (49.53) into (49.57) and neglecting a small term gives

$$
\frac{27\pi^2}{32000 a_0^6} \left(\frac{r_{m1}}{L^*}\right)^2 \frac{r}{r_{m1}} \frac{b^2}{r_{m1}^4} < 1. \tag{49.58}
$$

This gives

$$
b < \frac{20\sqrt{5}(2a_0)^3}{3\pi\sqrt{3}} \left(\frac{L^*}{r_{m1}}\right) \left(\frac{r_{m1}}{r}\right)^{\frac{1}{2}} r_{m1}^2. \tag{49.59}
$$

Thus, from (49.46), the rotation rate of inertial frames is

$$
|\Omega(t)| < \frac{20\sqrt{5}}{3\pi\sqrt{3}} \left(\frac{L^*}{r_{m1}}\right) \left(\frac{r_{m1}}{r}\right)^{\frac{1}{2}} \frac{r_{m1}^2}{[1+\alpha(t)]r(t)^3}. \tag{49.60}
$$

If we now take the Planck distance L^* to be 1.6×10^-33 cm, use the Hubble distance of 1.7×10^{28} cm for r and r_{m1}, and neglect α compared to 1, then we get

$$
|\Omega| < 1.4 \times 10^{-71} \text{ radians per year}, \tag{49.61}
$$

which is much less than the bound set by experiment (Hawking 1969)[164]. The rotation rate in (49.61) is so small because the Planck distance is so much smaller than the Hubble distance. The ratio of the Planck distance to the Hubble distance entered in (49.59) to give a small value for the ratio of b to r_{m1}^2. The small value for b in (49.59) comes in turn from using the action (49.50) in (49.56).

That the estimate of the action in (49.48) is nearly independent of the lower limit in the integral in (49.40) suggests that the semiclassical approximation for the action is valid. That is, there is no significant contribution to the action from the time when the universe was a size comparable to the Planck distance. Specifically, it suggests that the small allowed value for b (relative to r_{m1}^2) in (49.59) when L^* is much less than r_{m1} is probably valid.

That the small value of allowed rotation rate depends only on the universe being much larger than a Planck distance rather than on details of the model suggests that the same calculation for other parameters and other models will give similar results.

We notice also, that the selection criterion in (49.56) is so sharp that the initial wave function in the integration in (49.18) would have to be very sharply peaked to overcome it.

49.7 Empty and sparse universes

We use the action in (49.24) and General Relativity as an example, but the results obviously have more generality. For an empty universe, $T^{\alpha\beta}$ is zero and therefore L_{matter} is also zero. In the semiclassical calculation, the integration in (49.18) is restricted to actions evaluated for classical cosmologies. Therefore, all cosmologies in the integrations in (49.18) satisfy Einstein's equations, so that $T^{\alpha\beta}$ equal zero implies that $R^{\alpha\beta}$ also equal zero, and therefore $R = 0$, and therefore L_{geom} is also zero. If we further consider only static universes, then the surface term in the action in (49.24) is also zero. Under those conditions, the action in (49.24) is zero.

Therefore, the action is zero for all empty static universes. (It may be possible to show that the action for all empty universes is zero.) Thus, the integration in (49.18) will weight all empty universes equally (except for the initial wave function). Therefore, for empty universes, there are no saddlepoints for the integrations in (49.18), and therefore no selection of solutions as we had for the solutions with matter in the previous section. Therefore, the only way to have a single matter free static solution would be for it to be selected by the initial wave function. That is, the initial wave function would have to be sharply peaked.

We would thus not expect to find an empty universe, in agreement with our concept that such solutions are non Machian. Similar arguments apply to asymptotically flat solutions, which many agree are also not Machian.

Sparse universes (that is, universes with some, but not much, matter) are perhaps more interesting. We should expect that the action in (49.24) is a continuous function of the amount of matter in the universe, even as the amount of matter approaches zero. Therefore, we would expect the action to be arbitrarily small for an arbitrarily small amount of matter.

We would therefore expect the integrations in (49.18) to not be sharply peaked in such cases, but to be broad, even for the cases where a saddlepoint exists. We would therefore not expect the saddlepoint approximation to be valid, and many classical cosmologies would contribute significantly to the wave function in the final state.

Thus, for sparse universes also, there would be no selection among classical cosmologies that contribute to the wave function in (49.18). To test this idea quantitatively, we consider the Bianchi model VI_h in the previous section, and let the amount of matter approach zero. Specifically, let us consider the density of matter given by (49.41) for r equal to some small value r_0.

$$8\pi\rho = 3r_{m1}^{3\gamma_1-2}r_0^{-3\gamma_1} \tag{49.62}$$

For a fixed value of r_0, decreasing the amount of matter in the universe requires a decrease in ρ, which in turn requires a decrease in r_{m1}. If the amount of matter decreases enough, r_{m1} will eventually no longer be much larger than the Planck distance. In that case, the action will no longer be sharply peaked at the saddlepoint $b = 0$, and the allowed values for b in (49.59) will no longer be restricted to be so small, and in turn, the allowed rotation rates in (49.60) will be much larger.

49.8 Discussion

We see that considerations of quantum cosmology show how a range of classical cosmologies can be selected that contribute significantly to the wave function in the final state. The effect enters through the action. Using semiclassical calculations gives results that should not depend on particular features of the theory of quantum gravity.

For our universe (which is much larger than the Planck distance) the selection is very sharp. The wave function over 3-geometries would have to be extremely sharp (not a probable occurrence) to dominate over the effect of the action.

The selection process seems to occur very soon in the development of a cosmology. That is, for a broad wave function over 3-geometries in the initial state, the wave function becomes sharply peaked after the universe has become a few orders of magnitude larger than the Planck distance.

The quantum selection process seems to agree with previous ideas about Machian cosmologies. Thus, cosmologies with enough matter will behave in a Machian way by the quantum selection process, but not empty universes, asymptotically flat universes, or sparse (nearly empty) universes.

If we were to use the consistent histories (or decoherent histories) approach to quantum cosmology (Gell-Mann and Hartle 1993)[165], then the results presented here could possibly be extended to include boundary conditions in addition to initial conditions.

49.9 Acknowledgments

I would like to thank Julian Barbour, Bruno Bertotti, Gary Bornzin, Hubert Goenner, Douglas Gough, Stephen Hawking, Don Page, A. Jay Palmer, David Peterson, and Derek Raine for useful discussions.

49.10 Cover letter

18 April 1994

Thank you for giving me your comments on my manuscript, "Quantum Selection of a Classical Cosmology." They have helped me to express my ideas more carefully. I have revised the manuscript based on your comments, and am enclosing it here. As I recall, your comments were:

1. It is impossible to specify the matter distribution independent of the metric.
2. It is possible to specify initial conditions in a relative configuration space as a relative configuration plus a relative direction. The velocity not needed.
3. There is no time in quantum cosmology because there is no time in the Wheeler-DeWitt equation, and therefore there is no initial wave function, and therefore there is no integration over initial wave functions.

If I have misrepresented your comments, I hope that you will correct me.

I agree with the first point that it is not possible to specify the matter distribution without reference to a metric. I was being careless in using statements like "the matter distribution determines the geometry," and I have removed such phrases from the manuscript.

However, there are certain aspects of the matter distribution that can be specified independently of knowing the complete metric. For example, it is possible to specify that the matter distribution is spatially homogeneous without specifying anything about the metric beyond it being spatially homogeneous.

I also agree with your second point. I had mistakenly said in the introduction of my paper that the main difficulty was in being able to specify initial and boundary conditions on the geometry in terms of kinematically observable quantities. Since that statement was irrelevant to my manuscript anyway, I have removed it.

Your third point seems to assume that I need an *a priori* time, but I actually don't. All I really need is the concept of a propagator that gives the amplitude to go from a state with metric g_1 and matter fields ϕ_1 on a surface S_1 to a state with a metric g_2 and matter fields ϕ_2 on a surface S_2. Hawking and others use such propagators in the path-integral formulation of quantum cosmology. Although using such propagators may turn out not to be a correct approach, it seems viable at the present.

The existence of such propagators implies that any spacelike hypersurface will have an amplitude for various 3-geometries associated with it. Therefore, the spacelike hypersurface S_1 will also have an amplitude for various 3-geometries. Therefore, the method to calculate the amplitude for various 3-geometries on the hypersurface S_2 in terms of the amplitude for various 3-geometries on S_1 is to perform the integration in equation (16) [formerly (17)] of my manuscript.

That process does not require the concept of time. My use of the terms "initial wavefunction" and "final wavefunction" is simply shorthand for the amplitudes for various 3-geometries and matter fields on the surfaces S_1 and S_2, respectively.

I hope everything is going well with getting the proceedings ready for publication.

Best regards,

Michael Jones
enclosure: "Quantum Selection of a Classical Cosmology"

49.11 Afterthoughts - 2008

The details of the calculation of the action for the Bianchi VI_h model are given by Jones (1999)[166] and by chapter 67.

I should have used a different cosmological model than the Bianchi VI_h model, because it is an open rather than a closed model. Also, I should not have neglected the cosmological constant.

49.12 Afterthoughts - 2009

Newton noticed that his laws of motion applied in a frame that does not rotate relative to the "fixed stars," and concluded the existence of an "absolute space" with the stars fixed in that absolute space (what we now refer to as an "inertial frame").

Mach denied the existence of absolute space, pointing out that only relative motions are observable. Instead of absolute space, he postulated that inertia is caused by an interaction with the rest of the universe. He suggested that what we now refer to as an inertial frame is caused by the rest of matter in the universe (now referred to as frame dragging).

Einstein based his General Relativity partly on Mach's ideas, and coined the term "Mach's principle" for the idea that inertial frames should be determined by the matter distribution. The term "Mach's principle" now means many things to many people, and there is no general agreement on what it means or on the validity of the various versions.

Although Einstein's General Relativity does include frame dragging, frame dragging is not complete. There are many solutions of Einstein's field equations in which there is inertia in the absence of matter or where there is rotation of inertial frames relative to the average matter distribution. This is because initial and boundary conditions also contribute to inertia.

Some researchers have proposed that boundary and initial conditions be chosen so that inertia and/or the metric be determined by the matter distribution. Some researchers have proposed that Mach's principle be used as a selection principle to discard solutions that have relative rotation of matter and inertial frames or have inertia without matter.

I have argued that if the metric (gravitation) is determined solely by sources (the matter), with no dependence on initial or boundary conditions, then gravitation would be very different from electromagnetic theory, in which the electromagnetic field has its own degrees of freedom, and can have arbitrary initial and boundary conditions. Thus, I have rejected Mach's principle in such a restricted form.

That leaves us still with the problem of explaining why we observe no relative rotation of inertial frames and the visible matter in the universe. I suggested that if we consider quantum gravity, that we could allow all of the solutions to the field equations, and that the non-physical solutions (those with relative rotation of inertial frames and matter) would cancel each other out if we look at the change in the action with variation of the initial and boundary conditions on the gravitational field.

It was at this point that I made a mistake in the present manuscript. I suggested that we vary the initial and boundary conditions on the gravitational field while holding the matter distribution constant. It was pointed out that there is no general procedure for doing that because it is not possible to specify the matter distribution independent of the geometry.

What I should do instead (and what I actually did in my calculations) is consider a family of solutions to the field equations (such as one of the Bianchi cosmologies) in which there is an initial condition that determines the amount of relative rotation of inertial frames and matter, and consider the change in action as that parameter is varied. In this way, we have no problems of the kind I mentioned above. Then, a saddlepoint approximation to the path integral for that parameter will give the solution for zero relative rotation.

Chapter 50

Quantum Selection of a Classical Cosmology - 1.5[1]

abstract

We investigate the consequences of three principles.

1. The laws of physics must be expressed in terms of kinematically observable quantities. (Mach's principle)

2. The gravitational field is just as fundamental as matter. (Thus, we do not require <u>in principle</u> that matter determine the geometry.)

3. The observation that there is no relative rotation of inertial frames and the "fixed stars" (within experimental error) is not chance. (This implies a unique relation <u>in practice</u> between matter and geometry.)

These three principles are inconsistent within a framework of classical gravitation. The first two principles can be satisfied in a framework of quantum cosmology. The third can be satisfied as a classical approximation by requiring that the action be stationary with respect to variation of each parameter that specifies the initial 3-geometry. The first principle is satisfied if the action is expressed in terms of kinematically observable quantities.

50.1 Introduction

Ernst Mach (1872, 1933)[120, 102, 122, 15] criticized Newton's formulation of mechanics and gravitation because it involved quantities (such as inertial frames) that are not kinematically observable. That inertial forces (having no apparent origin) exist in frames that rotate or accelerate relative to certain "inertial frames" is no more than a definition of inertial frame since there is no other way to determine such frames. That inertial frames seem not to rotate or accelerate relative to the "fixed stars" has no operational use in mechanics unless there is a specific connection between inertial forces and the rotation and acceleration of the stars.

In spite of many tries (too numerous to mention here), no generally accepted solution to Mach's challenge that "our law of inertia is wrongly expressed" has been found. Further, in spite of Mach's ideas having guided Einstein in his formulation of General Relativity and been given the name

[1] Part of a revised version (20 October 1996) of the paper that was presented at (and submitted to the Proceedings of) the conference on Mach's Principle: "From Newton's Bucket to Quantum Gravity", held at Tübingen, Germany, July 26-30, 1993, File: 20oct96.spr

"Mach's Principle" by Einstein (1918)[152], there is no general agreement on what that principle is or should be.

One reasonable interpretation of Mach's ideas on inertia is "The laws of physics must be expressed in terms of kinematically observable quantities." Although it may be straightforward to interpret "kinematically observable" for a collection of particles, it is more difficult for fluids, fields, and the geometry itself. Barbour (this conference) puts forth convincing evidence that Einstein's field equations are already expressed in terms of kinematically observable quantities. More generally, his work shows how one can specify fields (and the geometry in particular) in terms of kinematically observable quantities.

Many attempts to implement Mach's ideas on inertia seem to have implicitly used interpretations that are at least similar to the one above. We consider some of those attempts here. The difficulty of applying Mach's ideas is not always obvious. The main difficulty is in being able to specify initial and boundary conditions on the gravitational field (in modern terminology, on the geometry) in terms of kinematically observable quantities.

Lynden-Bell (1992)[5] presents a formulation that would have probably satisfied Mach at first glance. However, even such a direct inter-particle formulation as his contains implicit initial and boundary conditions on the gravitational field.

In a field theory of gravitation (such as Einstein's General Relativity), initial and boundary conditions enter more explicitly. One approach to specify initial and boundary conditions is to require the matter to determine the geometry. This approach has two advantages. First, it seems easier to specify the matter distribution in terms of kinematically observable quantities. Second, it seems to solve the problem of why the relative rotation of inertial frames and the "fixed stars" seems zero (at least within experimental error). Thus, Wheeler (1964)[115] proposes using Mach's principle as a criterion to select solutions of Einstein's field equations. A related approach uses integral formulations of General Relativity (Al'tshuler, 1967[154]; Lynden-Bell, 1967[155]; Sciama, Waylen, and Gilman 1969[16]; Gilman, 1970[156]; Raine, 1975[109]).

Requiring the matter to determine the geometry may be too strict. It can be argued (Earman, Kuchar, Raine, and others, this conference) that the gravitational field should be considered as fundamental as matter. For example, should we consider the energy in gravitational waves a part of the matter? What about pair creation in a strong gravitational field? The difficulty of deciding where to draw the line suggests that we drop the requirement that matter determine the geometry and allow arbitrary initial conditions. It is not clear what the status of boundary conditions should be.

But, if we were to drop the requirement that matter determine the geometry, we have to either specify initial and/or boundary conditions in terms of kinematically observable quantities or explain that unlikely set of initial conditions which leads to zero relative rotation between inertial frames and the "fixed stars."

Allowing arbitrary initial conditions for the geometry solves the problem of specifying initial conditions on the geometry in terms of kinematically observable quantities by deleting it as a requirement. That leaves the second problem of explaining our observation that inertial frames do not rotate relative to the "fixed stars." In the framework of any classical gravitational theory, there would be no solution.

However, if we consider quantum cosmology, in which all initial conditions on the 3-geometry [that is, the part (the Weyl tensor) not restricted by the matter distribution] be allowed through an initial wave function over 3-geometries, then our observable world might be recovered under some approximation.

To be more explicit, let us assume that we have solutions of the field equations, 4-geometries $G^{(4)}(b)$ for the same matter distribution, where b is a parameter that specifies the initial 3-geometry. Although it is not always easy to show that several different spacetimes have the same matter distributions, it can be done in some cases, such as homogeneous spacetimes. It is only in such cases where the question arises about matter determining geometry arises, anyway. Our goal here is to show that at least in some cases, requiring the action for the 4-geometry to be stationary with respect to variation of the parameter b automatically selects a 4-geometry without having to impose conditions artificially. That is,

$$\frac{\delta S(b)}{\delta b} = 0 \; , \tag{50.1}$$

where S is the action.

Chapter 51

Gravity Without Geometry - I[1]

51.1 Introduction

Three of the four fundamental interactions have been unified. Part of the difficulty with unifying gravitation with the others depends on two things:

1. The gravitational interaction has not yet been quantized.

2. The gravitational interaction is treated as geometry.

Let us consider the effect of the latter. As long as the gravitational interaction is treated as a geometric arena upon which all other interactions are staged, it will be difficult to unify it with the other interactions on an equal basis. The first question to ask, is whether the gravitational interaction is intrinsically geometrical in nature or whether the apparent geometrical character is an accident of the fact that there is so much matter in the universe that the local geometry is induced by an interaction with rest of matter in the universe. If the answer is the former, then it will remain difficult to unify gravitation with the rest of the fundamental interactions. If, however, the answer is the latter, then expressing the gravitational interaction in non-geometrical terms may simplify the unification process. (It may also simplify the quantization of the gravitational interaction.)

Ernst Mach[120, 102] argued in the last century, using different terminology, that the latter was the case. That is, that inertia was not an intrinsic property of matter, but due to an interaction with the rest of matter in the universe. Although inertia and geometry are not the same, similar arguments that Mach used for inertia can also be applied to geometry. Mach argued that the law of inertia was wrongly expressed. We can argue similarly that expressing gravitation in terms of geometry leads us in a wrong directions when we try to generalize laws of physics.

If the local geometrical character of space-time is due to gravitational interaction with matter in the universe, then if there were a net charge on matter in the universe the local geometrical character of space-time would be due to an electromagnetic interaction with matter in the universe. To support this view, we consider two examples in which electromagnetic interaction induces mass.

The first example comes from radiowave propagation in the ionosphere. Neglecting the effects of the Earth's magnetic field and collisions of electrons with neutral air molecules, the dispersion relation for radiowave photons in such a cold plasma is the same as the dispersion relation for free massive particles, in which the photon has an induced mass m such that $mc^2 = hf$, where c is the speed of light, h is Planck's constant, and f is plasma frequency (which depends on the density of ionized electrons in the ionosphere). Since electron density is position dependent in the ionosphere, the effective mass of the photon is position dependent.

[1]essay, written before 1 April 1997

The second example is from motion of electrons in a crystalline lattice, in which the lattice induces an effective mass on electrons due to electromagnetic interaction between the electron and the lattice. In this case, the effective mass is highly position dependent.

Having set up the motivation for why we should look for a non-geometric formulation for gravitation, let us proceed. Trying to express gravitation in non-geometrical terms is difficult, for at least two reasons:

1. Present gravitational formulation is so closely linked with geometry, it seems nearly impossible to separate the two.

2. But, without a geometric background, how can we express any laws of physics?

We shall take a less radical approach, by simply assuming a geometry exists, without asking where it came from, expressing it in terms of the metric tensor as usual, but trying to express local gravitational interaction in ways that are more analogous to the other long-range interaction, the electromagnetic interaction.

To do this, we must rewrite the following aspects of the gravitational interaction to conform to the form of the other interactions, especially the other long-range interaction, the electromagnetic interaction:

1. The form of the action

2. The form of the field equations

3. The form of the local force law (the geodesic equation)

One method for generalizing physical laws can be done in three steps:

1. Rewrite the existing laws in a different but equivalent form.

2. Alter the new form to one which is not equivalent, but indistinguishable.

3. Possibly, rewrite the new form in another equivalent form.

51.2 Geodesic Equation

We begin by writing the geodesic equation including the Lorentz force equation.

$$m[g_{\mu\beta}\ddot{x}^{\beta} + \frac{1}{2}(g_{\mu\nu,\beta} + g_{\mu\beta,\nu} - g_{\beta\nu,\mu})\dot{x}^{\beta}\dot{x}^{\nu}] + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^{\nu} = 0 \tag{51.1}$$

The Lorentz force involves derivatives of the electromagnetic 4-vector potential multiplied by the 4-current. The geodesic equation has similar terms involving the derivative of the metric tensor multiplied by the energy-momentum tensor for a single particle, but also a term that is proportional to the second derivative of the 4-momentum. This inertial term, that does not have a form of a product of a derivative of a potential with a source term, has no counterpart in the Lorentz force. At a minimum, it will be necessary to re-express the force law without such a term.

If we transform to a frame moving with the particle, then that term disappears, leaving both the geodesic equation and the Lorentz force expressed as simply derivatives of the corresponding potential.

$$m(g_{\mu 0,0} - \frac{1}{2}g_{00,\mu}) + e(A_{\mu,0} - A_{0,\mu}) = 0 \tag{51.2}$$

Here, we can consider two possibilities:

1. The force law should be expressed only in the frame of the particle.

2. Can we transform this force law to an arbitrary frame without changing the desired form of the law?

Along the path of the particle, we have the identity

$$\ddot{x}^\mu = \dot{x}^\mu_{,\nu}\dot{x}^\nu \ . \tag{51.3}$$

If we define

$$\pi_\mu \equiv m g_{\mu\beta}\dot{x}^\beta \ , \tag{51.4}$$

then

$$\pi_{\mu,\nu} = g_{\mu\beta,\nu}\pi^\beta + m g_{\mu\beta}\dot{x}^\beta_{,\nu} \ , \tag{51.5}$$

so, the geodesic equation plus Lorentz force (51.1) becomes

$$(\pi_{\mu,\nu} - \frac{1}{2}g_{\beta\nu,\mu}\pi^\beta)\dot{x}^\nu + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^\nu = 0 \ . \tag{51.6}$$

We notice that

$$\pi_{\mu;\nu}\dot{x}^\nu = \pi_{\mu,\nu}\dot{x}^\nu - \frac{1}{2}g_{\beta\nu,\mu}\pi^\beta\dot{x}^\nu \ . \tag{51.7}$$

Thus, (51.6) can be written

$$\pi_{\mu;\nu}\dot{x}^\nu + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^\nu = 0 \tag{51.8}$$

Finally, The identity

$$-\pi_{\nu;\mu}\dot{x}^\nu + \frac{1}{2m}(g^{\alpha\beta}\pi_\alpha\pi_\beta)_{;\mu} = 0 \tag{51.9}$$

allows us to write (51.8) as

$$(\pi_{\mu;\nu} - \pi_{\nu;\mu})\dot{x}^\nu + \frac{1}{2m}(g^{\alpha\beta}\pi_\alpha\pi_\beta)_{;\mu} + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^\nu = 0 \tag{51.10}$$

which can be written as

$$(\pi_{\mu,\nu} - \pi_{\nu,\mu})\dot{x}^\nu + \frac{1}{2m}(g^{\alpha\beta}\pi_\alpha\pi_\beta)_{,\mu} + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^\nu = 0 \tag{51.11}$$

Back to the gravitational vector potential. It is equal to π (mechanical momentum) extended to be a vector field minus half the gradient of τ, where τ is the proper time for the particle extended to be a scalar field. Maybe it is possible to write π and τ as integrals over some source using the SWG integral formulation of General Relativity. – Yes, it is possible. Maybe Donald Lynden-Bell[5] has already done this. I need to check this. Yes, it works for momentum, including the side condition that makes momentum a vector field. The total momentum of the universe is zero. I'm not sure how to get τ, the proper-time scalar field, though. I don't know to what that corresponds.

τ is $\int i ds$, and in the frame of the particle, that is just $\int d\tau$, which equals $\int \sqrt{g_{00}}dt$. Since g_{00} is a constant along a geodesic (even including a Lorentz force), this is a very easy integral. Still, how do I relate τ to the distribution of matter? Somehow, I am setting up a coordinate system based on the trajectory of this particle. Maybe I can't relate this to the distribution of matter. Still, g_{00} is given as an integral over the matter distribution, so at least there is a scaling factor times $\int dt$. Of course, if we knew this scalar field τ as a function of position, we would know a family of solutions of the geodesic equation. If we knew that, we would not have to solve the geodesic equation.

OK, so that is why we can't have a gravitational vector potential. Can we at least get a force law that looks like an EM force law? The momentum part seems to be OK, and should make Sciama's 1953[11] calculation rigorous, but we still have the τ part. Can we make that right? One component of the gravitational vector potential was the gradient of τ. Locally, the gradient of τ should be in the direction of momentum. The magnitude should be about one, or maybe the square root of g, but also multiplied by $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. I can calculate these things from the distribution of matter. It looks OK. We really can get a gravitational vector potential.

Still a problem. Donald Lynden-Bell's paper[5] doesn't seem to give a formula for integrating the momentum of the matter distribution, but maybe I can still do it. I'm not sure.

The Sciama-Waylen-Gilman[16] result gives the metric as an integral of the stress-energy tensor, but Donald Lynden-Bell[5] says that gravitational energy and momentum should also contribute to inertia. How do I reconcile that?

Knowing the metric allows one to calculate the geodesic equation. No other information is needed, except the initial value of the momentum of the particle. If I can integrate the momentum of the universe, is that additional information? No, it is somehow the same information as the momentum of the particle. Lynden-Bell[5] says that the total of any conserved quantity is zero. So the total momentum of the universe is zero. So the momentum of the particle plus the momentum of the rest of the universe is zero. (That must be only in the frame of the center of mass of the universe.) So making the integral of the momentum of the universe is equivalent to knowing the momentum of the particle.

We can make an integral of the momentum of the universe. We start with the Sciama, Waylen, Gilman integral formulation of General Relativity[16]. Their equation (51.14) gives the $g^{\mu\nu}$ at one point as an integral of a propagator times T^α_β over a 4-volume plus a surface term. If we multiply each T^α_β by U^β, that gives the momentum density (the current). If we integrate that, we get the corresponding potential (the momentum of our particle) at the point in question. This gives part of our gravitational vector potential.

For the other term, we need τ, the local proper time. To get this, we return to the SWG equation, and multiply by $dx_\alpha dx^\beta$ before integrating. This should give us $(d\tau)^2$. We need to take the square root, and integrate. However, we need $\tau_{,\mu}$, so we have to take a gradient. This calculation is still a little funny. I may have to modify it some. It seems very complicated.

There is still the problem about whether I can use the same propagator for these integrals. Does the propagator depend on what is being integrated, or are there general propagators?

Donald Lynden-Bell[5] has also used a Lagrangian that has relative velocities instead of velocities relative to an arbitrary coordinate system. Possibly I can use that idea for the action in general relativity. In this way, it may be possible to formulate general relativity without reference to any frame, especially to inertial frames. However, this is still a mechanical, kinetic formulation, so doesn't do what I want here.

It seems that Lynden-Bell, Katz, and Bicak[167] have already done the calculation I need. Their equation (3.36) gives the current for momentum, angular momentum, etc. [There is a misprint, which can be corrected by looking at their equation (3.32)[3].]

$$\pi^\mu = -c\tau\int \hat{J}^\mu d^3x = -c\tau\int[(\hat{T}^\mu_\nu + \hat{t}^\mu_\nu)\xi^\nu + \hat{\Sigma}^{\mu\sigma}_\nu \bar{D}_\sigma\xi^\nu]d^3x \tag{51.12}$$

For momentum for the Robertson-Walker metric as a background, I need to use quasi-translations. They prove that the total quasi-momentum of the universe is exactly zero, so the local contribution due to the body in question plus that of the rest of the universe is zero, so the local momentum of the body equals the negative of that due to the rest of the universe. Thus, this is what I need.

Actually, their calculation is only for the 3-momentum (actually, 3-quasi-momentum). I still have the 4th component, namely, the energy. I'm not sure how I get that one. Since the Robertson-Walker background metric is time-varying, energy is not strictly conserved, so I have a problem

[3]Note added in 2008: I just discovered their errata paper, in which they correct this error along with others[168]

in getting that component in terms of an integral over the universe. That may be related to my problem with the extra term.

Now for the term with the τ, I am still not sure what to do. Does τ correspond to some symmetry that I can use to calculate using Their work? One factor in this part looks like the contraction of the stress-energy tensor for a single particle. Maybe I can relate that factor to the Action for the rest of the universe. The other factor is the gradient of τ, the proper time for the particle.

To continue extending Lynden-Bell's[5] calculations, I can put it into a form where the inertia part looks like a vector potential part. That is, we have a gradient of a vector times the local current, which looks like a momentum. So, active gravitational mass and active gravitational current is integrated over to give the local vector potential, which replaces (but is equal to) inertial mass or inertial current, which acts upon the local passive gravitational mass or current, which also looks like momentum.

Ordinary gravitational force is the gradient of the metric tensor, multiplied by the tensor current, which is the stress-energy tensor. Problems: The metric tensor does not include gravitational energy and momentum as a source. We have derivatives with respect to some coordinate system.

51.3 Action

Another way to look at the situation is the following. That term in the geodesic equation comes from the mgv^2 term in the action, $(d/dx)(d/d\dot{x})$. It is having the v^2 term. To get rid of that term means changing that term in the action somehow, not in value, but in form. We can use gT for that term. That is analogous to the AJ term in the action for the EM part. In both cases we have the product of the potential with the source.

It is not clear what comes next. In deriving Lagrange equations, we use path integrals. We alter the path slightly. We need to specify the path in a more general way than as position. I am not sure how.

Let's look at the action again. Does geometry come from gravity because it is a tensor interaction or because there is a lot of matter in the universe? Can we make a geometry from EM vector interaction? For a geometry, we need the motion to be independent of the charge. That will be true if we neglect all other interactions, and the stars are charged.

Somehow, the formula for the action must show that we could get a metric even without gravitation. EM interaction should be enough to get a geometry. Maybe we are back to a different metric for each particle. The problem is, without a geometry, we cannot define position, time, nor derivatives. Maybe we don't need that to define an action, but we do need those to define the force law. Of course, without a geometry, we don't need a force law. All we need is an action (or something equivalent, like Hoyle and Narlikar had for the infitesimal particle propagator, which was multiplicative and nonabelian).

The action is usually defined as the integral over 4D spacetime of a Lagrangian plus some surface turns. We can't define a 4D integral until we have a geometry. How do we get around that one? Maybe we do a sum over particles. That should be ok. But what do we do about the fields?

Let's do a calculation. The usual Lagrangian for a particle of mass m and charge e in General Relativity is

$$L = -\frac{1}{2}mg_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - eA_\mu\dot{x}^\mu \qquad (51.13)$$

This leads to the geodesic equation (51.1). With the insight we have gained so far, we can say that at least one of the $\dot{x}$s in (51.13) corresponds to a vector potential, not a kinetic term. Thus, we can imagine replacing one of the $\dot{x}$s by π/m. That Lagrangian will not give the correct law of motion

(51.11) because of the factor of a half. However, if we take the Lagrangian to be

$$L = -\pi_\mu \dot{x}^\mu - eA_\mu \dot{x}^\mu \tag{51.14}$$

then we do get (51.11) from Lagrange's equations.

51.4 Metric

Next, we need to have inertia from EM. That is ok. Next, we need to have a metric. In the usual gravitational case, we define the metric in terms of light distance. That is, the actual distance (squared) minus the distance (squared) light could travel in the time difference. This is useful because light has a constant speed in flat spacetime. For the EM case, we use a different particle to define the metric. In analogy with light, we need a chargless particle. This isn't quite right. Light has no rest mass, but has energy and momentum. In the EM case, we need a particle that has no rest charge, but has current. It is not clear what that would be. This may be OK if it moves at the speed of light because of the gamma factor that would be infinite at the speed of light. Maybe there would be such a particle if the universe were charged. Maybe light would not exist if the universe did not have a lot of mass.

Back to the origin of geometry. We recognize that there can be different kinds of metrics, one for each kind of particle. Why does the light metric give the gravitational potential? Light is EM. Light is also massless energy and momentum. A neutrino is also massless energy and momentum. A photon is the carrier of EM field. A neutrino is probably not the carrier of any field because it is a Fermion.

Maybe the speed of light is induced by the matter in the universe. Also all fields. Maybe time and distance are induced. Maybe the EM field really does not have independent degrees of freedom. Maybe it comes entirely from charges and currents. Maybe the same is true of gravitons. That is, maybe there really are no independent degrees of freedom of the gravitational field. In that case, Mach's principle is easier.

51.5 Currents

Actually, it may be different. Current is the transport of charge as momentum is the transport of mass, except that it is possible to have momentum without mass in the case of light, but it is not possible to have current without charge, except for possibly some kind of current rings? The energy-momentum tensor is the transport of momentum.

51.6 Dennis Sciama's Calculation

Also, in the EM case, we have two kinds of charges, plus and minus. The 1953 calculation of Sciama[11] applied to EM shows that in a positively charged universe, positive charges will attract, negative charges will repel, and opposite charges will accelerate in the same direction, with the positive charge in the lead.

51.7 Quantum effects

A graviton is the carrier or the gravitational interaction, and is massless with spin two. It has both energy and momentum. In the absence of a lot of matter in the universe, we don't have Minkowski space, we have no geometry. We have a lot of Minkowski spaces in a quantum superposition (Jones, 1979, 1980, 1993)[107, 169, 170]. Thus, the building up of a geometry is not a simple adding up of contributions. We must at least do a path integral. It may be enough to get the formula for the action right.

Also, how do we treat the wave properties of the particles? On the other hand, if the wave properties of the particles is simply an artifact of trying to fit a tensor gravitational field with a vector EM field, maybe we don't need to worry about the (artifact) wave properties of particles.

But if the wave properties of particles is an artifact, then what happens to the idea of wave interference among cosmologies whose action is not an extremum?

Here is a conflict. The action is useful only if there is an underlying quantum theory. But if the apparent wave properties of particles is only an artifact of trying to bring a tensor gravitational field in line with a vector EM field, where does that leave us? If the quantum behavior comes only from trying to make a gravitational vector potential?

There is still the idea that wave properties come from a linearization of some underlying nonlinear behavior. Wave properties look like first order linearization of a nonlinear calculation.

Chapter 52

Gravity Without Geometry - II[1]

52.1 Introduction

Three of the four fundamental interactions have been unified. Part of the difficulty with unifying gravitation with the others depends on two things:

1. The gravitational interaction is treated as geometry.

2. The gravitational interaction has not yet been quantized.

Let us consider the effect of the former. As long as the gravitational interaction is treated as a geometric arena upon which all other interactions are staged, it will be difficult to unify it with the other interactions on an equal basis. If the gravitational interaction is intrinsically geometrical in nature, then it will remain difficult to unify gravitation with the rest of the fundamental interactions. If, however, the apparent geometrical character is an accident of the fact that there is so much matter in the universe, that the local geometry is induced by an interaction with rest of matter in the universe, then expressing the gravitational interaction in non-geometrical terms may simplify the unification process. (It may also simplify the quantization of the gravitational interaction.)

Ernst Mach[120, 102] argued in the last century, using different terminology, that the latter was the case. That is, that inertia was not an intrinsic property of matter, but due to an interaction with the rest of matter in the universe. Although inertia and geometry are not the same, similar arguments that Mach used for inertia can also be applied to geometry. Mach argued that the law of inertia was wrongly expressed. We can argue similarly that expressing gravitation in terms of geometry leads us in a wrong directions when we try to generalize laws of physics.

If the local geometrical character of space-time is due to gravitational interaction with matter in the universe, then if there were a net charge on matter in the universe the local geometrical character of space-time would be due to an electromagnetic interaction with matter in the universe. To support this view, we consider two examples in which electromagnetic interaction induces mass.

The first example comes from radiowave propagation in the ionosphere. Neglecting the effects of the Earth's magnetic field and collisions of electrons with neutral air molecules, the dispersion relation for radiowave photons in such a cold plasma is the same as the dispersion relation for free massive particles, in which the photon has an induced mass m such that $mc^2 = hf$, where c is the speed of light, h is Planck's constant, and f is plasma frequency (which depends on the density of ionized electrons in the ionosphere). Since electron density is position dependent in the ionosphere, the effective mass of the photon is position dependent.

[1] essay2, written sometime before 1 April 1997

The second example is from motion of electrons in a crystalline lattice, in which the lattice induces an effective mass on electrons due to electromagnetic interaction between the electron and the lattice. In this case, the effective mass is highly position dependent.

Having set up the motivation for why we should look for a non-geometric formulation for gravitation, let us proceed. Trying to express gravitation in non-geometrical terms is difficult, for at least two reasons:

1. Present gravitational formulation is so closely linked with geometry, it seems nearly impossible to separate the two.

2. But, without a geometric background, how can we express any laws of physics?

We shall take a less radical approach, by simply assuming a geometry exists, without asking where it came from, expressing it in terms of the metric tensor as usual, but trying to express local gravitational interaction in ways that are more analogous to the other long-range interaction, the electromagnetic interaction.

Here, we can use one method for generalizing physical laws that can be done in three steps:

1. Rewrite the existing laws in a different but equivalent form.

2. Alter the new form to one which is not equivalent, but indistinguishable.

3. Possibly, rewrite the new form in another equivalent form.

To do this, we must rewrite the following aspects of the gravitational interaction to conform to the form of the other interactions, especially the other long-range interaction, the electromagnetic interaction:

1. The form of the local force law (the geodesic equation)

2. The form of the action

3. The form of the field equations

52.2 Geodesic Equation

We begin by writing the geodesic equation for a body of mass m and charge e including the Lorentz force. [e.g., [18, Weinberg, 1972 eq. (3.2.3) p. 71] & [20, Misner, Thorne, and Wheeler, 1973, pp. 73 and 88]] using standard notation.

$$m[g_{\mu\beta}\ddot{x}^{\beta} + \frac{1}{2}(g_{\mu\nu,\beta} + g_{\mu\beta,\nu} - g_{\beta\nu,\mu})\dot{x}^{\beta}\dot{x}^{\nu}] + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^{\nu} = 0 \tag{52.1}$$

[11, Sciama (1953)] showed that inertial forces (including coriolis and centrifugal forces) would result as an induction force from considering the matter of the universe as a very large current in the coordinate system of the body in question. Davidson (1957) showed that General Relativity already had that property. He showed this in a 3+1 formalism, as did Sciama[11]. We can show this in a 4-vector that matches the 4-velocity of the body on it's world line. The first term of (52.1) is the inertial part of the gravitational force. That the form is the same as the 3rd term suggests that inertia really does arise from an induction type force. It is still necessary, however, to show that a sub μ arises from gravitational current sources. Davidson showed this was true as an approximation, but here, I need to show that this is exactly true and can be used to reformulate gravitational theory.

The usual Lagrangian does not work because it has a velocity squared. We have to have a velocity times some vector function (a vector potential). This will lead to my old formula for a gravitational vector potential. Then, somehow, I have to get π and τ as integrals over all of spacetime.

Equation (52.1) can also be written in the form

$$\pi_{\mu,\nu}\dot{x}^{\nu} - \frac{1}{2}m\dot{x}^{\beta}\dot{x}^{\nu}g_{\beta\nu,\mu} + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^{\nu} = 0 \tag{52.2}$$

or

$$(\pi_{\mu,\nu} - \frac{1}{2}g_{\beta\nu,\mu}\pi^{\beta})\dot{x}^{\nu} + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^{\nu} = 0 \tag{52.3}$$

or

$$\pi_{\mu;\nu}\dot{x}^{\nu} + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^{\nu} = 0 \tag{52.4}$$

or

$$(\pi_{\mu;\nu} - \pi_{\nu;\mu})\dot{x}^{\nu} + \frac{1}{2m}(g^{\alpha\beta}\pi_{\alpha}\pi_{\beta})_{;\mu} + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^{\nu} = 0 \tag{52.5}$$

or

$$(\pi_{\mu,\nu} - \pi_{\nu,\mu})\dot{x}^{\nu} + \frac{1}{2m}(g^{\alpha\beta}\pi_{\alpha}\pi_{\beta})_{,\mu} + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^{\nu} = 0 \tag{52.6}$$

where

$$\pi_{\mu} \equiv mg_{\mu\beta}\dot{x}^{\beta} \tag{52.7}$$

is the momentum of the body, which we represent as a vector field. g is the metric tensor and A is the electromagnetic vector potential.

It is first necessary to point out that the "derivation" of (52.2) through (52.6) is not valid, because the derivation assumes that π is a vector field, which it is not. π defined in (52.7) is a vector, not a vector field, and therefore, it is not valid to consider gradient operations on it. Nevertheless, the various versions (52.2) through (52.6) are equivalent to the geodesic equation (52.1), although other apparently equivalent forms are not equivalent to (52.1). It seems that the non-valid parts of various terms cancel. In addition, some of the above forms of the geodesic equation can be derived from Hamilton's equations, which give also (52.7).

As a check, all versions (52.1) through (52.6) reduce to the same equation in the frame of the body. If we transform to a frame moving with the body, then the troublesome first term in (52.1) disappears, and all six versions of the geodesic equation reduce to

$$m(g_{\mu0,0} - \frac{1}{2}g_{00,\mu}) + e(A_{\mu,0} - A_{0,\mu}) = 0 \tag{52.8}$$

leaving both the geodesic equation and the Lorentz force expressed as simply derivatives of the corresponding potential.

Here, we can consider two possibilities:

1. The force law should be expressed only in the frame of the body.
2. Maybe we can transform this force law to an arbitrary frame without changing the desired form of the law.

Although none of these forms (52.1) through (52.6) are the final form we want, and all clearly have some problems, we are clearly getting close to something. Some of the above forms of the geodesic equation suggest that π might represent something like a vector potential rather than momentum in the usual sense. Let us consider this analogy further. We sometimes consider active gravitational mass, passive gravitational mass and inertial mass separately. Knowing that in General Relativity (and any acceptable gravitational theory nowadays), energy and momentum are also sources of gravitation, we would want to generalize that as active gravitational 4-momentum (the "gravitational currents" that are the sources of a gravitational field), passive gravitational 4-momentum (the "gravitational currents" upon which gravitational fields act), and inertial 4-momentum (actually, something like a vector potential) whose sources are the active gravitational 4-momenta.

It will clearly not be easy to separate passive gravitational 4-momenta from inertial 4-momenta. In addition, what are usually considered to be active gravitational 4-momenta may have some inertial component that may be difficult to separate. Since gravitational theory (especially since the geometric representation of Einstein) purposely makes them the same, it is difficult to separate them.

We can start by assuming that the π in (52.2) through (52.6) represents a vector potential, and try to calculate it from the active gravitational 4-momenta in the universe. It should be some kind of an integral over the whole of space time with a propagator of some sort. We can start with the integral formulation of General Relativity of Sciama, Waylen, and Gilman (1969)[16]. That gives a formula for the metric tensor at a point x_μ as an integral over all of space time of the stress-energy tensor times a Green's function plus a surface term. If we take the universe to be closed and integrate over all of space-time, then there is no surface term.

That takes care of getting $g_{\mu\nu}$, but we want a vector potential, which will be equal to the local momentum of the body of mass m. First, according to the result of Lynden-Bell et al. (1995)[167], The total momentum (actually quasi-momentum) of the universe is zero under some conditions that we satisfy. Let us calculate $g_{\mu\nu}$ at the body in question, in the frame of the body. We can do that with the SWG formula. Now, we want to get the vector potential. To do that, we multiply (contract) the stress-energy tensor by the velocity of the observer. We shall assume that the universe has the symmetries of the Robertson-Walker metric. Quasi-momentum corresponds to a rotation about an axis half way around to the other side of the galaxy (90 degrees away from us). In this case, the total quasi-momentum of the universe is zero, so the quasi-momentum of the body equals minus the quasi-momentum of the rest of the universe.

Somehow at this point, I need to show that the vector potential at the body equals minus the momentum of the body. The calculation should be something like the vector potential will be zero if the body has zero momentum, but will be boosted by the right amount when the body moves.

We start with the formula for $g_{\mu\nu}$ in a frame in which the quasi-momentum of the body is zero (and in which the quasi-momentum of the rest of the universe is zero). If we then multiply that $g_{\mu\nu}$ by $m\dot{x}^\nu$, we will get π_μ, equal to the momentum of the body. At the same time, the integral will give the vector potential of the universe, and will be equal to the same. (Somehow this will work. I just can't seem to get the details right now.)

It depends on the actual universe having the exact symmetry of the Robertson-Walker metric. In the real case, I have to make the argument more carefully, and add in the stress-energy tensor of the gravitational field, as Lynden-Bell et al. (1995)[167] did. This will be more difficult.

I read Lynden-Bell et al. paper[167] again. It seems that the total charge in the universe must be zero, and also the total mass-energy if the universe is closed, because no lines of force can get out. Therefore, having the total angular momentum zero is not so strange. This may not be necessarily true. I have to think about currents. I think currents are no problem. Currents can bring charge into or out of a volume, so that the charge in that volume changes with time. However, if the volume is the whole of a closed universe, then currents cannot take charge in or out of that volume.

It's time to put down some equations. The formula for the metric from Sciama et al. (1969)[16] is

$$g^{\acute{\alpha}\acute{\beta}} = 16\pi G \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{\mu}(T^{\mu}_{\nu} - \frac{1}{2}T^{\lambda}_{\lambda}\delta^{\mu}_{\nu})[-g(x)]^{\frac{1}{2}} d^4x = 16\pi G \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{\mu} R^{\mu}_{\nu}[-g(x)]^{\frac{1}{2}} d^4x \qquad (52.9)$$

We can multiply by $\dot{x}$ to give

$$\frac{\pi^{\acute{\alpha}}}{m} = \dot{x}^{\acute{\alpha}} = \dot{x}_{\acute{\beta}} g^{\acute{\alpha}\acute{\beta}} = 16\pi G \dot{x}_{\acute{\beta}} \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{\mu} R^{\mu}_{\nu} [-g(x)]^{\frac{1}{2}} d^4x \tag{52.10}$$

If we approximate the cosmology by a Robertson-Walker metric, then a rotation is a Killing vector. We would like a translation, but Robertson-Walker is not constant under a translation. However, we can make a quasi-translation, which is a rotation about an axis 90 degrees away (that is, an axis half way around to other side of the universe). That looks like a translation here. This allows us to take that factor inside the integral in (52.10) to give

$$\dot{x}^{\acute{\alpha}} = 16\pi G \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{\mu} \dot{x}_{\acute{\beta}} R^{\mu}_{\nu} [-g(x)]^{\frac{1}{2}} d^4x \tag{52.11}$$

If we don't have a Robertson-Walker metric, then the step from (52.10) to (52.11) is only approximate. However, since the Green function has the property that it does not change to first order for first-order changes in the stress-energy tensor or in the metric, the approximation should be good. In fact, it should not be measurable. However, I will take (52.11) to be the correct and fundamental equation, and (52.10) to be the approximation.

Now, we have π as an integral over currents, as we wanted. There are still a couple of problems, however. First, we have only non-gravitational energy and momentum currents as sources, but we believe that there should be gravitational energy sources also. I can't remember what the other problems were. Oh, yes, the extra term in (52.6).

Another way, is to start in the frame of the body, using (52.8). Then, we take a special case of (52.9) to give

$$g^{\acute{\alpha}0} = 16\pi G \int G^{-\acute{\alpha}0\nu}{}_{\mu} R^{\mu}_{\nu} [-g(x)]^{\frac{1}{2}} d^4x \tag{52.12}$$

where I have multiplied by the mass of the body.

This doesn't seem to be working. However, I can use the inhomogeneous part of the differential equation for the Green function to show that many parts of the Green function are zero for many combinations of indices because they don't get generated at the source and they don't get coupled into. This does allow me to transfer the velocity I multiplied by from the observer partly to the source.

So, I need to start with Gibbs formula for the current, and put that into the SWG integral as a source and see what I get.

short aside about geometry We can define geometry in terms of the travel time of any body. We must generalize laws from applying in any coordinate system to applying in any geometry. That is, we consider transformations from one geometry to another. To test this, I derived the metric for charged massive particles, starting with the Hamiltonian. I get the usual metric. I guess I have to wait until I have a new theory to get a different metric for each kind of particle.

Let us consider which terms in (52.1) through (52.6) seem OK and which not. We want to contrast and compare the gravitational part with the electromagnetic part. We shall consider the electromagnetic part as a model with which the gravitational part should emulate.

The Lorentz force involves derivatives of the electromagnetic 4-vector potential multiplied by the 4-current.

In (52.1) and (52.2), the gravitational part has similar terms involving the derivative of the metric tensor multiplied by the energy-momentum tensor for a single body (a momentum flux), which may be OK for now. However, in (52.1), there is also a term that is proportional to the second derivative of the 4-momentum. This inertial term, that does not have the form of a product of a derivative of a potential with a current or source term, has no counterpart in the Lorentz force. At a minimum, it will be necessary to re-express the force law without such a term.

The first term in (52.2) or (52.3) (which mostly represents the inertial force) has the same form as the first term in the Lorentz force, if we can interpret π as a vector potential instead of a momentum. The second term in (52.2) or (52.3) (which represents the non-inertial gravitational force) has some similarity with the second term in the Lorentz force, if we interpret g as a tensor gravitational potential, and the factor of the stress-energy tensor as analogous to the current in the Lorentz force.

The form of the geodesic equation in (52.4) expresses the non-inertial part of the gravitational force as a covariant derivative, in keeping with the geometric interpretation of gravitation, which we wish here to avoid. The advantage of (52.4), (52.5), and (52.6) is that they are clearly covariant equations.

Although the form of the geodesic equation in (52.2), (52.3), and (52.4) have some similarities to the Lorentz force, that depends on the vector π actually being something like a vector potential, rather than the momentum of the body.

It seems that we really can write π as a vector field. This uses a result of Lynden-Bell et al. (1995)[167]. It seems that Lynden-Bell, Katz, and Bicak[167] have already done the calculation I need. They show that something like momentum (quasi-momentum) is zero when integrated over the whole of space for any time slice if the universe is closed.

It is becoming clearer that most of the kinds of Mach's principle that we would like require a closed universe. Thus, I will here assume that the universe is closed. If it turns out that the universe is open, then my calculation will not apply. However, that would spoil such nice results that I doubt the universe is really open.

On a 2-sphere, quasi-momentum corresponds to a rotation about an axis 90 degrees away. For the Robertson-Walker metric, it corresponds to the same thing one dimension higher.

The idea is that if the total quasi-momentum integrated over 3-space is zero, then the contribution to the total from our body of mass m must be equal to the negative of that of the rest of the universe. Thus, using equations (3.36) (correcting an obvious misprint[4]) and (3.39), allows us to write

$$\pi^\mu = -c\tau \int \hat{J}^\mu d^3x = -c\tau \int [(\hat{T}^\mu_\nu + \hat{t}^\mu_\nu)\xi^\nu + \hat{\Sigma}^{\mu\sigma}_\nu \bar{D}_\sigma \xi^\nu] d^3x \tag{52.13}$$

For momentum for the Robertson-Walker metric as a background, I need to use quasi-translations. Lynden-Bell et al. (1995)[167] prove that the total quasi-momentum of the universe is exactly zero, so the local contribution due to the body in question plus that of the rest of the universe is zero, so the local momentum of the body equals the negative of that due to the rest of the universe. Thus, this is what I need. Actually, Lynden-Bell et al. (1995)[167] get only inertial frames, not inertia. $\sum p_\mu = 0$ does not help.

Actually, their calculation is only for the 3-momentum (actually, 3-quasi-momentum). I still have the 4th component, namely, the energy. I'm not sure how I get that one. Since the Robertson-Walker background metric is time-varying, energy is not strictly conserved, so I have a problem in getting that component in terms of an integral over the universe. That may be related to my problem with the extra term in the gravitational vector potential.

To continue extending Lynden-Bell's calculations[5], I can put it into a form where the inertia part looks like a vector potential part. That is, we have a gradient of a vector times the local current, which looks like a momentum. So, active gravitational mass and active gravitational current is integrated over to give the local vector potential, which replaces (but is equal to) inertial mass or inertial current, which acts upon the local passive gravitational mass or current, which also looks like momentum.

[4]Note added in 2008: I just discovered their errata paper, in which they correct this error along with others[168]

Ordinary gravitational force is the gradient of the metric tensor, multiplied by the tensor current, which is the stress-energy tensor.

Problems:

1. The metric tensor does not include gravitational energy and momentum as a source.

2. We have derivatives with respect to some coordinate system.

3. Equation (52.13) does not give a vector field, only a constant value. This means that I may have made a mistake. No, maybe it gives momentum as a function of time. I'm not sure how we get a function of position. Anyway, it does not give a vector field.

4. Equation (52.13) does not give π as an integral of currents with a Green function. That would give a vector field.

Instead, I should use the Sciama, Waylen, Gilman (1969)[16] formula for the metric tensor. By multiplying the $T_{\mu\nu}$ under the integral by the appropriate angular velocity or quasi-translation, I should be able to get the corresponding vector potential, which should equal the momentum. I'm not sure how to show that. At least there is a Green function, so we get a vector field.

52.3 Relative Newtonian Mechanics

Donald Lynden-Bell[5] has also used a Lagrangian that has relative velocities instead of velocities relative to an arbitrary coordinate system. Possibly I can use that idea for the action in general relativity. In this way, it may be possible to formulate general relativity without reference to any frame, especially to inertial frames. However, this is still a mechanical, kinetic formulation, so doesn't do what I want here.

52.4 Gravitational Vector Potential

Back to the gravitational vector potential.[6] It is equal to π (mechanical momentum) extended to be a vector field minus half the gradient of τ, where τ is the proper time for the body extended to be a scalar field. Maybe it is possible to write π and τ as integrals over some source using the SWG[16] integral formulation of General Relativity. – Yes, it is possible. Maybe Donald Lynden-Bell[5] has already done this. I need to check this. Yes, it works for momentum, including the side condition that makes momentum a vector field. The total momentum of the universe is zero. I'm not sure how to get τ, the proper-time scalar field, though. I don't know to what that corresponds.

τ is the integral of ids, and in the frame of the body, that is just the integral of $d\tau$, which equals the integral of $\sqrt{g_{00}}dt$. Since g_{00} is a constant along a geodesic (even including a Lorentz force), this is a very easy integral. Still, how do I relate τ to the distribution of matter? Somehow, I am setting up a coordinate system based on the trajectory of this body. Maybe I can't relate this to the distribution of matter. Still, g00 is given as an integral over the matter distribution, so at least there is a scaling factor times the integral of dt. Of course, if we knew this scalar field τ as a function of position, we would know a family of solutions of the geodesic equation. If we knew that, we would not have to solve the geodesic equation.

OK, so that is why we can't have a gravitational vector potential. Can we at least get a force law that looks like an EM force law? The momentum part seems to be OK, and should make Sciama's 1953 calculation[11] rigorous, but we still have the τ part. Can we make that right? One component of the gravitational vector potential was the gradient of τ. Locally, the gradient of

[6] An earlier draft of the gravitational vector potential was made on 18 December 1974.

τ should be in the direction of momentum. The magnitude should be about one, or maybe the square root of g, but also multiplied by $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$. I can calculate these things from the distribution of matter. It looks OK. We really can get a gravitational vector potential.

For the other term, we need τ, the local proper time. To get this, we return to the SWG[16] equation, and multiply by $dx^\alpha dx^\beta$ before integrating. This should give us $(d\tau)^2$. We need to take the square root, and integrate. However, we need $\tau_{,\mu}$, so we have to take a gradient. This calculation is still a little funny. I may have to modify it some. It seems very complicated.

There is still the problem about whether I can use the same propagator for these integrals. Does the propagator depend on what is being integrated, or are there general propagators?

Now for the term with the τ, I am still not sure what to do. Does τ correspond to some symmetry that I can use to calculate using their work? One factor in this part looks like the contraction of the stress-energy tensor for a single body. Maybe I can relate that factor to the Action for the rest of the universe. The other factor is the gradient of τ, the proper time for the body.

52.5 Action

Another way to look at the situation is the following. That term in the geodesic equation comes from the mgv squared term in the action, $(d/dx)(d/d\dot{x})$. It is having the v^2 term. To get rid of that term means changing that term in the action somehow, not in value, but in form. We can use gT for that term. That is analogous to the AJ term in the action for the EM part. In both cases we have the product of the potential with the source.

It is not clear what comes next. In deriving Lagrange equations, we use path integrals. We alter the path slightly. We need to specify the path in a more general way than as position. I am not sure how.

Let's look at the action again. Does geometry come from gravity because it is a tensor interaction or because there is a lot of matter in the universe? Can we make a geometry from EM vector interaction? For a geometry, we need the motion to be independent of the charge. That will be true if we neglect all other interactions, and the stars are charged.

Somehow, the formula for the action must show that we could get a metric even without gravitation. EM interaction should be enough to get a geometry. Maybe we are back to a different metric for each body. The problem is, without a geometry, we cannot define position, time, nor derivatives. Maybe we don't need that to define an action, but we do need those to define the force law. Of course, without a geometry, we don't need a force law. All we need is an action (or something equivalent, like Hoyle and Narlikar had for the infitesimal particle propagator, which was multiplicative and nonabelian).

The action is usually defined as the integral over 4D spacetime of a Lagrangian plus some surface terms. We can't define a 4D integral until we have a geometry. How do we get around that one? Maybe we do a sum over bodies. That should be ok. But what do we do about the fields?

Let's do a calculation. The usual Lagrangian for a body of mass m and charge e in General Relativity is

$$L = -\frac{1}{2}mg_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - eA_\mu\dot{x}^\mu \tag{52.14}$$

This leads to the geodesic equation (52.1). With the insight we have gained so far, we can say that at least one of the $\dot{x}$s in (52.14) corresponds to a vector potential, not a kinetic term. Thus, we can imagine replacing one of the $\dot{x}$s by π/m. That Lagrangian will not give the correct law of motion (52.5) or (52.6) because of the factor of a half. However, if we take the Lagrangian to be

$$L = -\pi_\mu\dot{x}^\mu - eA_\mu\dot{x}^\mu \tag{52.15}$$

then we do get part of (52.5) or (52.6) from Lagrange's equations.

52.6 Metric

Next, we need to have inertia from EM. That is ok. Next, we need to have a metric. In the usual gravitational case, we define the metric in terms of light distance. That is, the actual distance (squared) minus the distance (squared) light could travel in the time difference. This us useful because light has a constant speed in flat spacetime. For the EM case, we use a different particle to define the metric. In analogy with light, we need a chargless particle. This isn't quite right. Light has no rest mass, but has energy and momentum. In the EM case, we need a particle that has no rest charge, but has current. It is not clear what that would be. This may be OK if it moves at the speed of light because of the gamma factor that would be infinite at the speed of light. Maybe there would be such a particle if the universe were charged. Maybe light would not exist if the universe did not have a lot of mass.

Back to the origin of geometry. We recognize that there can be different kinds of metrics, one for each kind of particle. Why does the light metric give the gravitational potential? Light is EM. Light is also massless energy and momentum. A neutrino is also massless energy and momentum. A photon is the carrier of EM field. A neutrino is probably not the carrier of any field because it is a Fermion.

Maybe the speed of light is induced by the matter in the universe. Also all fields. Maybe time and distance are induced. Maybe the EM field really does not have independent degrees of freedom. Maybe it comes entirely from charges and currents. Maybe the same is true of gravitons. That is, maybe there really are no independent degrees of freedom of the gravitational field. In that case, Mach's principle is easier.

52.7 Currents

Actually, it may be different. Current is the transport of charge as momentum is the transport of mass, except that it is possible to have momentum without mass in the case of light, but it is not possible to have current without charge, except for possibly some kind of current rings? The energy-momentum tensor is the transport of momentum.

52.8 Dennis Sciama's Calculation

Also, in the EM case, we have two kinds of charges, plus and minus. The 1953 calculation of Sciama[11] applied to EM shows that in a positively charged universe, positive charges will attract, negative charges will repel, and opposite charges will accelerate in the same direction, with the positive charge in the lead.

52.9 Quantum effects

A graviton is the carrier or the gravitational interaction, and is massless with spin two. It has both energy and momentum. In the absence of a lot of matter in the universe, we don't have Minkowski space, we have no geometry. We have a lot of Minkowski spaces in a quantum superposition (Jones, 1979, 1980, 1993, 2002)[107, 169, 170, 171]. Thus, the building up of a geometry is not a simple adding up of contributions. We must at least do a path integral. It may be enough to get the formula for the action right.

Also, how do we treat the wave properties of the particles? On the other hand, if the wave properties of the particles is simply an artifact of trying to fit a tensor gravitational field with a vector EM field, maybe we don't need to worry about the (artifact) wave properties of particles.

But if the wave properties of particles is an artifact, then what happens to the idea of wave interference among cosmologies whose action is not an extremum?

Here is a conflict. The action is useful only if there is an underlying quantum theory. But if the apparent wave properties of particles is only an artifact of trying to bring a tensor gravitational field in line with a vector EM field, where does that leave us? If the quantum behavior comes only from trying to make a gravitational vector potential?

There is still the idea that wave properties come from a linearization of some underlying nonlinear behavior. Wave properties look like first order linearization of a nonlinear calculation.

Chapter 53

Gravitation is Not Geometry - I[1]

53.1 Introduction

Three of the four fundamental interactions have been unified. Part of the difficulty with unifying gravitation with the others depends on two things:

1. The gravitational interaction is treated as geometry.

2. The gravitational interaction has not yet been quantized.

Let us consider the effect of the former. As long as the gravitational interaction is treated as a geometric arena upon which all other interactions are staged, it will be difficult to unify it with the other interactions on an equal basis. If the gravitational interaction is intrinsically geometrical in nature, then it will remain difficult to unify gravitation with the rest of the fundamental interactions. If, however, the apparent geometrical character is an accident of the fact that there is so much matter in the universe that the local geometry is induced by an interaction with rest of matter in the universe, then expressing the gravitational interaction in non-geometrical terms may simplify the unification process. (It may also simplify the quantization of the gravitational interaction.)

Ernst Mach[120, 102] argued in the last century, using different terminology, that the latter was the case. That is, that inertia was not an intrinsic property of matter, but due to an interaction with the rest of matter in the universe. Although inertia and geometry are not the same, similar arguments that Mach used for inertia can also be applied to geometry. Mach argued that the law of inertia was wrongly expressed. We can argue similarly that expressing gravitation in terms of geometry leads us in a wrong direction when we try to generalize laws of physics.

Einstein (1905)[172] pointed out that Newtonian mechanics had Galilean invariance whereas electrodynamics had Lorentz invariance, and that this was inconsistent. Reformulating mechanics in terms of Lorentz invariance made them consistent and led to the special theory of relativity.

Here we have a similar situation. Standard quantum theory has Poincare invariance whereas our cosmology has an approximate invariance which is that of the Robertson-Walker spacetime. I need to look into what Bernard Kay and others have done in developing quantum theory on a curved background to see if they have a quantum theory on a Robertson-Walker metric. The goal here is not to replace present gravitational theory by a different one (although that may be the result), but to express present theory in a form in which the gravitational interaction does not have a preferred position among the fundamental interactions as is now the case where gravitation is the geometry that forms the arena in which the other fundamental interactions reside.

I can think of only two ways of resolving this difficulty:

[1] essay3, April 1997

1. We discover a way to separate gravitation from geometry.

2. We discover a way in which geometry can be influenced (or determined) by any of the fundamental interactions.

In the first way, we think of geometry as an arena upon which all four fundamental interactions reside, but not influenced by any of them. Here, we would have to imagine a way to express General relativity in which geometry and gravitation are somehow separate. This would probably require, for example, a separation of the metric tensor g that raises and lowers indices from the g that represents the gravitational potential. Or alternatively, it might require a representation which avoids geometrical concepts.

One possible representation might be to calculate the force on any body only in a coordinate system fixed with that body. Thus, the sum of the forces on any body (gravitational plus electromagnetic plus others) would be zero. The metric tensor would be calculated in the frame of the body using possibly an integral formulation such as that used by Sciama, Waylen, and Gilman (1969)[16]. The metric tensor would be used only in that way as the gravitational interaction, and would have no geometric meaning, and could not be used to raise or lower indices. There would be no transformation of equations from one frame to another. Some other method would have to be found to make sure equations are consistent. I do not know if such a system could be made consistent.

In the second way, we again think of geometry as an arena upon which all four fundamental interactions reside, but a geometry that is determined (at least in principle) by all four interactions and which acts through all four interactions. In this case, the formulation must be such that it does not favor one particular interaction at the outset. For example, because energy and momentum are sources of gravitational fields and are acted upon by gravitational fields, energy and momentum are gravitational quantities. Therefore, to base a theory on energy and momentum is to bias that theory at the outset by treating gravitation as special. Lagrangians, are usually based on energy and momentum, even when they are used to both gravitational and electromagnetic interactions. Similarly, The stress-energy tensor (which can be used to derive both the gravitational interaction in terms of the geodesic equation and the Lorentz force) is based on the transport of energy and momentum.

One possible requirement, therefore, might be to consider the electromagnetic analog of energy and momentum, that is, charge and electric current, and their transport. We thus imagine the electromagnetic analogy of the stress-energy tensor to be a tensor that represents the transport of charge and electric current (not the energy or momentum of a charged particle or the energy or momentum of an electromagnetic field). We would similarly require an extension of the Lagrangian (if a Lagrangian formulation is used).

Let us consider an argument against this line of thought. The concept of force was originally invented by Newton (I think) and defined as what produces acceleration. Mach (1933)[122] pointed out that Newton's second law was really very circular. We now know that mass times acceleration (and other inertial forces) are real gravitational forces. So we are left with a choice (or dilemma). We can either think of there being different kinds of forces for different interactions and the sum of those forces in the frame of the body under question must sum to zero, or we can think of force in general as being a gravitational quantity, and other interactions may possess that gravitational quantity. So, for example, is the Lorentz force an electromagnetic quantity or is it a gravitational aspect of the electromagnetic interaction? Energy and momentum of the electromagnetic field and of electrons are gravitational quantities, for example, but electric charges and currents are electromagnetic quantities. I don't yet know the answer to this question, but I don't think we are free to make a choice here. If the Lorentz force is a gravitational quantity, then the usual way of calculating it either from a (gravitational) Lagrangian or from a stress-energy tensor, would be perfectly OK. But if the Lorentz force is an electromagnetic quantity, then we must formulate things differently.

Actually, this gets back to the main point. If the Lorentz force (and force in general) is a gravitational quantity, then all types of force interaction is a gravitational quantity, even when the force is from another interaction. In that case, it is appropriate for gravitation as geometry to be the arena for all other interactions, because those interactions are gravitational (geometrical) in character. In that case, gravitation is a special kind of interaction, and either it cannot be unified with the other three, or if it is unified, it will be unified in a different way, possibly on a different footing with the other three.

On the other hand, if the Lorentz force is not gravitational in character, then it is appropriate reformulate gravitation in such a way that geometry is not associated with gravitation any more than with any of the other three forces.

What kind of experiment can I do or what kind of thought experiment can I do to tell which is the case? I can't think of any experiment to do. The only thing I can think of to do is just try assuming that force in general is not just a gravitational concept and that the four forces should be on an equal basis and see if I can get such a representation. The geodesic equation including Lorentz force come from $\nabla \cdot \mathbf{T} = 0$. The $\nabla \cdot \mathbf{T}$ for a mass m gives the rate of change of momentum for the body, which leads to the geodesic equation. The $\nabla \cdot \mathbf{T}$ for the EM field gives the rate of change of momentum for the field. combining with Maxwell's equations gives the Lorentz force. Taken together, $\nabla \cdot \mathbf{T} = 0$ expresses conservation of momentum for the body plus field together. The equation does not express that momentum is conserved because gravitational interaction with the rest of matter in the universe leads to the inertial force that causes conservation of momentum locally.

$\nabla \cdot \mathbf{T} = 0$ comes from Einstein's field equations, $G = T$. $\nabla \cdot \mathbf{G} = 0$ is an identity. It then requires $\nabla \cdot \mathbf{T} = 0$ to give conservation of momentum. Einstein's equations tell how matter curves spacetime. They do not tell how to orient flat spacetime. The only orientation comes from scientists' habit of choosing solutions of the field equations to have the same symmetries as those of T. The equations do not require that, since they are differential equations. That is the same as adding boundary conditions.

We usually say that Gamma gives the gravitational force. $\Gamma = 0$ for flat spacetime, but there is still one term in the geodesic equation, namely $\ddot{x}$.

So, we need to change or supplement Einstein's equations to handle the orientation of spacetime and possibly more.

Notice that the field equations require that G have the same symmetries as T, but they do not require g to have the same symmetries as T. We could require in addition to the field equations that g have the same symmetries as T. But, what if T doesn't have any symmetries? There is still the orientation problem. This is like a boundary condition. Maybe it is equivalent to a boundary condition. If the universe is closed, we don't have any boundary conditions, and we can write g as an integral of T over all of spacetime. Maybe that takes care of the orientation problem.

We still have the problem that $G = T$ requires local conservation of momentum without a mechanism that shows that local conservation of momentum comes from interaction with the rest of matter in the universe. I think that $G = T$ must be an approximation for the true equations that is valid only when there is a lot of matter. I think Hoyle and Narlikar did something like that a long time ago.

Part of the problem is that G gives the curvature of spacetime, and T gives the matter. This assumes that no curvature is the natural state without matter.

In addition, T gives the local kinetic energy and momentum of matter, but is not in terms of relative coordinates as gravitational potential energy is. Donald Lynden-Bell[5] fixed that up for Newtonian gravitation a few years ago. He got kinetic energy (and the Lagrangian) in terms of relative coordinates. I need to do that for General Relativity. It will be harder because of the curved spacetime, but it should be possible. I may need to define some new Green's functions.

The next thing to do is to use the Lagrangian of Donald Lynden-Bell[5] to estimate the first Fresnel zone for a frame in which the momentum and angular momentum of the universe is not zero. I would consider an expanding universe for either the open or closed case (total energy positive or negative).

After that, try to use the insight to alter Einstein's field equations. Try to imagine what Einstein would have done if we had had Lynden-Bell's equations[5] instead of Newton's available.

In addition, try to imagine how quantum theory would have developed if we had had Lynden-Bell's equations[5] back then. For the nonrelativistic case, I think it would work like this. $(dr_{ij}/dt)^2$ would become $M(p_{ij})^2/(m_i m_j)$ I think. Then p_{ij} would become partial derivative with respect to r_{ij}. It would all be in terms of relative coordinates. In any case, there is a straightforward way to define canonical momenta and the Hamiltonian.

That is OK for the non-rotating frame, but for the rotating frame, it will be more difficult. It should still not be more difficult than for quantum theory in the presence of a magnetic field. It might be more difficult. This time, it adds some gradient terms, not just force or potential terms.

For the relativistic case, it is not so easy. Special relativity is probably not valid except as an approximation. It would depend on how Maxwell's equations generalize in the relative particle formulation. The speed of light might not be a constant.

I think the symmetry group should be for rotations of a 3-sphere in 4 dimensions. I think the group is O(4), which is equal to SU(2) $\times$ SU(2). So, what does that signify about special relativity? SU(2) is the special unitary group, which is the set of all 2-by-2 unitary matrices (matrices with determinant = 1). These correspond to the angular momentum spin matrices. The Casimir operator is J^2, which has the eigenvalue $J(J+1)$. So, for O(4), instead of mass and spin for Casimir operators, we have two spins.

The above assumes that the Robertson-Walker metric will still be a solution for whatever theory takes the place of General Relativity. Well, let's assume that for now. So, what will replace 4-vectors with $E^2 - P^2 = m^2$?

Since special relativity is valid to a high degree of precision, I have to make sure the formulas agree approximately with special relativity.

I continue some of these lines of reasoning in the file "symm" (Chapter 69) on the unix computer terra.

I then realized that I could start with Donald Lynden-Bell's[5] extension of Newtonian mechanics in Julian Barbour's proceedings of the 1993 Mach's principle conference in Tübingen. I had to change them somewhat, and found out that I had independently derived the formulas of Reissner (1914)[2, 3] and Schroedinger (1925)[4], except that Lynden-Bell[5] had added rotation.

From these equations, I realized that I could ask what would have happened if these relative-coordinate formulations of Newtonian theory had been available to Einstein in 1905 and 1916, and to Schroedinger and Heisenberg in 1925. I examine these ideas in the next essay, Essay4 [which has been superseded by Essay5, Chapter 60].

Chapter 54

Can the Universe be Open?[1]

Recent evidence [173, 174, 175, 176, 177][2] indicates that the universe may be open. Frankly, this news is disconcerting to me, as it probably is others who have been influenced by the ideas of Ernst Mach. The problem with an open universe is that solutions to Einstein's field equations require boundary conditions in addition to initial conditions for an open universe, whereas a closed universe requires no boundary conditions because it has no boundary. Such boundary conditions at infinity can give rise to gravitational effects, including inertia, just as much as matter. We would then require for our universe particular boundary conditions to achieve our observed universe in which local inertial frames in the Solar system appear not to rotate or accelerate relative to the visible universe.

Most people who believe in some form of Mach's principle assume the universe to be closed, as did Einstein, who was influenced by Mach's ideas. Some particular calculations based on Mach's principle also lead to a conclusion of a closed universe [12, 156, 109, 159, 5].

I consider here the consequences of that observation. The main consequence is that the cosmological constant Λ can no longer be considered to be zero. When Λ was considered to be zero, there were only three possibilities for a homogeneous, isotropic universe:

1. The universe is finite, has positive spatial curvature, will reach a maximum expansion, and collapse back to a final crunch.

2. The universe is infinite, is spatially flat, and will expand forever.

3. The universe is infinite, has negative spatial curvature, and will expand forever.

That is, the universe will reach a maximum expansion if and only if it is finite. The two go together if $\Lambda = 0$. That is no longer true if $\Lambda \neq 0$. In that case, whether the universe is finite or whether it will expand forever or not are two independent conditions. However, it is now generally accepted that

- Our universe is open, is spatially flat, will continue to expand forever, has an infinite volume, and has an infinite amount of matter.

[1] 31 December 1997, revised 27 October 2022

[2] Citations added later.

Chapter 55

What is gravitation?[1]

What is gravitation? A popular answer is:

$$\text{Gravitation is the curvature of spacetime.} \tag{55.1}$$

If that is so, then there is no gravitation in Minkowski spacetime. Then, inertial forces (which certainly exist in a flat spacetime) are not gravitational forces. One could certainly argue, however, that Minkowski spacetime is a bad example for a cosmology, although it seems to be a good approximation locally (as far as inertial forces are concerned). But even if we choose a more realistic cosmology, say the Robertson-Walker metric (either open or closed), the curvature is very small and insignificant, especially with regard to its effect on inertia.

The point is, as is well-known, that inertial forces are not manifestations of spacetime curvature, but of spacetime itself. If we wish to regard inertial forces as gravitational forces, then we must abandon (55.1) as a definition of gravitation. Instead, let us consider:

$$\text{Gravitation is any force that acts on masses.} \tag{55.2}$$

This is a little better. It includes inertial forces in addition to the usual gravitational forces due to spacetime curvature. However, it does not include gravitational forces on massless particles, such as the photon. Thus, it would not include bending of light by the sun, for example. Let us then consider:

$$\text{Gravitation is any force that acts on energy or momentum.} \tag{55.3}$$

At first glance, this seems to handle it. One problem, however, is that energy and momentum are not as easy to define. They are not scalar quantities. They depend on the coordinate system. In that sense, mass was better for massive bodies, because mass is a scalar. Also, there may be some circularity creeping in.

It is useful at this point to consider analogy with the electromagnetic case. We should not expect the analogy to be completely helpful, however, because our laws of electromagnetism (Maxwell's equations plus the Lorentz force equation) depend on an inertial background that is determined outside of the EM framework. (The Green's functions depend on the inertial background.)

In the electromagnetic case, it is perfectly reasonable to define electromagnetism as the analogy to (55.2) That is:

$$\text{Electromagnetism is what acts on charges.} \tag{55.4}$$

Well, that is not quite right. What about currents? OK, then:

$$\text{Electromagnetism is what acts on charges or currents.} \tag{55.5}$$

[1] 29 January 1998

That looks better, and seems to be the analogy of (55.3) In fact, as we all know, charge/current is simply a 4-vector proportional to 4-velocity in the same way that 4-momentum is proportional to 4-velocity. The corresponding scalars are "rest charge" and "rest mass."

But, let's consider the differences between gravitation and electromagnetism so far. First, there are both positive and negative charges. That gives the possibility of having a current (say in a wire) even if the net charge is zero. As far as we know, that is not possible in the gravitational case. But that is a macroscopic example. If we restrict our examples to single particles, then we cannot have current without charge.

Another difference is the one that led to this comparison: A massless particle can have energy and momentum, whereas a particle that has zero "rest charge" cannot have charge or current. It is not clear why not. A photon must travel a the speed of light to have energy and momentum. A chargeless particle would have to travel at the speed of light to have charge and current. So, why don't massless, chargeless particles have charge and current if they travel at the speed of light?

My guess comes back to Mach's principle. There is a lot of mass in the universe, but as far as we know, the net charge of the universe is zero. The mechanism, however, is illusive.

After that interlude, we come back to the question, "What is gravitation?" It seems that (55.3) is the correct answer, because it is analogous to (55.5) However, there is still something wrong. Our notions of energy and momentum may be circular.

I think, however, that we finally come back (partly) to Einstein's position:

$$\text{Gravitation is geometry.} \tag{55.6}$$

Let us consider the implications of (55.6). It implies first, that absence of gravitation requires absence of geometry. Not just absence of curvature, but absence of geometry, for as we have seen earlier, geometry has inertial forces, and as we agreed, any force that acts on mass is considered a gravitational force, including inertial forces. It is difficult to imagine what "absence of geometry" means, but we shall return to that point later.

Now, let us consider Einstein's field equations

$$G = T \ , \tag{55.7}$$

where I have suppressed the two indices and some constants. The stress-energy tensor T is usually considered to be the source of gravitation. However, setting T to zero (which should result in no gravitation) does not lead to "absence of geometry", although it can lead to absence of curvature. (It does not necessarily lead to absence of curvature, because T is a source of curvature, but that curvature can propagate away from its source.) Thus, the meaning of (55.7) is that T (which is normally considered a source of gravitation) is actually a source of curvature. Thus, gravitation according to Einstein's field equations (55.7) is not geometry, but curvature. Thus, we have a contradiction between (55.2) or (55.3) or (55.6) and (55.7). It is this contradiction that I want to explore further.

It is clear that (55.7) can never have as its solution "absence of geometry." In fact, I cannot imagine any equation that involves geometry in simple terms as having the solution "absence of geometry." I think that the proper interpretation of (55.7) is that it gives the change in geometry for sources that can be described in terms of a stress-energy tensor T **in the presence of a background of matter in the universe**, but it cannot satisfactorily describe all of gravitation.

With that step, we seem to be getting close to some of Ernst Mach's ideas, but probably not in a way that he had imagined. Some questions may give us some clues, such as, "Is gravitation intrinsically geometry or is it geometry by default because there is a lot of mass in the universe that induces geometry?" That is, if the universe were electrically charged, then would electromagnetism be geometry? Or, if we had neither a lot of mass nor a lot of charge in the universe what would the universe be like? I doubt that we would have either inertia or geometry.

That leads us back to trying to imagine what "absence of geometry" can mean.

Chapter 56

Gravitation is geometry, not curvature[1]

Consider a particle of mass m and charge q. The motion of the particle is found by summing the geodesic equation (multiplied by the mass of the particle) with the Lorentz force equation (multiplied by the charge of the particle) and setting that sum to zero.[20, eq. (20.41), p. 474]

$$(g_{ij}\ddot{x}^j + \Gamma_{ijk}\dot{x}^j\dot{x}^k)m + [(A_{i,j} - A_{j,i})\dot{x}^j]q = 0, \tag{56.1}$$

where A is the 4-vector EM potential, and i, j, and k range from 0 to 3.

We can consider the part that multiplies the mass (the geodesic equation) to be the gravitational part of the interaction. (Is it possible to disagree with that?) Absence of that term (or its having a value of zero) would correspond to an absence of gravitation. If at least an electric field is present, then the part of the equation multiplying the charge will be nonzero, so that the gravitational part must also be nonzero for the sum to be zero. If we consider a geometry with no curvature (such as Minkowski space), then the gravitational part will still be nonzero, showing that gravitation does not require curvature. In fact, absence of gravitation corresponds to absence of geometry, not absence of curvature. What is meant by "absence of geometry" will be considered later.

Now let us consider Einstein's field equations

$$G = T \tag{56.2}$$

where I have suppressed the two indices and dropped some constants. Absence of gravitational sources corresponds to the stress-energy tensor T being zero. That leads to at most an absence of curvature, not an absence of geometry. In fact, no equation involving the geometry in a simple way such as (56.2) could lead to an "absence of geometry," such as we discussed above.

Thus Einstein's equations (or anything like them) are inadequate to describe gravitation in general (i.e., including absence of gravitation). Instead, an interpretation of (56.2) is the equation that describes gravitation when gravitation is strong enough. Possibly, it is the mass associated with the stars, galaxies, etc. that make gravitation strong enough to be described by geometry, such as (56.2).

That (56.2) cannot apply to "absence of geometry" and therefore not to "absence of gravitation" suggests that there is a transition region where gravitation is not absent, but still cannot be described by (56.2) or by a geometry. Presumably, the general theory that includes "absence of gravitation" would indicate where such a transition would occur.

This leads to two questions:

1. What do we mean by, "absence of geometry"?

[1] 30 January, 2 February 1998, file what2.tex

and

2. What is the theory that describes gravitation in general?

Let us first consider the first question. The only thing I can think of to imagine "absence of geometry" is some kind of quantum geometry, something like what Wheeler refers to as quantum-geometrodynamics. This does not mean that the correct generalization of gravitation must be a quantum description; only that my imagination can not think of anything else.

This suggests a possibility for an answer to the second question. Namely, some kind of quantum gravity. Of course, we don't yet have a theory of quantum gravity. There seem to be problems with divergences of perturbation expansions, etc. Quantum cosmology seems to be even more difficult, but this looks like the most likely way to go. Perhaps by thinking along the lines of thought expressed here, we can solve these problems.

Chapter 57

Is gravitation geometry or curvature?[1]

Is gravitation geometry or curvature? If we judge by the geodesic equation, then geometry is the answer, not just curvature. The geodesic equation has two parts, the x dot dot part, which represents inertia, and the Γ part, that represents the usual gravitational forces. The geodesic equation represents inertia balancing the other gravitational forces.

Often, we think of the Γs as being not quite real because they are not tensors. Thus, the fact that they can be transformed away (a frame can always be found in which the Γs are zero) makes them seem less valid. However, the geodesic equation is a tensor equation. The sum of all the terms in the geodesic equation is a tensor, and if that sum is zero in one frame, then that sum will be zero in all frames.

More specifically, consider the frame fixed on the body itself. In that frame, the x dot dot term is identically zero. (A body does not accelerate in its own frame.) For the geodesic equation to hold in that frame, the sum of the Γ terms must be zero. As it turns out, all of the Γs are zero in that frame. This is not because the Γs have been "transformed way," but because the body is constrained to move in such a way that that the Γs are zero in the frame of the body. (The sum of the forces on a body in its own reference must be zero.)

Now consider the motion of a charged, massive particle. In that case, The Lorentz force must be included in the geodesic equation. This gives[20, eq. (20.41), p. 474]

$$(\text{gravitational part})m + (\text{Lorentz part})e = 0\ . \tag{57.1}$$

The "gravitational" part is the sum of the $\ddot{x}$ term and the Γ terms from the geodesic equation. Equation (57.1) just says that the total gravitational force (including inertial forces) plus the electromagnetic forces must sum to zero. It does not seem difficult to justify including inertial forces with gravitational forces since they are all multiplied by the mass of the body.

If now we transform to the frame of the body, the $\ddot{x}$ term is still zero. (The body does not accelerate in its own frame.) However, the Γs will no longer be zero. The body will move in just such a way that the gravitational forces plus the electromagnetic forces will be zero. (For simplicity, I am not considering possible magnetic dipole interactions on the body.)

In general, then, "gravitational" part of the forces in (57.1) is the part that multiplies the mass m. Setting the "gravitational" part to zero would give the geodesic equation. Even in the absence of curvature, there are gravitational forces. In Minkowski space, for example, that has no curvature, a charged, massive particle will move so that the x dot dot term balances the Lorentz force.

The point is, that all of the terms that multiply the mass are gravitational terms, including inertia. Thus, gravitation is geometry, not just curvature.

[1] 3 February 1998, file what3.tex

57.1 Einstein's field equations

Consider now Einstein's field equations,

$$G = T \ , \tag{57.2}$$

where I have left off the two subscripts and some constants. We normally consider the stress-energy tensor T to be the source gravitational fields. However, the simplest solution for T=0 is Minkowski space, which has no curvature. There are other solutions, that have curvature, but we can still think of T as a source of curvature.

Thus, Einstein's field equations describe sources of curvature, not sources of geometry. But if gravitation is geometry rather than curvature, then Einstein's equations (57.2) do not describe sources of gravitation. In fact, no equation that does not describe sources of geometry could correctly describe sources of gravitation.

57.2 Absence of Gravitation

A correct description of sources of gravitation must be able to include the case of "absence of gravitation." That is, the case where all of the gravitational fields are zero (even in the presence of electromagnetic forces). In that case, the body would move in such a way that in the frame of the body, the electric forces would be zero. (There are no magnetic forces on a body in its own frame.) "Absence of gravitation" means that the gravitational forces are zero in all frames.

However, since the existence of geometry implies the existence of gravitational forces (the existence of frames where the gravitational forces are not zero), then the absence of gravitation requires the absence of geometry.

"Absence of geometry" is hard to imagine. The only thing I can image is what Wheeler calls quantum-geometrodynamics, actually some kind of quantum gravity. There may be other possibilities for "absence of geometry," but I cannot imagine any.

Thus, the theory we need to fully describe sources of gravitation is probably some sort of quantum gravity theory. A fully classical (geometrical) theory is not adequate to describe gravitation completely.

Is gravitation, then, fundamentally different from other interactions, such as electromagnetic, which seem not to require a quantum description? Probably not. Our present electromagnetic theory is a theory superimposed on a geometrical background. The Green's functions for E&M depend very much on the existence of local inertial frames. Without geometry (that is, without gravitation), E&M would also presumably be much different. That may give us some clues for how quantum gravity must be.

Chapter 58

Is gravitation geometry?[1]

58.1 Introduction

The geodesic equation including Lorentz force that gives the motion of a body of mass m and charge e in the presence of an electromagnetic field is:[20, eq. (20.41), p. 474]

$$(\text{gravitational part})m + \ (\text{Lorentz part})e = 0 \tag{58.1}$$

If there are no electric or magnetic fields, then

$$(\text{Lorentz part}) = 0 \tag{58.2}$$

identically for all 4 components in all frames of reference independent of the motion of the body. In that case (in the absence of an electromagnetic field), the motion of the body will be governed by the pure geodesic equation:

$$(\text{gravitational part}) = 0 \tag{58.3}$$

In this case, the motion of the body is governed by pure gravitational effects, and is independent of the charge of the body.

The opposite case (a pure electromagnetic field with zero gravitational field) would correspond to

$$(\text{gravitational part}) = 0 \tag{58.4}$$

identically for all 4 components in all frames of reference independent of the motion of the body, so that the motion of the body would be governed by purely Lorentz forces:

$$(\text{Lorentz part}) = 0\ . \tag{58.5}$$

This second case does not exist, for any experiment that can be done in the existing universe, not in theory, because it is not possible to find any situation in which the geodesic equation is identically zero in all frames of reference independent of the motion of the body, because that is not a property of any geometry that I know of.

Thus, Einstein's theory in particular, and geometrical theories in general, can not be a complete description of general gravitational effects, because they do not allow the case of zero gravitational fields.

[1]file what4.tex, written between 3 and 10 February 1998

58.2 Comments - 2008

I didn't explicitly say so, but since the terms in (58.1), (58.3), and (58.5) depend on the motion of the body (relative to the coordinate system), the body will move in such a way that the sum of the terms add to zero in each of the four components. Even if the coordinate system is chosen to be on the body, the coordinate system will be required to be such that the sums of the terms in each of the four components is zero.

For the case (which does not exist for our universe) that there are no gravitational forces and no electromagnetic forces, then there will be nothing to determine the motion of the body. We might approach that case for a sparse universe. From the wave mechanics view, we might have very large de Broglie wavelengths, and not a very classical universe.

Thus, wave properties of bodies or particles might follow naturally from just thinking carefully about the situation.

Chapter 59

Is gravitation geometry? - II[1]

59.1 Introduction

Einstein postulated that the equation of motion for bodies under the influence of gravitation, but no other forces, is the geodesic equation. That is, that bodies under the influence of only gravitation follow a geodesic in a curved spacetime. I argue here that there are some cases where gravitation cannot be represented by geodesics.

The geodesic equation including Lorentz force that gives the motion of a body of mass m and charge e in the presence of an electromagnetic field is:[20, equation (20.41), p. 474]

$$(\text{gravitational part})m + (\text{Lorentz part})e = 0 \tag{59.1}$$

If there are no electric or magnetic fields, then

$$(\text{Lorentz part}) = 0 \tag{59.2}$$

identically for all 4 components in all frames of reference independent of the motion of the body. In that case (in the absence of an electromagnetic field), the motion of the body will be governed by the pure geodesic equation:

$$(\text{gravitational part}) = 0 \tag{59.3}$$

In this case, the motion of the body is governed by pure gravitational effects, and is independent of the charge of the body.

The opposite case (a pure electromagnetic field with zero gravitational field) would correspond to

$$(\text{gravitational part}) = 0 \tag{59.4}$$

identically for all 4 components in all frames of reference independent of the motion of the body, so that the motion of the body would be governed by purely Lorentz forces:

$$(\text{Lorentz part}) = 0\ . \tag{59.5}$$

This second case does not exist, for any experiment that can be done in the existing universe, nor in theory, because it is not possible to find any situation in which the geodesic equation is identically zero in all frames of reference independent of the motion of the body, because that is not a property of any geometry that I know of.

Thus, Einstein's theory in particular, and geometrical theories in general, cannot be a complete description of general gravitational effects, because they do not allow the case of zero gravitational

[1] file what5.tex, 10 February 1998

fields. (No gravitational fields means also no inertial fields. I take gravitational fields to be any force that acts on the mass of a body. This is in the spirit of Einstein's representation of gravitation by a geodesic equation. The geodesic equation includes inertial forces.)

Question: are we in agreement so far?

59.2 Comments - 2008

I didn't explicitly say so, but since the terms in (59.1), (59.3), and (59.5) depend on the motion of the body (relative to the coordinate system), the body will move in such a way that the sum of the terms add to zero in each of the four components. Even if the coordinate system is chosen to be on the body, the coordinate system will be required to be such that the sums of the terms in each of the four components is zero.

For the case (which does not exist for our universe) that there are no gravitational forces and no electromagnetic forces, then there will be nothing to determine the motion of the body. We might approach that case for a sparse universe. From the wave mechanics view, we might have very large de Broglie wavelengths, and not a very classical universe.

Thus, wave properties of bodies or particles might follow naturally from just thinking carefully about the situation.

Chapter 60

Gravitation is Not Geometry - II[1]

60.1 Introduction

Three of the four fundamental interactions have been unified. Part of the difficulty with unifying gravitation with the others depends on two things:

1. The gravitational interaction is treated as geometry.

2. The gravitational interaction has not yet been quantized.

Let us consider the effect of the former. As long as the gravitational interaction is treated as a geometric arena upon which all other interactions are staged, it will be difficult to unify it with the other interactions on an equal basis. If the gravitational interaction is intrinsically geometrical in nature, then it will remain difficult to unify gravitation with the rest of the fundamental interactions. If, however, the apparent geometrical character is an accident of the fact that there is so much matter in the universe that the local geometry is induced by an interaction with rest of matter in the universe, then expressing the gravitational interaction in non-geometrical terms may simplify the unification process. (It may also simplify the quantization of the gravitational interaction.)

Ernst Mach[120, 102] argued in the last century, using different terminology, that the latter was the case. That is, that inertia was not an intrinsic property of matter, but due to an interaction with the rest of matter in the universe. Although inertia and geometry are not the same, similar arguments that Mach used for inertia can also be applied to geometry. Mach (1933)[122] argued that the law of inertia was wrongly expressed. We can argue similarly that expressing gravitation in terms of geometry leads us in a wrong direction when we try to generalize laws of physics.

Einstein (1905)[172] pointed out that Newtonian mechanics had Galilean invariance whereas electrodynamics had Lorentz invariance, and that this was inconsistent. Reformulating mechanics in terms of Lorentz invariance made mechanics and electrodynamics consistent and led to the special theory of relativity.

Here we have a similar situation. Standard quantum theory has Poincare invariance whereas our cosmology has an approximate invariance which is that of the Robertson-Walker spacetime. I need to look into what Bernard Kay and others have done in developing quantum theory on a curved background to see if they have a quantum theory on a Robertson-Walker metric.

The goal here is not to replace present gravitational theory by a different one (although that may be the result), but to express present theory in a form in which the gravitational interaction does not have a preferred position among the fundamental interactions as is now the case where gravitation is the geometry that forms the arena in which the other fundamental interactions reside.

[1] file essay5.tex, written sometime before 1 April 1998, possibly 19 February 1998, Copied from essay4 on 23 January 1998 and starting to alter on that date.

I can think of only two ways of resolving this difficulty:

1. We discover a way to separate gravitation from geometry.
2. We discover a way in which geometry can be influenced (or determined) by any of the fundamental interactions.

In the first way, we think of geometry as an arena upon which all four fundamental interactions reside, but not influenced by any of them. Here, we would have to imagine a way to express General relativity in a way in which geometry and gravitation are somehow separate. This would probably require, for example, a separation of the metric tensor g that raises and lowers indices from the g that represents the gravitational potential. Or alternatively, it might require a representation which avoids geometrical concepts.

60.1.1 Reinterpretation of energy

Einstein's field equations can be written in symbolic form as

$G = T$,

where G represents spacetime curvature, and T is the stress-energy tensor. However, energy is a gravitational quantity as explained below. Therefore, the field equations equate spacetime curvature (deviation of geometry from Minkowski spacetime) with a gravitational quantity. We also know that the electromagnetic interaction can generate forces proportional to acceleration, which cannot be distinguished from inertial forces. We might therefore imagine that electromagnetic interaction by itself could generate an inertial frame if we had a universe that was positively charged. The formula $G = T$ does not allow for this, because T represents the transport of energy-momentum, not charge. We can imagine a T_e that represents the transport of charge instead of energy-momentum. Therefore, it seems either that $G = T$ is unsatisfactory or it must be reinterpreted.

Energy is a gravitational concept in the following way. We normally write Newton's second law as $f = ma$. Following Mach, we should write it as $f - ma = 0$, where we now regard $-ma$, an inertial force, as a gravitational force. We then should rewrite Newton's second law as $F = 0$, where F is the total force (gravitational plus electromagnetic + other), and gravitational force includes inertial forces.

In doing this, we need to make some reinterpretations of normal energy and momentum concepts. Let us consider only gravitational plus electromagnetic interactions, because first, these are the only forces of everyday life, and second, understanding them will allow us to understand everything.

In the usual interpretation of energy and momentum, we are able to use these concepts to make calculations, because under some conditions these are conserved, and we can think of converting from one form of energy to another (potential and kinetic, for example) or between gravitational and electromagnetic, for example. Or, when we have forces, the forces will change energy and momentum according to known rules.

I want to change these concepts now. I will consider energy and momentum to be gravitational quantities.

F has several terms.

Momentum, $p : -dp/dt$ is a gravitational force, so p must be a vector potential. However, I still need to get the sign right. Also, it may be π instead of p. Next, I need to relate π to sources, either through differential equations or an integral. π or p can be part of a tensor rather than a vector. That part doesn't matter.

gravitational potential energy, $v : -\nabla v$ is a gravitational force.

electromagnetic potential energy, $v_e : -\nabla v_e$ is an electromagnetic force.

electromagnetic vector potential, $A : -dA/dt$ is an electromagnetic force.

also, curl A is an electromagnetic force.

Kinetic energy, T: I know this has something to do with a gravitational interaction, but why am I having trouble with this one? Let us consider the energy-momentum 4-momentum. This is related to the zeroth component of the 4-force. What does the zeroth component of the 4-force mean? I will come back to this. Clearly, T must be the zeroth component of a 4-vector potential, that is, a gravitational scalar potential. Yes! That's it. Since $f = -\nabla v$, where v is the scalar potential, and force times distance gives the change in kinetic energy, T must be a scalar potential. I still need to get the sign right. Also, it may be π_0 rather than T. The next step is to relate T to gravitational sources. I can do this with differential equations or an integral. It can be part of a tensor rather than part of a 4-vector. That part doesn't matter. Actually, I still don't have it quite right. Yes, I should treat T as a scalar potential, but because it is velocity dependent, I need to put it into Lagrange's equations to get the right equations of motion. That is, it is not just a usual scalar potential. I have to rethink just how Lagrange's equations work. Normally, the velocity-dependent part is the kinetic energy, but here we have a velocity-dependent scalar potential. Isn't this already handled correctly in the electromagnetic case? In that case, we have a vector potential, and part of the vector potential comes from the time derivative of the vector potential, but there is no velocity dependence of the scalar potential. Looking at the electromagnetic case should help resolve how to handle it. The velocity-dependent part may be like inertial mass, that depends on speed.

Going on further: Here are some arguments against such a reinterpretation: I said that we could generate a geometry from purely electromagnetic interactions, because there are acceleration-dependent forces that act like inertial forces. If the universe had a positive charge, we could generate such a geometry. However, if the universe is closed (and that is the only case I consider), then the total charge must be zero, so that cannot happen.

But why doesn't this argument hold to stop generation of a geometry from gravity? The same argument shows that the total energy must be zero. However, there are many sources of energy, not just mass-energy. But there are no other sources of charge. Particles have charge. You can simply add it up. You can't create it. You can create electric fields that don't end on charges, but such field lines cannot end; they must close on themselves. Electric field lines can end only on charges.

For energy it is different. We have mass-energy plus kinetic energy plus potential energy. We can create energy so that the total energy is zero even though we have all of those stars out there with so much mass. So, maybe I can argue that it is impossible to consider a charged universe. We do have baryon conservation, which is similar, but we do not have the sum of baryons equal zero. It has something to do with whether the flux is the gradient of a conserved quantity.

60.1.2 A reinterpretation of momentum as a vector potential

When Sciama (1953)[11] got inertia from a calculation analogous to an electromagnetic interaction, he had to make his calculation in the rest frame of the body. This can be avoided by using the relative formulas for kinetic energy similar to Lynden-Bell[5] (1995)[167], Hofmann (1904)[178], Reissner (1914, 1915)[2, 3], Schroedinger (1925)[4], or the Weber potentials (1846, 1848)[179, 180, 181], Assis (1995)[181]. Actually, one wants to use relative formulas for the vector potential to replace the momentum. Then, Sciama's[11] calculation becomes frame independent. This can easily be done, and it becomes clear how momentum is really a vector potential.

The problem then, is that there are no retardation effects, because these formulas are just two-body direct action formulas. The next step, then is to carefully rewrite Maxwell's equations in a relative-position formulation, avoiding absolute space. This might be difficult, but we shall see. We should still have a theory equivalent to Maxwell's, but in relative-position formulation. There might still be a problem with the effect of inertial frames on photons, but maybe not. In any case, this should allow us to see how it is done, and then apply the same thing to gravitation to get the retardation right there. We should then have a gravitational formulation in which vector potential has replaced momentum and scalar potential has replaced kinetic energy.

That may be Newtonian gravitation only. We still have to revise general relativity. That may be harder.

There is a problem here. Electromagnetic theory doesn't work this way. If we have a bunch of charge around, then in a coordinate system fixed with that charge, there will be no vector potential, and no inertia term for moving charges. That is, the force between charges does not depend on the relative motion of the charges. We need currents on both ends. The problem is that photons have ordinary inertial effects, so we have three frames to consider. (1) The frame of the charge that produces the field (2) the frame of the charge being acted on, and (3) the inertial frame in which photons travel in straight lines. That makes it more difficult.

Maybe I have to do it this way. I need to find some potential that satisfies a local differential equation. It doesn't have to be a vector potential, however, I could try my gravitational vector potential g_μ to see if it satisfies some local differential equation with some kind of sources.

60.1.3 The real problems

1. Separate gravitation from geometry and coordinate effects.

2. Separate momentum current as source of gravitation from momentum as potential as a result of gravitational interaction.

3. Express kinetic energy and momentum as currents or potentials as appropriate.

4. The asymmetry of deriving the geodesic equation and the Lorentz force equation from $\nabla \cdot \mathbf{T} = 0$

   $\nabla \cdot \mathbf{G} = 0$ identically. Therefore, if $\mathbf{G} = \mathbf{T}$, then $\nabla \cdot \mathbf{T} = 0$. But T is the source of gravitation. Or is it?

5. The asymmetry of deriving both gravitation and electromagnetic force equations from the gravitation equations

I need partial derivatives, not total derivatives in the force equation. The intrinsic derivative looks in form a lot like the formula for a covariant derivative. I need to look carefully at how the geodesic equation is derived from $\nabla \cdot \mathbf{T} = 0$.

In the coulomb gauge, scalar potential has no retardation. Maybe there is a gauge where gravitational scalar potential and kinetic energy potential has no retardation. In gravitation, gauge transformations are coordinate transformations. Will that still be true if I can separate gravitation from geometry?

Here are a few more things. There are 3 references frames to consider. (1) the frame of the currents that are the source of the field, (2) the frame of the currents that are being acted on, and (3) the frame where the propagator for the field takes on a simple form. For both the gravitational and the electromagnetic case, the graviton propagator and the photon are simple in inertial frames. For the gravitational case, the inertial frames seem to coincide with the center of mass frame if Mach's principle holds. Of course, how do you define the center of mass frame until you have a geometry?

60.1.4 What is wrong with the geodesic equation?

The geodesic equation does not act like the kind of force equation I would expect from looking at the Lorentz force equation. That is, it is not the product of a current with some derivative of some kind of potential. Supposedly, $T^{\mu\nu}$ represents the source for gravitational fields, and $g_{\mu\nu}$ represents the potential. So, the force law should look like $T^{\mu\nu}$ times some derivative of $g_{\mu\nu}$. This seems to be true only in the rest frame of the body. In other frames, the force law does not take that form. In one representation, it is $\nabla \cdot T^{\mu\nu}$. That looks more like a derivative of the current rather than a current times a derivative of a potential.

So, maybe I should start in the frame of the body and try to transform to an arbitrary frame. Γ represents a derivative of $g_{\mu\nu}$, but it is not a tensor. Can I define a tensor that is the same as Γ in the frame of the body? That just might work. In the geodesic equation, the term for the covariant derivative that is a product of Γ times $T^{\mu\nu}$ does have the desired form for a force law. It is only the $\ddot{x}$ term that does not follow the proper form.

On the other hand, what is wrong with an $\ddot{x}$ term? In one form of writing the geodesic equation, we have $T^{\mu\nu}_{;\nu} = 0 = T^{\mu\nu}_{,\nu} - T_{\alpha\beta}\Gamma^{\alpha\beta\mu} = 0$. Maybe in this equation, $T_{\alpha\beta}$ is a current, but $T^{\mu\nu}$ is a potential.

It seems that $T^{\mu\nu}$ in the $T^{\mu\nu}_{,\nu}$ term must be a potential instead of a current, because it is potentials that we take the derivative of, not currents in the force equation. But we need something more concrete than that.

Representing gravitation by geometry forces a choice of units in which inertial mass and passive gravitational mass are equal. Thus, within the geometrical framework it is not possible to discuss the distinction between inertial and passive gravitational mass. I don't think allowing the gravitational constant G to vary will help, because there is still active gravitational mass to consider.

For a spin 2 gravitational system, we have more that just mass; we have tensor currents. In Einstein's system, $T^{\mu\nu}$ represent the tensor currents. To replace inertial mass, we must have a tensor potential. The metric $g_{\mu\nu}$ seems to do part of the job, but not the whole job. The $T^{\mu\nu}$ in the $T^{\mu\nu}_{,\nu}$ term in the geodesic equation seems to do the rest of the job. Thus, this $T^{\mu\nu}$ acts more like a potential than like a current.

How can I argue conclusively that this $T^{\mu\nu}$ must be a potential rather than a current?

How to separate geometry from gravitation: Transform to some arbitrary coordinate system which satisfies no pre-conceived conditions. For example, G=T is not necessarily satisfied for that coordinate system. Also, the geodesic equation is not necessarily satisfied for that coordinate system, and the metric $g_{\mu\nu}$ does not necessarily give distances in that coordinate system. But, there will be a geometry in which all of these things are satisfied, so we must express all of those things in terms of this arbitrary coordinate system.

For the geodesic equation locally, I could choose a coordinate system that coincides with the geometry except that we have $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$. I hope that will work.

Lynden-Bell, Katz, and Bicak[167] do something using a separate coordinate system. Maybe that takes care of separating gravitation from geometry. In their mapping, the same point should have the same coordinates in both systems. For the case where both systems are orthogonal, that means the transformation will have $\partial x/\partial \acute{x} = 1$ I think. That doesn't seem very interesting.

60.2 1967 (Gough), 1953 (Sciama)

Douglas Gough showed me Dennis Sciama's[11] 1953 article in 1967. That got me started. Since quantum theory is so connected with inertia (because of inertial frames), I realized that Mach's ideas would apply to quantum theory also.

60.3 1916 deficiencies (Einstein)

60.4 1687 (Newton), 17 — (Lagrange)

60.5 1883 (Mach)[1]

60.6 1914 (Reissner)[2, 3], 1925 (Schroedinger)[4], 1992 & 1994 (Lynden-Bell)[5, 6]

60.7 1905 revisited

Would special relativity be different if Einstein had had Lynden-Bell's[5, 6] relative version of Mechanics instead of Newton't absolute version?

60.8 1926 revisited

Would quantum mechanics be different if they had had Lynden-Bell's[5, 6] relative version of Mechanics instead of Newton't absolute version?

To try to answer that question, we start with the relative-mechanics Lagrangian [5, 6]

$$L = \frac{1}{4M} \sum_i \sum_j m_i m_j \dot{\mathbf{r}}_{ij}^2 - V \ , \tag{60.1}$$

where

$$M = \sum_i m_i \tag{60.2}$$

is the total mass of the universe,

$$\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j \ , \tag{60.3}$$

and

$$V = -G \sum_i \sum_{j>i} \frac{m_i m_j}{|r_{ij}|} \tag{60.4}$$

is the potential.

To start, we use the standard formula for the canonical momentum

$$\mathbf{p}_i \equiv \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \ . \tag{60.5}$$

This gives

$$\mathbf{p}_i = \frac{m_i}{M} \sum_j m_j \dot{\mathbf{r}}_{ij} = -\frac{m_i}{M} \sum_j \mathbf{p}_{ji} = -\frac{m_i}{M} \overline{\mathbf{p}}_i = m_i \frac{-\overline{\mathbf{p}}_i}{M} \ , \tag{60.6}$$

where

$$\mathbf{p}_{ji} \equiv m_j \dot{\mathbf{r}}_{ji} \ , \tag{60.7}$$

and

$$\overline{\mathbf{p}}_i \equiv \sum_j m_j \dot{\mathbf{r}}_{ji} = \sum_j \mathbf{p}_{ji} \tag{60.8}$$

is the total momentum of all of the other particles in the universe in a coordinate system fixed with the particle of mass m_i. Then

$$\overline{\mathbf{v}}_i = \frac{\overline{\mathbf{p}}_i}{M} \tag{60.9}$$

is the velocity of the center of mass of all of the other particles in the universe relative to the particle of mass m_i, and $-\overline{\mathbf{v}}_i$ is the velocity of the mass m_i relative to the center of mass of all of the other particles. Therefore, from (60.6), $\mathbf{p}_i$ is the momentum of m_i relative to the center of mass of all of the other particles.

The above development is not correct because (60.5) takes a derivative with respect to an absolute coordinate instead of a relative coordinate. Instead, we try

$$\mathbf{p}_{ij} \equiv \frac{\partial L}{\partial \dot{\mathbf{r}}_{ij}} \ . \tag{60.10}$$

That gives

$$\mathbf{p}_{ij} = \frac{m_i}{M} m_j \dot{\mathbf{r}}_{ij} \ , \tag{60.11}$$

To get a Hamiltonian, we use

$$H = \frac{1}{2}\sum_i \sum_j \mathbf{p}_{ij}\dot{\mathbf{r}}_{ij} - L \ , \tag{60.12}$$

where the factor of a half cancels the double counting in the double sum. Substituting (60.11) and (60.1) into (60.12) gives

$$H = \sum_i \sum_j \frac{M\mathbf{p}_{ij}^2}{4m_i m_j} + V(\mathbf{r}_{ij}) \ . \tag{60.13}$$

We can eliminate double counting by writing

$$H = \sum_i \sum_{j>i} \frac{M\mathbf{p}_{ij}^2}{2m_i m_j} + V(\mathbf{r}_{ij}) \ . \tag{60.14}$$

Using Hamilton's equations gives

$$\dot{\mathbf{r}}_{ij} = \frac{\partial H}{\partial \mathbf{p}_{ij}} = \frac{M\mathbf{p}_{ij}}{m_i m_j} \tag{60.15}$$

and

$$\dot{\mathbf{p}}_{ij} = -\frac{\partial H}{\partial \mathbf{r}_{ij}} = -\frac{\partial V}{\partial \mathbf{r}_{ij}} = -G\frac{m_i m_j}{|r_{ij}|^3}\mathbf{r}_{ij} \ , \tag{60.16}$$

in which (60.15) agrees with (60.11). Taking the time derivative of (60.15) gives

$$\ddot{\mathbf{r}}_{ij} = \frac{M\dot{\mathbf{p}}_{ij}}{m_i m_j} = -\frac{M}{m_i m_j} G\frac{m_i m_j}{|r_{ij}|^3}\mathbf{r}_{ij} = -G\frac{M}{|r_{ij}|^3}\mathbf{r}_{ij} \ . \tag{60.17}$$

Or,

$$\frac{m_i m_j}{M}\ddot{\mathbf{r}}_{ij} = -G\frac{m_i m_j}{|r_{ij}|^3}\mathbf{r}_{ij} \ . \tag{60.18}$$

Summing gives

$$\sum_j \frac{m_i m_j}{M}\ddot{\mathbf{r}}_{ij} = \sum_j \frac{m_i m_j}{M}\ddot{\mathbf{r}}_i - \sum_j \frac{m_i m_j}{M}\ddot{\mathbf{r}}_j = m_i\ddot{\mathbf{r}}_i - \sum_j \frac{m_i m_j}{M}\ddot{\mathbf{r}}_j = -\sum_j G\frac{m_i m_j}{|r_{ij}|^3}\mathbf{r}_{ij} \ . \tag{60.19}$$

Equation (60.19) is equivalent to equation (1.9) in the 1992 paper by Lynden-Bell [5, 6], which he shows can be converted to agree with the standard Newtonian formula by subtracting an amount that varies linearly with time from each $\mathbf{r}_{ij}$.

It seems that $j = i$ should have been excluded in the sum in (60.19), but Lynden-Bell [5, 6] did not do so, and that would have made the part after the second equals sign in (60.19) untrue. However, if there are many particles, that may not be a problem because the approximation would be very good. The main problem seems to be the validity of (60.18) if it is not summed. Lynden-Bell [5, 6] used Lagrange's equations instead of Hamilton's equations to derive (60.19), so that was not a problem. The question now is whether Hamilton's equations as I have used them here are valid. For a flat space, the number of variables and the number of equations $3N(N-1)$ in (60.18) is much greater than the number of independent degrees of freedom $3N$, where N is the number of particles. However, if the space is not constrained to be flat, the extra degrees of freedom could be an implicit measure of curvature.

At this point, it appears that we might have a viable relative-coordinate classical theory except for the problems mentioned above. Now we want to develop the corresponding quantum theory and compare it with Schrödinger's equation. It looks like we can get something that looks like a Schrödinger equation if we let $\mathbf{p}_{ij} \to -i\hbar\nabla_{ij}$. The length units might be special.

What I shall actually do is let $\mathbf{p}_{ij} \to -i\beta\nabla_{ij}$, where β is a constant to be determined. Using $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ gives

$$i\hbar\frac{\partial\psi(t,\mathbf{r}_{ij})}{\partial t} = H\psi(t,\mathbf{r}_{ij}) = -\beta^2\sum_i\sum_{j>i}\frac{M}{2m_im_j}\nabla^2_{ij}\psi(\mathbf{r}_{ij}) + V(t,\mathbf{r}_{ij})\psi(t,\mathbf{r}_{ij}) \ , \tag{60.20}$$

where ∇_{ij} is a gradient with respect to $\mathbf{r}_{ij}$, and $\psi(t,\mathbf{r}_{ij})$ and $V(\mathbf{r}_{ij})$ are a functions of all of the $\mathbf{r}_{ij}$.

In some cases, we can make a separation of variables. One such case is where the potential can be written as

$$V(t,\mathbf{r}_{ij}) = \sum_i\sum_{j>i}V_{ij}(t,\mathbf{r}_{ij}) \ , \tag{60.21}$$

where $V_{ij}(t,\mathbf{r}_{ij})$ depends on only one $\mathbf{r}_{ij}$. That is, we consider only 2-particle interactions. Sometimes in that case, we can write $\psi(t,\mathbf{r}_{ij})$ as a product.

$$\psi(t,\mathbf{r}_{ij}) = \prod_i\prod_{j>i}\psi_{ij}(t,\mathbf{r}_{ij}) \ , \tag{60.22}$$

where $\psi_{ij}(t,\mathbf{r}_{ij})$ depends on only one $\mathbf{r}_{ij}$. Substituting (60.21) and (60.22) into (60.20) and dividing by $\psi(t,\mathbf{r}_{ij})$ gives

$$i\hbar\sum_i\sum_{j>i}\frac{1}{\psi_{ij}(t,\mathbf{r}_{ij})}\frac{\partial\psi_{ij}(t,\mathbf{r}_{ij})}{\partial t} = -\beta^2\sum_i\sum_{j>i}\frac{M}{2m_im_j}\frac{\nabla^2_{ij}\psi_{ij}(t,\mathbf{r}_{ij})}{\psi_{ij}(t,\mathbf{r}_{ij})} + \sum_i\sum_{j>i}V_{ij}(t,\mathbf{r}_{ij}) \ . \tag{60.23}$$

Each term in the sums depends on only one $\mathbf{r}_{ij}$. That allows us to separate each term in the sums to give

$$i\hbar\frac{1}{\psi_{ij}(t,\mathbf{r}_{ij})}\frac{\partial\psi_{ij}(t,\mathbf{r}_{ij})}{\partial t} = -\beta^2\frac{M}{2m_im_j}\frac{\nabla^2_{ij}\psi_{ij}(t,\mathbf{r}_{ij})}{\psi_{ij}(t,\mathbf{r}_{ij})} + V_{ij}(t,\mathbf{r}_{ij}) \ . \tag{60.24}$$

Multiplying by $\psi_{ij}(t,\mathbf{r}_{ij})$ gives

$$i\hbar\frac{\partial\psi_{ij}(t,\mathbf{r}_{ij})}{\partial t} = -\beta^2\frac{M}{2m_im_j}\nabla^2_{ij}\psi_{ij}(t,\mathbf{r}_{ij}) + V_{ij}(t,\mathbf{r}_{ij})\psi_{ij}(t,\mathbf{r}_{ij}) \ . \tag{60.25}$$

For the case that there are only two particles, say an electron and a proton, with masses m_e and m_p, we have

$$i\hbar\frac{\partial\psi(t,\mathbf{r})}{\partial t} = -\hbar^2\frac{m_e+m_p}{2m_em_p}\nabla^2\psi(t,\mathbf{r}) + V(t,\mathbf{r})\psi(t,\mathbf{r}) \ , \tag{60.26}$$

where $\mathbf{r}$ is the relative coordinate between the electron and the proton, we have taken $\beta = \hbar$, and $\frac{m_em_p}{m_e+m_p}$ is the correct reduced mass for the electron, and we have the correct Schrödinger equation in terms of relative coordinates [182, Schiff, 1955, p. 81]. Notice that in the case there are only two particles, then (60.23) has only one term in the sum.

In the real situation, there are many particles, but when considering the wave function for a hydrogen atom, say, we consider only two of those particles. Equation (60.25) should be the equation to describe that situation, but is it really correct? If we were to choose

$$\beta = \hbar\sqrt{\frac{m_i + m_j}{M}}\ , \tag{60.27}$$

then (60.25) would reduce to (60.26), but that would mean β in (60.23) would not be a constant, but would be different for each term in the sum. That is unacceptable. Therefore, there is something wrong with this derivation.

There are several possible explanations for the problem, but there is a feature of the problem with (60.27) that we need to look at. The reduced mass for the electron in (60.26) results from considering a coordinate system fixed to the center of mass of the electron and proton, but the effective reduced mass in (60.25) results from considering a coordinate system fixed to the center of mass of all particles in the system. The latter would be the correct choice if the inertial frame were determined by all of the particles (as it would in a correct theory), but here, the inertial frame is given, so the former is correct for this situation. There may be other problems, such as those below:

1. The above procedure for separating variables in (60.20) may not be valid when there are more than two particles.

2. Equation (60.14) may not be the correct equation to quantize.

3. Even though (60.14) is written in terms of relative coordinates between the particles, it still represents particles superimposed on a given background inertial frame, and therefore not a completely relative system.

4. A more appropriate system to quantize might be General Relativity if we could write it in terms of relative coordinates.

Before continuing, it is useful to consider here the nature of particles and waves. Equation (60.26) is not really an equation for the wave function of an electron, but actually for some kind of relative wave function for the electron and proton. The independent variable for the wave function is the relative coordinate between the two particles. So, the wave function is actually a wave function for that relative coordinate.

Even though (60.20) is probably not a correct wave equation, we can still discuss some of its features which may be correct. The wave function in (60.20) is a function of all of the relative coordinates of all of the particles. The correct wave equation will probably be a function of other things as well, but let us consider that aspect of relative coordinates. If the number of particles is N, then the number of degrees of freedom for the system should be $3N$. But in (60.20), there are $N(N-1)$ relative coordinates, which gives $3N(N-1)$ degrees of freedom, which equals the former only for $N = 2$. If space were flat, the apparent extra degrees of freedom would be redundant. However, we already know from General Relativity that spacetime is not flat, and having a theory based on relative coordinates would be an implicit way to indicate the curvature of spacetime.

In (60.20) and (60.26), the particle nature enters through the dependence of the potential on the relative coordinates of the particles. The wave nature results from the solutions to those equations. However, as we now realize, the wave function solutions in ordinary quantum mechanics do not represent actual fields, but only ensemble averages of the actual fields. We know that the actual fields fluctuate, and the magnitude of those fluctuations are roughly one quantum. For the electromagnetic field, that would be one photon. For a hydrogen atom (or any other system of Fermions), the meaning of fluctuations would be more complicated.

Possibly we can say that a particle corresponds to how much a wave function can fluctuate. Possibly a particle is 3 degrees of freedom.

Taking $\beta = \hbar$ in (60.25) gives

$$i\hbar \frac{\partial \psi_{ij}(t, \mathbf{r}_{ij})}{\partial t} = -\hbar^2 \frac{M}{2m_i m_j} \nabla_{ij}^2 \psi_{ij}(t, \mathbf{r}_{ij}) + V_{ij}(t, \mathbf{r}_{ij}) \psi_{ij}(t, \mathbf{r}_{ij}) \ . \tag{60.28}$$

A WKB approximation to (60.28) is

$$\psi_{ij}(t, \mathbf{r}_{ij}) = \left[\frac{2m_i m_j}{\hbar^2 M} \left(\hbar\omega_{ij} - V_{ij}(t, \mathbf{r}_{ij}) \right) \right]^{-1/4} e^{-i\omega_{ij} t} e^{i \int \mathbf{k}_{ij} \cdot \, \mathrm{d}\mathbf{r}_{ij}} \ , \tag{60.29}$$

where the Hamiltonian is

$$H_{ij} = E_{ij} = \hbar\omega_{ij} = \frac{\hbar^2 M}{2m_i m_j} \left(k_{ijx}^2 + k_{ijy}^2 + k_{ijz}^2 \right) + V_{ij}(t, \mathbf{r}_{ij}) \ , \tag{60.30}$$

$$\mathbf{k}_{ij} = \mathbf{p}_{ij} / \hbar \ , \tag{60.31}$$

and

$$k_{ijx}^2 + k_{ijy}^2 + k_{ijz}^2 = \frac{2m_i m_j}{\hbar^2 M} \left[\hbar\omega_{ij} - V_{ij}(t, \mathbf{r}_{ij}) \right] \ . \tag{60.32}$$

Therefore, using (60.29) in (60.22) gives the WKB solution to (60.20) as

$$\psi(t, \mathbf{r}_{ij}) = e^{-i\omega t} e^{i \sum_i \sum_{j>i} \int \mathbf{k}_{ij} \cdot \, \mathrm{d}\mathbf{r}_{ij}} \prod_i \prod_{j>i} \left[\frac{2m_i m_j}{\hbar^2 M} \left(\hbar\omega_{ij} - V_{ij}(t, \mathbf{r}_{ij}) \right) \right]^{-1/4} \ , \tag{60.33}$$

where

$$\omega \equiv \sum_i \sum_{j>i} \omega_{ij} \ . \tag{60.34}$$

I think the meaning of $\psi(t, \mathbf{r}_{ij})$ is that it gives the amplitude that each $\mathbf{r}_{ij}$ has the corresponding value.

It was really not necessary to make a WKB approximation for a solution to (60.28), because we can write down the solution as hydrogen atom wave functions, with $e^2 \to G m_i m_j$, $\mu \to m_i m_j / M$, and $E \to E_{ij}$.

If we then apply the equations to the solar system, neglect all of the moons and Pluto, but keep all of the stars in the universe, but for the moment neglect that stars are bound in galaxies, and neglect planets bound in other star systems. Then we know the solution to (60.28) for each pair of bodies in the system, and we can write down the total wave function in (60.22). Only eight of the wave functions for pairs of bodies will be bound states, those for the sun with each of the eight planets. All of the rest will be unbound states. Actually, for the solar system, we want to use the WKB approximation, and then we get (60.19), which agrees with Newtonian theory.

For an example of using the wave equation to get a wave function, we apply the equations to a hydrogen atom. In that case,

$$V_{ij} = -G \frac{m_i m_j}{|r_{ij}|} \tag{60.35}$$

gives the potential between each pair of particles except for the proton and electron. The potential between the proton and electron is

$$V_{12} = -\frac{e^2}{|r_{12}|} - G \frac{m_i m_j}{|r_{12}|} \ , \tag{60.36}$$

in which the second term is negligible. There is only one pair of particles which are in bound states, namely the proton and electron, and I am neglecting bound states in the solar system, etc. We still get hydrogen atom wave functions, with $e^2 \to Gm_im_j$, $\mu \to m_im_j/M$, and $E \to E_{ij}$ for all pairs of particles except the proton-electron pair, where we have $\mu \to m_1m_2/M$, and $E \to E_{12}$.

The energy levels in a hydrogen atom are [182, Schiff, 1955. p. 84]

$$E_n = -\frac{\mu e^4}{2\hbar n^2} , \tag{60.37}$$

where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass of the electron. In out case, we get

$$E_{12n} = -\frac{m_1m_2e^4}{2\hbar Mn^2} , \tag{60.38}$$

where M is the total mass of the universe. However, the correct energy level is

$$E_{12n} = -\frac{m_1m_2e^4}{2\hbar(m_1+m_2)n^2} . \tag{60.39}$$

Maybe I need to re-normalize. The charge in (60.38) is a bare charge, whereas the charge in (60.39) is the observed charge. If we take

$$e^4_{\text{bare}} = \frac{M}{m_1+m_2} e^4_{\text{observed}} , \tag{60.40}$$

then we have solved our problem. The wave function will be correct also.

Because of an apparent coincidence, without renormalization, the energy levels in a hydrogen atom due to gravitational attraction would be about the same as they are for electrical attraction.

60.8.1 Special relativity with Newtonian gravity

When we include relativistic effects, it is not so easy to write the dynamics in terms of relative coordinates because velocities do not add (or subtract) linearly in special relativity.

Although energy and momentum add, there are square roots involved, which make calculations difficult. For example, the relativistic formula for the total energy E of a body of mass m and momentum p is

$$E = \sqrt{p^2c^2 + m^2c^4} . \tag{60.41}$$

The total energy of a number of particles, including a potential V would be

$$E = \sum_i \sqrt{p_i^2c^2 + m_i^2c^4} + V . \tag{60.42}$$

We could convert to relative coordinates by, say, subtracting the center-of-mass coordinate, whose velocity is [10, Leighton, 1959, exercises 1-44 and 1-48]

$$\beta_{\text{cm}} = \frac{c\sum_{i=1}^N \mathbf{p}_i}{\sum_{i=1}^N E_i} = \frac{c\sum_{i=1}^N \mathbf{p}_i}{E} = \frac{\sum_{i=1}^N \beta_i E_i}{E} , \tag{60.43}$$

where

$$E \equiv \sum_{i=1}^N E_i . \tag{60.44}$$

However, doing that with the sum in (60.42) would lead to a mess.

Instead, we start with

$$E_i^2 = m_i^2 c^4 \left(1 - \beta_i^2\right)^{-1} \quad . \tag{60.45}$$

We write (60.45) as

$$E_i^2 = m_i^2 c^4 \left(1 - \beta_i^2\right)^{-1} = m_i^2 c^4 \left(1 - \beta_{ix}^2 - \beta_{iy}^2 - \beta_{iz}^2\right)^{-1} \quad . \tag{60.46}$$

Next, we subtract the velocity of the center of mass from each velocity in (60.46) to give

$$E_{i\mathrm{cm}}^2 = m_i^2 c^4 \left[1 - (\beta_{ix} - \beta_{\mathrm{cm}x})^2 - (\beta_{iy} - \beta_{\mathrm{cm}y})^2 - (\beta_{iz} - \beta_{\mathrm{cm}z})^2\right]^{-1} \quad . \tag{60.47}$$

Substituting (60.43) into (60.47) gives

$$E_{i\mathrm{cm}}^2 = m_i^2 c^4 \left[1 - \left(\beta_{ix} - \frac{\sum_{j=1}^N \beta_{jx} E_j}{E}\right)^2 - \left(\beta_{iy} - \frac{\sum_{j=1}^N \beta_{jy} E_j}{E}\right)^2 - \left(\beta_{iz} - \frac{\sum_{j=1}^N \beta_{jz} E_j}{E}\right)^2\right]^{-1} \quad . \tag{60.48}$$

Or,

$$E_{i\mathrm{cm}}^2 = m_i^2 c^4 \left[1 - \left(\frac{E\beta_{ix} - \sum_{j=1}^N \beta_{jx} E_j}{E}\right)^2 - \left(\frac{E\beta_{iy} - \sum_{j=1}^N \beta_{jy} E_j}{E}\right)^2 - \left(\frac{E\beta_{iz} - \sum_{j=1}^N \beta_{jz} E_j}{E}\right)^2\right]^{-1} \quad . \tag{60.49}$$

Substituting (60.44) into (60.49) gives

$$E_{i\mathrm{cm}}^2 = m_i^2 c^4 \left[1 - \left(\frac{\sum_{j=1}^N (\beta_{ix} - \beta_{jx}) E_j}{E}\right)^2 - \left(\frac{\sum_{j=1}^N (\beta_{iy} - \beta_{jy}) E_j}{E}\right)^2 - \left(\frac{\sum_{j=1}^N (\beta_{iz} - \beta_{jz}) E_j}{E}\right)^2\right]^{-1} \quad . \tag{60.50}$$

We can write (60.50) as

$$E_{i\mathrm{cm}}^2 = m_i^2 c^4 \left[1 - \left(\frac{\sum_{j=1}^N \beta_{ijx} E_j}{E}\right)^2 - \left(\frac{\sum_{j=1}^N \beta_{ijy} E_j}{E}\right)^2 - \left(\frac{\sum_{j=1}^N \beta_{ijz} E_j}{E}\right)^2\right]^{-1} \quad , \tag{60.51}$$

where

$$\boldsymbol{\beta}_{\mathbf{ij}} \equiv \boldsymbol{\beta}_{\mathbf{i}} - \boldsymbol{\beta}_{\mathbf{j}} \ . \tag{60.52}$$

Expanding the squares in (60.51) gives

$$E_{i\mathrm{cm}}^2 = m_i^2 c^4 \left(1 - \sum_{j=1}^N \sum_{k=1}^N \frac{E_j E_k}{E^2} \boldsymbol{\beta}_{ij} \cdot \boldsymbol{\beta}_{ik}\right)^{-1} \quad . \tag{60.53}$$

Taking the square root of (60.53) and summing to get the total energy doesn't look useful, partly because the right side doesn't have quite the right symmetry. To give it the right symmetry, we write (60.53) as

$$\left(\frac{m_i^2 c^4}{E_{i\mathrm{cm}}^2} - 1\right) E_i^2 = -\sum_{j=1}^N \sum_{k=1}^N \frac{E_i^2 E_j E_k}{E^2} \boldsymbol{\beta}_{ij} \cdot \boldsymbol{\beta}_{ik} \ . \tag{60.54}$$

The symmetry of (60.54) is more obvious if we write (60.54) as

$$\left(\frac{m_i^2 c^4}{E_{i\mathrm{cm}}^2} - 1\right) E_i^2 = -E^2 \sum_{j=1}^N \sum_{k=1}^N \boldsymbol{\beta}'_{ij} \cdot \boldsymbol{\beta}'_{ik} \ , \tag{60.55}$$

where

$$\beta'_{ij} \equiv \frac{E_i E_j}{E^2} \beta_{ij} \ . \tag{60.56}$$

We can rewrite (60.55) as

$$\left(1 - \frac{m_i^2 c^4}{E_{i\mathrm{cm}}^2}\right) E_i^2 = E^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \beta'_{ij} \cdot \beta'_{ik} \ , \tag{60.57}$$

Equation (60.57) is valid in any coordinate system. If we choose to write it in the center-of-mass frame, then we have

$$E_{i\mathrm{cm}}^2 - m_i^2 c^4 = E_{\mathrm{cm}}^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \beta'_{ij\mathrm{cm}} \cdot \beta'_{ik\mathrm{cm}} \ , \tag{60.58}$$

where

$$\beta'_{ij\mathrm{cm}} \equiv \frac{E_{i\mathrm{cm}} E_{j\mathrm{cm}}}{E_{\mathrm{cm}}^2} \beta_{ij} \ . \tag{60.59}$$

From here on, however, I shall drop the cm subscripts, and assume we are in the center-of-mass coordinate system. Then, we can write (60.58) as

$$E_i^2 - m_i^2 c^4 = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{E_i E_j}{E} \frac{E_i E_k}{E} \beta_{ij} \cdot \beta_{ik} \ . \tag{60.60}$$

We can rewrite (60.60) as

$$E_i - m_i c^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{E_i E_j E_i E_k \beta_{ij} \cdot \beta_{ik}}{E^2 (E_i + m_i c^2)} \ . \tag{60.61}$$

The total kinetic energy T of the system is

$$T = \sum_{i=1}^{N} (E_i - m_i c^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{E_i E_j E_i E_k \beta_{ij} \cdot \beta_{ik}}{E^2 (E_i + m_i c^2)} \ . \tag{60.62}$$

We take for the Lagrangian

$$L = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{E_i E_j E_i E_k \dot{\mathbf{r}}_{ij} \cdot \dot{\mathbf{r}}_{ik}}{E^2 c^2 (E_i + m_i c^2)} - V \ . \tag{60.63}$$

The momentum canonical to $\dot{\mathbf{r}}_{ij}$ is

$$\mathbf{p}_{ij} = \frac{\partial L}{\partial \dot{\mathbf{r}}_{ij}} = \frac{2 E_i E_j}{E^2 c^2 (E_i + m_i c^2)} \sum_{k=1}^{N} E_i E_k \dot{\mathbf{r}}_{ik} \ . \tag{60.64}$$

The Hamiltonian is

$$H = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{p}_{ij} \cdot \dot{\mathbf{r}}_{ij} - L = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{E_i E_j E_i E_k \dot{\mathbf{r}}_{ij} \cdot \dot{\mathbf{r}}_{ik}}{E^2 c^2 (E_i + m_i c^2)} + V \ . \tag{60.65}$$

However, we need to write the Hamiltonian in terms of momenta. From (60.64), we have

$$\sum_{k=1}^{N} E_i E_k \dot{\mathbf{r}}_{ik} = \frac{E^2 c^2 (E_i + m_i c^2)}{2 E_i E_j} \mathbf{p}_{ij} = \frac{E^2 c^2 (E_i + m_i c^2)}{2 E_i E_m} \mathbf{p}_{im} \ . \tag{60.66}$$

So, we can write (60.65) as

$$H = \sum_{i=1}^{N} \frac{E^2 c^2 (E_i + m_i c^2)}{4 E_i E_m E_i E_n} \mathbf{p}_{im} \cdot \mathbf{p}_{in} + V \ . \tag{60.67}$$

It is not clear that the above development is correct. For one thing, (60.64) does not appear to have the correct symmetry between i and j. Second, it is not clear that all of (60.66) can be correct. I am not sure how to correct this.

The problem may be that, even though we have formulas in terms of relative velocities and relative momenta, the formulas are still in terms of absolute energies rather than relative energies.

60.8.2 4 dimensions, relative energies, Newtonian gravity

We start with

$$\left(E_i - \sum_{k \neq i}^{N} V_{ik} \right)^2 - \mathbf{p}_i^2 c^2 - m_i^2 c^4 = 0 \ , \tag{60.68}$$

where V_{ij} is the potential between the ith and jth body. The average momentum is

$$\overline{\mathbf{p}} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{p}_i \ . \tag{60.69}$$

Similarly, we define the average mass

$$\overline{m} \equiv \frac{1}{N} \sum_{i=1}^{N} m_i \ , \tag{60.70}$$

the average energy

$$\overline{E} \equiv \frac{1}{N} \sum_{i=1}^{N} E_i \ , \tag{60.71}$$

and the average potential

$$\overline{V} \equiv \frac{1}{N} \sum_{i=1}^{N} \sum_{k \neq i}^{N} V_{ik} \ . \tag{60.72}$$

Subtracting the average from each factor in (60.68) gives an equation whose factors average to zero.

$$\left(E_i - \overline{E} - \sum_{k \neq i}^{N} V_{ik} + \overline{V} \right)^2 - \left(\mathbf{p}_i - \overline{\mathbf{p}} \right)^2 c^2 - \left(m_i - \overline{m} \right)^2 c^4 = 0 \tag{60.73}$$

Substituting (60.69) through (60.72) into (60.73) gives

$$\left(E_i - \frac{1}{N} \sum_{j=1}^{N} E_j - \sum_{k \neq i}^{N} V_{ik} + \frac{1}{N} \sum_{j=1}^{N} \sum_{k \neq j}^{N} V_{jk} \right)^2 - \left(\mathbf{p}_i - \frac{1}{N} \sum_{j=1}^{N} \mathbf{p}_j \right)^2 c^2 - \left(m_i - \frac{1}{N} \sum_{j=1}^{N} m_j \right)^2 c^4 = 0 \ . \tag{60.74}$$

Or,

$$\frac{1}{N^2} \left[\left(N E_i - \sum_{j=1}^{N} E_j - N \sum_{k=1}^{N} V_{ik} + N V_{ii} + \sum_{j=1}^{N} \left(\sum_{k=1}^{N} V_{jk} - V_{jj} \right) \right)^2 - \left(N \mathbf{p}_i - \sum_{j=1}^{N} \mathbf{p}_j \right)^2 c^2 - \left(N m_i - \sum_{j=1}^{N} m_j \right)^2 c^4 \right] = 0 \ . \tag{60.75}$$

Or,

$$\frac{1}{N^2}\left[\left(\sum_{j=1}^{N}(E_i-E_j)+\sum_{j=1}^{N}\left(-\sum_{k=1}^{N}V_{ik}+V_{ii}+\sum_{k=1}^{N}V_{jk}-V_{jj}\right)\right)^2 - \left(\sum_{j=1}^{N}(\mathbf{p}_i-\mathbf{p}_j)\right)^2 c^2-\left(\sum_{j=1}^{N}(m_i-m_j)\right)^2 c^4\right]=0\ . \tag{60.76}$$

Or,

$$\frac{1}{N^2}\left[\left(\sum_{j=1}^{N}\left(E_i-E_j-\sum_{k=1}^{N}(V_{ik}-V_{jk})+V_{ii}-V_{jj}\right)\right)^2 - \left(\sum_{j=1}^{N}(\mathbf{p}_i-\mathbf{p}_j)\right)^2 c^2-\left(\sum_{j=1}^{N}(m_i-m_j)\right)^2 c^4\right]=0\ . \tag{60.77}$$

Or,

$$\frac{1}{N^2}\left[\left(\sum_{j=1}^{N}\left(E_i-E_j-\sum_{m=1}^{N}(V_{im}-V_{jm})+V_{ii}-V_{jj}\right)\right)\left(\sum_{k=1}^{N}\left(E_i-E_k-\sum_{n=1}^{N}(V_{in}-V_{kn})+V_{ii}-V_{kk}\right)\right) - \left(\sum_{j=1}^{N}(\mathbf{p}_i-\mathbf{p}_j)\right)\cdot\left(\sum_{k=1}^{N}(\mathbf{p}_i-\mathbf{p}_k)\right)c^2-\left(\sum_{j=1}^{N}(m_i-m_j)\right)\left(\sum_{k=1}^{N}(m_i-m_k)\right)c^4\right]=0\ . \tag{60.78}$$

Or,

$$\frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\left(E_i-E_j-\sum_{m=1}^{N}(V_{im}-V_{jm})+V_{ii}-V_{jj}\right)\left(E_i-E_k-\sum_{n=1}^{N}(V_{in}-V_{kn})+V_{ii}-V_{kk}\right) - (\mathbf{p}_i-\mathbf{p}_j)\cdot(\mathbf{p}_i-\mathbf{p}_k)\,c^2-(m_i-m_j)\,(m_i-m_k)\,c^4\right]=0\ . \tag{60.79}$$

To explicitly show that we have an equation in terms of relative coordinates, we write (60.79) as

$$\frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[(E_{ij}-V_{i-j})\,(E_{ik}-V_{i-k})-\mathbf{p}_{ij}\cdot\mathbf{p}_{ik}c^2-m_{ij}m_{ik}c^4\right]=0\ , \tag{60.80}$$

where

$$E_{ij}\equiv E_i-E_j\ , \tag{60.81}$$

$$V_{i-j}\equiv\sum_{m=1}^{N}(V_{im}-V_{jm})-V_{ii}+V_{jj}=\sum_{m\neq i}^{N}V_{im}-\sum_{m\neq j}^{N}V_{jm}\ , \tag{60.82}$$

$$\mathbf{p}_{ij}\equiv\mathbf{p}_i-\mathbf{p}_j\ , \tag{60.83}$$

and

$$m_{ij} \equiv m_i - m_j \ . \tag{60.84}$$

To get a Hamiltonian (or super-Hamiltonian in Wheeler's terminology), we sum (60.80) over i to give

$$H = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[(E_{ij} - V_{i-j})(E_{ik} - V_{i-k}) - \mathbf{p}_{ij}\cdot\mathbf{p}_{ik}c^2 - m_{ij}m_{ik}c^4\right] \ . \tag{60.85}$$

Hamilton's equations give

$$\frac{\mathrm{d}\mathbf{x}_{ij}}{d\tau} = \frac{\partial H}{\partial \mathbf{p}_{ij}} = -\frac{2}{N^2}\sum_{k=1}^{N}\mathbf{p}_{ik}c^2 \ , \tag{60.86}$$

$$\frac{\mathrm{d}\mathbf{p}_{ij}}{d\tau} = -\frac{\partial H}{\partial \mathbf{x}_{ij}} = -\frac{2}{N^2}\frac{\partial V_{ij}}{\partial \mathbf{x}_{ij}}\sum_{k=1}^{N}\sum_{m=1}^{N}(2E_{km} - E_{im} - E_{jm} + 2V_{ij} - V_{kj} - V_{ki}) \ , \tag{60.87}$$

$$\frac{\mathrm{d}t_{ij}}{d\tau} = -\frac{\partial H}{\partial E_{ij}} = -\frac{2}{N^2}\sum_{k=1}^{N}(E_{ik} - V_{i-k}) = -\frac{2}{N^2}\sum_{k=1}^{N}\left(E_{ik} - \sum_{m\neq i}^{N}V_{im} + \sum_{m\neq k}^{N}V_{km}\right) \ , \tag{60.88}$$

$$\frac{\mathrm{d}E_{ij}}{d\tau} = \frac{\partial H}{\partial t_{ij}} = 0 \ , \tag{60.89}$$

where τ depends on the definition of H, and has no physical significance. Equation (60.89) depends on assuming the potentials do not depend on time. Dividing by (60.88) gives

$$\frac{\mathrm{d}\mathbf{x}_{ij}}{dt_{ij}} = \frac{\sum_{k=1}^{N}\mathbf{p}_{ik}c^2}{\sum_{k=1}^{N}\left(E_{ik} - \sum_{m\neq i}^{N}V_{im} + \sum_{m\neq k}^{N}V_{km}\right)} \tag{60.90}$$

and

$$\frac{\mathrm{d}\mathbf{p}_{ij}}{dt_{ij}} = \frac{\partial V_{ij}}{\partial \mathbf{x}_{ij}}\frac{\sum_{k=1}^{N}\sum_{m=1}^{N}(2E_{km} - E_{im} - E_{jm} + 2V_{ij} - V_{kj} - V_{ki})}{\sum_{k=1}^{N}\left(E_{ik} - \sum_{m\neq i}^{N}V_{im} + \sum_{m\neq k}^{N}V_{km}\right)} \ . \tag{60.91}$$

I think I have an error here. I should not have subtracted the average from V_{ij}.

60.8.3 Avoiding subtracting the average from V_{ij}

Subtracting the average from each factor in (60.68) (except for $\overline{V}$) gives an equation whose factors average to zero (except for $\overline{V}$).

$$\left(E_i - \overline{E} - \sum_{k\neq i}^{N}V_{ik}\right)^2 - (\mathbf{p}_i - \overline{\mathbf{p}})^2 c^2 - (m_i - \overline{m})^2 c^4 = 0 \tag{60.92}$$

Substituting (60.69) through (60.72) into (60.92) gives

$$\left(E_i - \frac{1}{N}\sum_{j=1}^{N}E_j - \sum_{k\neq i}^{N}V_{ik}\right)^2 - \left(\mathbf{p}_i - \frac{1}{N}\sum_{j=1}^{N}\mathbf{p}_j\right)^2 c^2 - \left(m_i - \frac{1}{N}\sum_{j=1}^{N}m_j\right)^2 c^4 = 0 \ . \tag{60.93}$$

Or,

$$\frac{1}{N^2}\left[\left(NE_i - \sum_{j=1}^{N}E_j - N\sum_{k=1}^{N}V_{ik} + NV_{ii}\right)^2 - \left(N\mathbf{p}_i - \sum_{j=1}^{N}\mathbf{p}_j\right)^2 c^2 - \left(Nm_i - \sum_{j=1}^{N}m_j\right)^2 c^4\right] = 0 \ . \tag{60.94}$$

Or,

$$\frac{1}{N^2}\left[\left(\sum_{j=1}^{N}(E_i - E_j) + \sum_{j=1}^{N}\left(-\sum_{k=1}^{N}V_{ik} + V_{ii}\right)\right)^2 - \left(\sum_{j=1}^{N}(\mathbf{p}_i - \mathbf{p}_j)\right)^2 c^2 - \left(\sum_{j=1}^{N}(m_i - m_j)\right)^2 c^4\right] = 0\ . \tag{60.95}$$

Or,

$$\frac{1}{N^2}\left[\left(\sum_{j=1}^{N}\left(E_i - E_j - \sum_{k=1}^{N}(V_{ik}) + V_{ii}\right)\right)^2 - \left(\sum_{j=1}^{N}(\mathbf{p}_i - \mathbf{p}_j)\right)^2 c^2 - \left(\sum_{j=1}^{N}(m_i - m_j)\right)^2 c^4\right] = 0\ . \tag{60.96}$$

Or,

$$\frac{1}{N^2}\left[\left(\sum_{j=1}^{N}\left(E_i - E_j - \sum_{m=1}^{N}(V_{im}) + V_{ii}\right)\right)\left(\sum_{k=1}^{N}\left(E_i - E_k - \sum_{n=1}^{N}(V_{in}) + V_{ii}\right)\right) - \left(\sum_{j=1}^{N}(\mathbf{p}_i - \mathbf{p}_j)\right)\cdot\left(\sum_{k=1}^{N}(\mathbf{p}_i - \mathbf{p}_k)\right)c^2 - \left(\sum_{j=1}^{N}(m_i - m_j)\right)\left(\sum_{k=1}^{N}(m_i - m_k)\right)c^4\right] = 0\ . \tag{60.97}$$

Or,

$$\frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\left(E_i - E_j - \sum_{m=1}^{N}(V_{im}) + V_{ii}\right)\left(E_i - E_k - \sum_{n=1}^{N}(V_{in}) + V_{ii}\right) - (\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}_i - \mathbf{p}_k)\,c^2 - (m_i - m_j)\,(m_i - m_k)\,c^4\right] = 0\ . \tag{60.98}$$

To explicitly show that we have an equation in terms of relative coordinates, we write (60.98) as

$$\frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[(E_{ij} - V_i)\,(E_{ik} - V_i) - \mathbf{p}_{ij}\cdot\mathbf{p}_{ik}c^2 - m_{ij}m_{ik}c^4\right] = 0\ , \tag{60.99}$$

where

$$E_{ij} \equiv E_i - E_j\ , \tag{60.100}$$

$$V_i \equiv \sum_{m=1}^{N}(V_{im}) - V_{ii} = \sum_{m\neq i}^{N} V_{im}\ , \tag{60.101}$$

$$\mathbf{p}_{ij} \equiv \mathbf{p}_i - \mathbf{p}_j\ , \tag{60.102}$$

and

$$m_{ij} \equiv m_i - m_j \ . \tag{60.103}$$

To get a Hamiltonian (or super-Hamiltonian in Wheeler's terminology), we sum (60.99) over i to give

$$H = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\left(E_{ij} - V_i\right)\left(E_{ik} - V_i\right) - \mathbf{p}_{ij}\cdot\mathbf{p}_{ik}c^2 - m_{ij}m_{ik}c^4\right] \ . \tag{60.104}$$

Hamilton's equations give

$$\frac{\mathrm{d}\mathbf{x}_{ij}}{d\tau} = \frac{\partial H}{\partial \mathbf{p}_{ij}} = -\frac{2}{N^2}\sum_{k=1}^{N}(\mathbf{p}_{ik} + \mathbf{p}_{kj})c^2 = -\frac{2}{N}(\mathbf{p}_i - \mathbf{p}_j)c^2 = -\frac{2}{N}\mathbf{p}_{ij}c^2 \ , \tag{60.105}$$

$$\frac{\mathrm{d}\mathbf{p}_{ij}}{d\tau} = -\frac{\partial H}{\partial \mathbf{x}_{ij}} = -\frac{2}{N^2}\frac{\partial V_{ij}}{\partial \mathbf{x}_{ij}}\sum_{k=1}^{N}\sum_{m=1}^{N}\left(-E_{im} - E_{jm} + 2V_{ij}\right) = \frac{2}{N}\frac{\partial V_{ij}}{\partial \mathbf{x}_{ij}}\sum_{m=1}^{N}\left(E_{im} + E_{jm} - 2V_{ij}\right) \ , \tag{60.106}$$

$$\frac{\mathrm{d}t_{ij}}{d\tau} = -\frac{\partial H}{\partial E_{ij}} = -\frac{2}{N}\left(E_i - V_i - E_j + V_j\right) = -\frac{2}{N}\left(E_i - \sum_{m\neq i}^{N} V_{im} - E_j + \sum_{m\neq j}^{N} V_{jm}\right) \ , \tag{60.107}$$

and

$$\frac{\mathrm{d}E_{ij}}{d\tau} = \frac{\partial H}{\partial t_{ij}} = 0 \ , \tag{60.108}$$

where $t_{ij} \equiv t_i - t_j$, τ depends on the definition of H, and has no physical significance. Equation (60.108) depends on assuming the potentials do not depend on time. Dividing (60.105) and (60.106) by (60.107) gives

$$\frac{\mathrm{d}\mathbf{x}_{ij}}{dt_{ij}} = \frac{\mathbf{p}_{ij}c^2}{E_i - \sum_{m\neq i}^{N} V_{im} - E_j + \sum_{m\neq j}^{N} V_{jm}} \tag{60.109}$$

and

$$\frac{\mathrm{d}\mathbf{p}_{ij}}{dt_{ij}} = -\frac{\partial V_{ij}}{\partial \mathbf{x}_{ij}}\frac{\sum_{m=1}^{N}\left(E_{im} + E_{jm} - 2V_{ij}\right)}{E_i - \sum_{m\neq i}^{N} V_{im} - E_j + \sum_{m\neq j}^{N} V_{jm}} \ . \tag{60.110}$$

We can write (60.107) as

$$\frac{\mathrm{d}t_{ij}}{d\tau} = \frac{\mathrm{d}t_i}{d\tau} - \frac{\mathrm{d}t_j}{d\tau} \ , \tag{60.111}$$

where

$$\frac{\mathrm{d}t_i}{d\tau} = -\frac{2}{N^2}\sum_{k=1}^{N}\left(E_i - \sum_{m\neq i}^{N} V_{im}\right) = -\frac{2}{N}\left(E_i - \sum_{m\neq i}^{N} V_{im}\right) \tag{60.112}$$

and

$$\frac{\mathrm{d}t_j}{d\tau} = -\frac{2}{N^2}\sum_{k=1}^{N}\left(E_j - \sum_{m\neq j}^{N} V_{jm}\right) = -\frac{2}{N}\left(E_j - \sum_{m\neq j}^{N} V_{jm}\right) \ . \tag{60.113}$$

In comparing with the usual formulas, we need to use t_i or t_j, not the relative time t_{ij}. Therefore, dividing (60.105) and (60.106) by (60.112) gives

$$\frac{\mathrm{d}\mathbf{x}_{ij}}{dt_i} = \frac{\mathbf{p}_{ij}c^2}{E_i - \sum_{m\neq i}^{N} V_{im}} = \frac{\mathbf{p}_{ij}c^2}{E_i - V_i} \tag{60.114}$$

and

$$\frac{\mathrm{d}\mathbf{p}_{ij}}{dt_i} = -\frac{\partial V_{ij}}{\partial \mathbf{x}_{ij}}\frac{\sum_{m=1}^{N}\left(E_{im} + E_{jm} - 2V_{ij}\right)}{E_i - \sum_{m\neq i}^{N} V_{im}} = -\frac{\partial V_{ij}}{\partial \mathbf{x}_{ij}}\frac{\sum_{m=1}^{N}\left(E_{im} + E_{jm} - 2V_{ij}\right)}{E_i - V_i} \ . \tag{60.115}$$

For the case where there are only two particles, we get

$$\frac{\mathrm{d}\mathbf{x}_1}{dt_1} - \frac{\mathrm{d}\mathbf{x}_2}{dt_1} = \frac{\mathrm{d}\mathbf{x}_{12}}{dt_1} = \frac{c^2 p_{12}}{E_1 - V_{12}} = \frac{c^2(p_1 - p_2)}{E_1 - V_{12}} \tag{60.116}$$

and

$$\frac{\mathrm{d}\mathbf{p}_1}{dt_1} - \frac{\mathrm{d}\mathbf{p}_2}{dt_1} = \frac{\mathrm{d}\mathbf{p}_{12}}{dt_1} = \frac{4V_{12}}{E_1 - V_{12}} \frac{\partial V_{12}}{\partial \mathbf{x}_{12}} \ . \tag{60.117}$$

60.8.4 The usual way, for comparison

$$H = \sum_{i=1}^{N} \left[(E_i - V_i)^2 - p_i^2 c^2 - m_i^2 c^4 \right] \ . \tag{60.118}$$

Hamilton's equations give

$$\frac{\mathrm{d}\mathbf{x}_i}{d\tau} = \frac{\partial H}{\partial \mathbf{p}_i} = -2\mathbf{p}_i c^2 \ , \tag{60.119}$$

$$\frac{\mathrm{d}\mathbf{p}_i}{d\tau} = -\frac{\partial H}{\partial \mathbf{x}_i} = 2\sum_{m=1}^{N} (E_i - V_i + E_m - V_m) \frac{\partial V_{im}}{\partial \mathbf{x}_{im}} = 2\,(E_i - V_i) \frac{\partial V_i}{\partial \mathbf{x}_i} + 2\sum_{m=1}^{N} (E_m - V_m) \frac{\partial V_{im}}{\partial \mathbf{x}_{im}} \ , \tag{60.120}$$

$$\frac{\mathrm{d}t_i}{d\tau} = -\frac{\partial H}{\partial E_i} = -2\,(E_i - V_i) = -2\left(E_i - \sum_{m\neq i}^{N} V_{im} \right) \ , \tag{60.121}$$

$$\frac{\mathrm{d}E_i}{d\tau} = \frac{\partial H}{\partial t_i} = 0 \ , \tag{60.122}$$

where τ depends on the definition of H, and has no physical significance. Equation (60.122) depends on assuming the potentials do not depend on time. Dividing by (60.121) gives

$$\frac{\mathrm{d}\mathbf{x}_i}{dt_i} = \frac{\mathbf{p}_i c^2}{E_i - V_i} = \frac{\mathbf{p}_i c^2}{E_i - \sum_{m\neq i}^{N} V_{im}} \tag{60.123}$$

and

$$\frac{\mathrm{d}\mathbf{p}_i}{dt_i} = -\sum_{m=1}^{N} \frac{E_i - V_i + E_m - V_m}{E_i - V_i} \frac{\partial V_{im}}{\partial \mathbf{x}_{im}} = -\frac{\partial V_i}{\partial \mathbf{x}_i} - \sum_{m=1}^{N} \frac{E_m - V_m}{E_i - V_i} \frac{\partial V_{im}}{\partial \mathbf{x}_{im}} \ . \tag{60.124}$$

Similarly, we get

$$\frac{\mathrm{d}\mathbf{x}_j}{dt_i} = \frac{\mathbf{p}_j c^2}{E_i - V_i} = \frac{\mathbf{p}_j c^2}{E_i - \sum_{m\neq i}^{N} V_{im}} \tag{60.125}$$

and

$$\frac{\mathrm{d}\mathbf{p}_j}{dt_i} = -\sum_{m=1}^{N} \frac{E_j - V_j + E_m - V_m}{E_i - V_i} \frac{\partial V_{jm}}{\partial \mathbf{x}_{jm}} = -\frac{E_j - V_j}{E_i - V_i} \frac{\partial V_j}{\partial \mathbf{x}_j} - \sum_{m=1}^{N} \frac{E_m - V_m}{E_i - V_i} \frac{\partial V_{jm}}{\partial \mathbf{x}_{jm}} \ . \tag{60.126}$$

Subtracting (60.125) from (60.123) gives

$$\frac{\mathrm{d}\mathbf{x}_{ij}}{dt_i} = \frac{\mathrm{d}\mathbf{x}_i}{dt_i} - \frac{\mathrm{d}\mathbf{x}_j}{dt_i} = \frac{(\mathbf{p}_i - \mathbf{p}_j)c^2}{E_i - V_i} = \frac{\mathbf{p}_{ij} c^2}{E_i - V_i} = \frac{\mathbf{p}_{ij} c^2}{E_i - \sum_{m\neq i}^{N} V_{im}} \ . \tag{60.127}$$

Subtracting (60.126) from (60.124) gives

$$\frac{\mathrm{d}\mathbf{p}_{ij}}{dt_i} = \frac{\mathrm{d}\mathbf{p}_i}{dt_i} - \frac{\mathrm{d}\mathbf{p}_j}{dt_i} = -\sum_{m=1}^{N} \frac{(E_i - V_i + E_m - V_m)\frac{\partial V_{im}}{\partial \mathbf{x}_{im}} - (E_j - V_j + E_m - V_m)\frac{\partial V_{jm}}{\partial \mathbf{x}_{jm}}}{E_i - V_i} \ . \tag{60.128}$$

Equation (60.127) agrees with (60.109), but (60.128) does not agree with (60.110). Equation (60.128) corresponds to

$$\frac{\mathrm{d}\mathbf{p}_{ij}}{d\tau} = -\frac{\partial H}{\partial \mathbf{x}_{ij}} = 2\sum_{m=1}^{N}\left[(E_i - V_i + E_m - V_m)\frac{\partial V_{im}}{\partial \mathbf{x}_{im}} - (E_j - V_j + E_m - V_m)\frac{\partial V_{jm}}{\partial \mathbf{x}_{jm}}\right] , \qquad (60.129)$$

which does not agree with (60.106), nor does it make sense, because it should involve derivatives with respect to $\mathbf{x}_{ij}$, not with respect to $\mathbf{x}_{im}$ or $\mathbf{x}_{jm}$.

It appears that trying to derive relative-coordinate equations in this way is not working. If I cannot even get classical equations of motion in terms of relative coordinates using Hamilton's equations, then it does not make sense to go further to try to develop relative-coordinate wave equations. Thus, it appears that revisiting 1926 to see how quantum theory would have developed if we had had relative-coordinate classical physics at that time is not so far useful because I have not been able to even develop a relative-coordinate classical physics.

A possible reason for the difficulty in developing a relativistic relative coordinate classical physics is that maybe we need to include inertial force directly instead of through an inertial frame.

Sciama's [11, 1953] calculation of inertia works only in the frame of the body in question. Maybe here, also, we must start by using relative coordinates.

60.8.5 4 dimensions, curved spacetime

To see more, I think we need to go to 4 dimensions. To start with, I need to do the same kind of relative mechanics for special relativity. That means we start with the paper [167, Donald Lynden-Bell, J. Katz, and J. Bičák in 1995]. In that paper, they develop a similar kind of relative-coordinate representation for General Relativity that Donald Lynden-Bell [5, 6] did for Newtonian mechanics.

In that paper [167, Donald Lynden-Bell, J. Katz, and J. Bičák in 1995], they show (in the case of the Robertson-Walker metric) that the metric is invariant under rotations (as expected) and instead of being invariant under translations, the metric is invariant under quasi-translations (rotations at 90 degrees from here).

I thought I had remembered that they then did the same kind of operation on the Lagrangian that Donald Lynden-Bell [5, 6] did for Newtonian mechanics. However, on rereading their paper, I cannot find a place where they do that.

60.9 1916 revisited

Would General Relativity be different if Einstein had had Lynden-Bell's[5, 6] relative version of Mechanics instead of Newton't absolute version?

Possibly, but I have so far not been able to derive a relativistic relative-coordinate classical physics. Maybe Einstein could have done it.

Chapter 61

Einstein's first great blunder[1]

61.1 Introduction

Albert Einstein considered the introduction of the cosmological constant into his field equations to be his greatest blunder. For him personally, it probably was, because it robbed him of predicting the expansion of the universe.

For the advance of science as a whole, however, a greater blunder was to formulate gravitation as a metric theory because that robs us of a mathematical language to even consider an origin for inertia. First, because inertial and gravitational mass are the same in a geometrical theory, and second, because it is not possible to consider the absence of inertia in a geometrical theory.

Einstein based his Theory of General Relativity partly on two principles: the principle of equivalence and Mach's principle. Unfortunately for Mach's principle, Einstein made the equivalence principle an integral part of the theory rather than a secondary consequence. That decision led to a theory that would be equivalent to a geometrical theory in the aspects mentioned above that have unfortunate consequences for Mach's principle, such as being able to consider the origin of inertia, a possible variation in the ratio of inertial and gravitational masses, and the absence of inertia.

It would have been better had Einstein used the equivalence principle simply as a guide to indicate that inertia should be considered to be a gravitational force. Now that job is left to us. That is, it is up to us to reformulate General Relativity in a non-geometrical formulation so that we have the possibility of generalizing it.

Reformulating Einstein's theory of gravitation is especially difficult because of some other unfortunate consequences of a geometrical theory. By making the metric tensor play the dual role of the gravitational potential and the tensor that raises and lowers indices in equations means that every tensor equation may depend on gravitation even though the metric tensor does not appear explicitly.

Let's start with the action that leads to General Relativity. The $\frac{1}{2}mv^2$ term in the Lagrangian is not in terms of relative coordinates, but it can be made so by doing the same trick that Lynden-Bell[5] did for Newtonian dynamics. However, that does not solve all of the problems, because the basic form for that term in the Lagrangian is based on being calculated in an inertial frame. In fact, it depends on there being an inertial frame.

An inertial frame in General Relativity is a freely falling frame, a frame where gravitational forces sum to zero. This is based on the equivalence principle. Then, in that frame we do local physics.

[1] 8 March 1998

61.2 Newtonian dynamics

It is just about as bad in Newtonian dynamics. Again, we have the $\frac{1}{2}mv^2$ term in the Lagrangian, which can be made relative as Lynden-Bell [5] did, but still depends on being calculated in an inertial frame and it depends on there being an inertial frame.

An inertial frame in Newtonian dynamics is different from an inertial frame in General Relativity. In Newtonian dynamics, an inertial frame is one where there are no inertial forces. There are still local gravitational forces. Otherwise, the difficulty is the same. In either case, the inertial part of the difficulty is the hard part, so let's see if I can solve the problem in Newtonian mechanics. If I can do it there, I might be able to do it in General Relativity.

So, we start by doing Lynden-Bell's[5] transformation on the $\frac{1}{2}mv^2$ term to get relative coordinates. That calculation must be done in an inertial frame, and there must be an inertial frame. Now, what we want to do here is somehow change this so that the inertial frame part and the inertia part can be considered part of a (gravitational) force rather than the background.

Let's think specifically about Lynden-Bell's[5] calculations. His Lagrangian for L in [5, (1.8)] is equivalent to the original Lagrangian. The derivation of that Lagrangian was done in an inertial frame. Applying that Lagrangian implies that the velocity of the center of mass of the universe (in that inertial frame) will be a constant. That should not be a surprise.

Continuing to apply that Lagrangian gives [5, (1.9)] for the equation of motion of the relative coordinates. If we instead use L* instead of L that includes only the relative coordinates, then we still get [5, (1.9)] for the equation of motion for the relative coordinates. That is, the equation of motion for the relative coordinates is independent of whether we include the term $\frac{1}{2}Mu^2$ in the Lagrangian (where M is the mass of the universe, and u is the velocity of the center of mass of the universe in the coordinate system).

Let's take a different interpretation of that result. $-u$ is the velocity of the inertial frame relative to the center of mass of the universe. Equation [5, (1.9)] gives the equation of motion for the relative positions of the bodies for either L or L*. Using L says that inertial frames move with constant velocity with respect to the center of mass of the universe. Using L* gives no equation for the movement of inertial frames relative to the center of mass. Thus, if L* is a valid Lagrangian, the inertial frame can have any motion whatever relative to the center of mass, and also that [5, (1.9)] gives the motion of the relative coordinates no matter what the motion of the inertial frame is relative to the center of mass. Is that reasonable? Is the inertial frame observable independently of the relative motion of the bodies? How else can we observe the inertial frame? It really does agree with observation that the center of mass of the universe seems to be an inertial frame. How do we observe that? Obviously, we have only relative motions of bodies available for us to observe. We can use astronomy, of course, so we can look at the stars.

I'll have to think about this. In the meantime, let's look at rotation.

61.2.1 rotation

For rotation, we get another term in the Lagrangian, something like $\frac{1}{2}I\Omega^2 =$ something like $\frac{1}{2}I^{-1}J^2$, where J is the angular momentum of the universe about its center of mass, and I is the momentum of inertia about its center of mass. The equations of motion make J a constant, but not necessarily zero. Thus we have the possibility of the universe rotating relative to the inertial frame. That is, we have the possibility of the inertial frame rotating relative to the universe. In that case, we get different equations of motion for the relative coordinates. Only if J is zero do we get the same equations of motion for the relative coordinates for the Lagrangians with and without the $\frac{1}{2}I\Omega^2$ term. Since the original Lagrangian was valid only in an inertial frame, it make sense that allowing rotation would change the result.

We can observe rotations in the universe, so we have an observation here.

However, the equations were valid only in an inertial frame, not a rotating one, so we haven't really learned much.

Come to think of it, showing that the center of mass of the universe moves with constant velocity is not very new either.

This is getting nowhere. I can't figure out how to separate out the inertial frame part. That is, I can't figure out how to write the formulas so they don't need inertial frames. Let's try the action.

61.3 Newtonian action

Here is the problem. I want to generalize the action to include the case where there is not much matter in the universe. As we approach that limit, we probably won't have inertial frames, so the usual formulas for the Lagrangian won't apply, because they are based on inertial frames. To get around that, I am going to assume that it does not take much matter to set up an inertial frame. Then, as we calculate the action, the contribution to the action when the amount of matter in the universe was small will be incorrect, but the error in that contribution won't matter because the contribution from that part will be small anyway.

No, that is not right. I want the calculation to apply also when the total amount of matter in the universe is small. I can try it this way. If I get it roughly right, I will have correct dependence when the amount of matter is large, and, hopefully for a small amount of matter I will get the action approaching zero in some reasonable way as the amount of matter approaches zero.

Let's go back to Lynden-Bell's[5] formulas for Newtonian action in terms of relative coordinates. Specifically, we take his equation [5, (1.8)]. The mass in the kinetic energy terms in that equation is inertial mass, while the mass in the potential energy term is gravitational mass. I would like to write everything in terms of gravitational mass. Let us take $m_I = f(M)m$, where m_I is inertial mass, m is gravitational mass, and f(M) is some function of the total mass distribution in the universe. In standard units, f(M) has the value of one for our universe now.

Then Lynden-Bell's [5] equation [5, (1.8)] becomes

$$L = T - V = \frac{1}{2} f(M) M u^2 + \frac{f(M)}{2M} \sum_i \sum_j m_i m_j (\dot{r}_i - \dot{r}_j)^2 + G \sum_i \sum_j \frac{m_i m_j}{|r_i - r_j|} \tag{61.1}$$

We would like the action to approach zero as the amount of matter in the universe approaches zero. This can happen in two ways: the Lagrangian can go to zero, or the 4-volume of integration can go to zero. Let's try the Lagrangian first.

We have a general feeling that f(M) should be proportional to M. So, try

$$f(M) = Mg(M) \tag{61.2}$$

This gives

$$L = T - V = \frac{1}{2} g(M)(Mu)^2 + \frac{g(M)}{2} \sum_i \sum_j m_i m_j (\dot{r}_i - \dot{r}_j)^2 + G \sum_i \sum_j \frac{m_i m_j}{|r_i - r_j|} \tag{61.3}$$

As long as $g(M)$ remains finite as M approaches zero, it looks like L will also approach zero. We still don't know if the action will approach zero, but it should, because in this formulation, the action should be just summing rather than integrating.

There is still another consideration. We usually think of distant matter contributing a smaller amount (per unit mass) to inertia. In that case, $g(M)$ must somehow show that dependence. Following Sciama[11], Reissner[2, 3], Schrödinger[4], and others, we try

$$f(M) = \sum_i \frac{m_i}{|r_i|} \; . \tag{61.4}$$

This implies that

$$g(M) = \frac{1}{M} \sum_i \frac{m_i}{|r_i|} \ . \tag{61.5}$$

Then the center of mass of the universe is given by

$$r_0 = \frac{\sum_i m_i r_i}{M} = \frac{\sum_i m_i r_i}{\sum_i m_i} \ . \tag{61.6}$$

Then

$$f(M) = \sum_i \frac{m_i}{|r_0 - r_i|} = \sum_i \frac{m_i}{|\frac{\sum_j m_j r_j}{M} - r_i|} = \sum_i \frac{m_i M}{|\sum_j m_j r_j - \sum_j m_j r_i|} = \sum_i \frac{m_i M}{|\sum_j m_j (r_j - r_i)|} \ . \tag{61.7}$$

This implies that

$$g(M) = \sum_i \frac{m_i}{|\sum_j m_j (r_j - r_i)|} \ . \tag{61.8}$$

61.4 quantum gravity

If geometric theories are inadequate to describe gravitation (because they do not allow the possibility of zero inertial force), then it is inadequate to express quantum gravity as a quantum superposition of 3-geometries that evolves in time. Therefore, it is necessary to first find an adequate representation of classical gravitation before quantizing. Quantizing will not solve the problem.

Chapter 62

The trouble with General Relativity[1]

62.1 Introduction

Einstein's theory of General Relativity has been enormously successful. Still, there are some nagging questions about it that make us doubt that it is the correct theory, but only an approximation that must be generalized appropriately. Here, I point out some of these nagging problems, suggest properties that a more general theory should have, and propose some strategies for looking for a new theory.

As a matter of notation, I use "gravitation" to refer to any interaction involving mass, energy, or momentum, including inertial forces. This is in keeping with the notion based on the equivalence principle that it is impossible to tell from a local measurement an inertial force from any other kind of gravitational force (e.g., Misner, Thorne, and Wheeler, 1973, Box 6.2, pages 164-164)[20].

Sometimes I use electromagnetic theory as an analogy, suggesting that a revised theory of gravitation might behave in some ways the way that electromagnetic theory behaves. Of course, this is only a guideline. There are some ways in which the two must differ, but part of that may not be intrinsic, but only due to the large amount of matter in the universe.

62.1.1 The gravitational field

In electromagnetic theory, the tensor $F_{\mu\nu}$ represents the electromagnetic field, including both electric and magnetic fields. In the gravitational case, there is no similar quantity that corresponds to a gravitational field (e.g., Misner, Thorne, and Wheeler, 1973, pages 399-400)[20].

I consider the absence of a gravitational field to be a shortcoming of Einstein's theory of gravitation. Gravitational forces, however, can be defined in Einstein's theory. For example, consider the geodesic equation including the Lorentz force that gives the motion of a body of mass m and charge e in the presence of an electromagnetic field (e.g. Misner, Thorne, and Wheeler, 1972, equation 20.41, page 474)[20]:

$$\text{(gravitational part)}m + \text{(Lorentz part)}e = 0 \tag{62.1}$$

The term that includes the mass m can certainly be considered the gravitational force just as the term that includes the charge e (the Lorentz force) can be considered the electromagnetic force. Equation (62.1) simply says that the sum of the forces (including inertial forces) are zero. Unfortunately, the form of the geodesic equation does not allow the gravitational force to be factored into a gravitational field times a velocity-dependent term times the mass in a manner similar to the way that the Lorentz force is factored.[2]

[1] 28 February 1998 with some comments by Jay Palmer and David Peterson

[2] Jay Palmer thinks that the last sentence in this paragraph is a good summary of the point I am trying to make here, and that I don't need the build up before it to get that point.

Einstein postulated that the equation of motion for bodies under the influence of gravitation, but no other forces, is the geodesic equation. That is, that bodies under the influence of only gravitation follow a geodesic in a curved spacetime. I argue here that representing gravitational interaction by geodesics has shortcomings. Of course, I am using the term "gravitation" to include inertia as mentioned above.

One way we might consider modifying the geodesic equation, is to multiply the $\ddot{x}$ term by m_I/m, where m_I is the inertial mass and m is the passive gravitational mass. Then we can consider m_I/m to be one component of the total gravitational field in addition to the Γs. Of course, this would have to be fixed up, because it doesn't have the right symmetries or the right transformation properties, but it would be a start. Also, we would have to connect that new factor to sources. We would also have to connect the Γs to sources.

62.1.2 The "zero gravitational field" case

If there are no electric or magnetic fields, then a factor in the second term in (62.1)

$$(\text{Lorentz part}) = 0 \tag{62.2}$$

identically for all 4 components in all frames of reference independent of the motion of the body. In that case (in the absence of an electromagnetic field), the motion of the body will be governed by the pure geodesic equation:

$$(\text{gravitational part}) = 0 \tag{62.3}$$

In this case, the motion of the body is governed by pure gravitational effects, and is independent of the charge of the body.

The opposite case (a pure electromagnetic field with zero gravitational field) would correspond to

$$(\text{gravitational part}) = 0 \tag{62.4}$$

identically for all 4 components in all frames of reference independent of the motion of the body, so that the motion of the body would be governed by purely Lorentz forces:

$$(\text{Lorentz part}) = 0. \tag{62.5}$$

This second case does not exist because it is not possible to find any situation in which the geodesic equation is identically zero in all frames of reference independent of the motion of the body. This is because General Relativity assumes that a background inertia exists for massive bodies. This inertia should be derived rather than assumed. This seems to be related to the property that there is no gravitational analog to the electromagnetic field $F_{\mu\nu}$. Γ is not really an analog, except possibly in the frame of the body.

If inertia is really due to a gravitational interaction with the bulk of matter in the universe, then any experiment that can be done in the existing universe will never yield zero gravitational field. If we imagine a more general theory, however, that would apply also to a universe that has not enough matter to cause significant inertial force, then we will need a theory able to represent zero gravitational field.

Einstein's theory in particular, and geometrical theories in general, cannot be a complete description of general gravitational effects, because they do not allow the case of zero gravitational fields. (Absence of gravitational fields includes absence of inertial fields. I take gravitational fields to be any force that acts on the mass of a body. This is in the spirit of Einstein's representation of gravitation by a geodesic equation. The geodesic equation includes inertial forces.)

If we consider that factor of m_I/m that I proposed in the previous section, however, then we can require that factor to be zero for the zero gravitational case. As I mentioned, however, this factor is not satisfactory unless it can be fixed up.

62.1.3 Einstein's equation represent gravitation as curvature

In Einstein's field equations,

$$G = T \tag{62.6}$$

the stress-energy tensor T represents the sources, while G represents curvature. Thus, in Einstein's theory, sources of gravitation are really sources of curvature. There is no explicit representation for sources of inertia. It should be no surprise that Einstein's equations give only part of the sources for the gravitational forces in the geodesic equation.

What we need instead, is an equation in which the natural state is no gravitational forces (including inertial forces), and the presence of masses, energy, and momentum would produce gravitational forces where there were none before.

I think that propagators based on an action will always lead to the geodesic equation, which we want to avoid except when there is a lot of matter. Maybe we can use the infinitesimal particle propagator that was used by Hoyle and Narlikar.

62.1.4 The Mach problem

You may recognize a common thread running through these various "problems," since they depend on the relationship of the vast amount of matter in the universe to local physics, and some of the ideas of Ernst Mach (1872, 1911, 1960)[120, 102, 15].

If inertia is really the result of some kind of induction interaction with the rest of matter in the universe (e.g., Sciama, 1953[11]; Sciama, et al., 1969[16]), then inertia should vanish for the case where there is no matter or other energy sources. Although Minkowski space, which has inertia but no matter, seems to pose a difficulty, as a solution of Einstein's field equations, a worse difficulty seems to be that there is no solution of the field equations in which inertia is absent. In fact, a metric representation of gravitation cannot have an absence of inertia.

62.1.5 The quantization problem

There are still problems with trying to quantize gravitation. There is a possibility that these problems are related to the fact that we represent gravitation as geometry. This makes gravitation a background arena upon which all other interactions reside. We might find it easier to quantize gravitation if we could find a representation of gravitation that was not intrinsically a background for other interactions.

62.1.6 The background problem

The great amount of matter in the universe may or may not affect the apparent laws of physics locally, but it is difficult to imagine that it does not. We know pretty well what the laws of physics are in a background of a lot of matter in the universe. We can only guess what these laws would be in a universe that has a lot less matter. Looking for such a generalization should give us insight into physics on a more fundamental level. Of course, matter can include both visible and dark matter and other forms of energy and momentum, such as radiation and gravitation energy and momentum.

A particularly intriguing question is, "Is gravitation inherently a background for other interactions because it is inherently a geometrical theory, or is its role as a background only a result of there being a lot of matter in the universe?"[4]

[4] Jay Palmer likes this paragraph.

62.1.7 The role of coordinate systems

Einstein argued that being able to write the laws of physics in any coordinate system, including accelerating frames, was sufficient to guarantee a generalized relativity. A more restrictive criterion, however, would be to require that only relative positions and motions be allowed to enter physical laws. For example, Newton's law of gravitation satisfies that restricted requirement because it is given in terms of the relative positions of bodies. However, his laws of dynamics do not satisfy that requirement because they are in terms of the positions of bodies relative to an absolute space. Lynden-Bell (1992)[5] alters Newton's laws of dynamics to be written in terms of relative positions and motions in a very plausible way. Specifically, he expresses kinetic energy as sums of 2-body terms instead of 1-body terms.

General relativity, however, has moved in the opposite direction. All of the quantities involve positions and motions relative to coordinate systems, and none in terms of relative coordinates. Of course, Einstein's theory includes retarded effects, which are more difficult to express in terms of relative coordinates. I have been wondering if it might be possible to express the stress-energy tensor as a sum of 2-body terms in a way similar to what Lynden-Bell did for Newtonian dynamics.

Lynden-Bell et al. (1995)[167] do something similar for General relativity. They show that by restricting solutions to closed universes they get solutions in terms of relative coordinates, and the problem of flat empty space or empty space without inertia does not occur. However, they do not get inertia to depend on the amount of matter, but only inertial frames to depend on the matter, and Einstein's equations themselves are still not in terms of relative coordinates.

Question: are we in agreement so far?

62.2 Properties that a more general theory should have

1. The theory should allow for cases of zero gravitational field (including zero inertial force), and allow for a smooth transition to the zero field case. The term "field" here includes the background construction of space-time due to immersion of a body in the rest of the universe.

2. The theory should allow for the case where the universe does not have enough matter to produce a significant amount of inertia, and should also allow for a universe that has enough charge to produce inertia.

3. The theory should probably not contain the positions, velocities or accelerations of bodies relative to some arbitrary reference frame, but be written only in terms of relative positions, velocities, or accelerations.

4. In searching for such a theory, the units used can be a help or a hindrance. For example, setting the speed of light or the gravitational constant equal to one does not allow for the possibilities that those are not universal constants. As another example, Units that set E and D numerically equal or B and H numerically equal in electromagnetic theory, such as the cgs system, should be avoided for similar reasons. Therefore, we should use systems of units that allow the most flexibility.

5. The equivalence principle should not be taken as an absolute principle beyond the idea that inertial force should be considered a gravitational force and to satisfy experiments.

62.3 Some consequences of the above properties

- Property 1. in the previous section seems to exclude geometric theories of gravitation because of the discussion in 1.5.

- Property 3. in the previous section excludes Einstein's theory of General Relativity, at least as presently written.
- Is some generalization of the Lorentz force from a vector to a tensor potential a viable replacement for the geodesic equation? This would be something like $A_{ij,k}dx^j/ds \times dx^k/ds$ where A is a second rank anti-symmetric tensor. I doubt if this would work, since it is anti-symmetric rather than symmetric.

62.4 Strategies

1. Change units so that a specific inertial mass occurs in the geodesic equation.
2. Try to write the stress-energy tensor in terms of relative motions.
3. In the frame of the body, we can write the geodesic equation, then transform to an arbitrary frame as though we had a tensor equation.

62.5 Comments by David Peterson, 24 February 1998:

No, No. A whole new concept is needed: the construction of something resembling a "covariant aether" due to the presence of all the mass-energy in the universe. "Fields" usually tend to be about unbalanced forces - but we probably mean a balanced force analogous to the field in the center of the earth - an immersion that simply yields a lot of interaction of something we don't know about yet. Like Cook, I bet it is about the communications of all the fundamental vibrations of all the energies of the universe. But these vibration communications are somehow additive. This is not quite like the concept of a potential - it is something else, something new (and yet old and primordial). We really need a good name for it: inductable potential, gravitational induction field, inductable "stuff", cosmic substance, cosmological scalar potential, pregeometry, Vacuum, - I think I like the word "fabric."

We are somewhat at a "20 questions game" stage: I am thinking of something (that doesn't have a name yet). It doesn't have a taste, smell, color, or feel; it offers no resistance to motion (velocity - but does offer resistance to acceleration). It fills all of space-time and may be the stuff that space-time is derived from, it may hold all the laws and constants of physics, it is linearly additive with matter additions - what is it? How does it work? It is not gravitation. It is more primitive and basic than gravity and gravity may be derivable from it.

Chapter 63

Notes from the 1998 Samos meeting[1]

63.1 Saturday 29 August 1998 - Armenistas, Ikaria

The geodesic equation is nothing more than Newton's first law ($\ddot{x} = 0$) generalized to curved spacetime. Einstein interpreted spacetime curvature as gravitation. Einstein's field equations implicitly take absence of curvature as the natural state. Energy & momentum is necessary to give spacetime curvature. The absence of curvature, however, does not mean the absence of inertia. Inertia is present in any geometrical representation of gravitation, whether or not curvature is present. The natural state, therefore, in Einstein's formulation, always has inertia. Since inertia cannot be absent in Einstein's formulation, it is impossible to express a source for inertia within his system. Whether that correctly represents the gravitational interaction is still undecided. We can speculate about whether absence of inertia is possible in a correct formulation of gravitation.

Using EM theory as a model for gravitation may not be a good model for gravitation. $F_{\mu\nu} = 0$ does not set all of the force on a charged particle to zero. There is still radiation reaction for an accelerated charged particle, including also a term that depends on the rate of change in acceleration. This is probably not a good model for inertia, since this effect depends on acceleration of charge relative to an inertial frame, not to a charged universe. Radiation of accelerated charges comes from moving charge in a current, which produces a magnetic field. Changing current produces a changing magnetic field., which produces an electric field, etc.

Sciama's calculation should apply to E&M with a charged universe, but it does not, because the calculation takes place not in an inertial frame, but an accelerated frame.

Maybe it comes back to my idea of quantum selection of classical cosmologies. But if $G = T$ is not a valid representation of gravitation, then it cannot be the basis for a quantum representation.

63.2 Sunday 30 August 1998

Newton discovered the physics of inertial frames, although he did not use that terminology. He noticed that inertial frames were fixed with the stars and announced the existence of absolute space. Einstein generalized the physics of inertial frames to include curvature of spacetime to represent gravitation. His field equations gave a prescription for the relationship between matter and curvature. Neither of them addressed the origin of inertial frames nor provided a representation general enough to allow the absence of inertia, although Einstein's formulation allowed inertial frames to be altered by matter. Ernst Mach argued that there must be a causal relationship to explain the coincidence of inertial frames with the stars. Some solutions of Einstein's equations have inertial frames coinciding with the bulk of matter in the universe, but only for models of high

[1]Samos 30 August - 4 Sept. 1998

symmetry, in which the same symmetry is assumed for the matter and the spacetime. It is known that gravitational energy contributes to curvature in Einstein's equations due to the nonlinearty, but I don't know if this has been tested experimentally. [I talked with Pete Bender on Sept. 9, and he said he thinks the perihelion shift of Mercury and lunar ranging may test this.] One might imagine a case where inertial frames could be formed with only gravitational energy, and empty Minkowski space might be an example.

63.3 Wednesday 2 September 1998

Jim York and Arlen Anderson presented some very nice results with a new improved formulation of the evolution equations in terms of curvature and Bianchi identities. Jim York told me we should use curvature to give relative motion of bodies instead of using the geodesic equation to give the motion of a body relative to a coordinate system. I wonder how that works for bodies that are very distant. Later. I just asked him. He says it works only for the case of close geodesics. Suppose there is a spectrum of geometric or gravitational structure. Then there will be a spectrum of curvature. Curvature will give the relative motion of bodies if they are separated by no more that the spectral size of the curvature of interest. For large-scale structure, we could calculate the relative motion of bodies across the universe. In particular, choose the Robertson-Walker metric. Get the curvature for that. Use that curvature to calculate the relative motion of bodies across the universe. That will not give the detailed motion of the earth in the solar system relative to some planet across the universe. It will give the relative motion of a body here in the absence of local gravitational effects. Specifically, the motion of a local inertial frame. (Not a locally freely falling frame.) This works, but I still need to change Einstein's theory from a metric theory.

63.4 Thursday 3 September 1998

I am getting closer to a change in Einstein's field equations. I use curvature to give the difference of geodesics of any 2 bodies using the constant curvature of the Robertson-Walker metric. Then I get the motion of any 1 body relative to the center of mass of the universe.

I need to use momenta instead of velocities. Also, I can use an exact formula instead of curvature.

I can take the geodesic equation for a body and subtract from that the geodesic equation for that body as it would be without local bodies, but for only a Robertson-Walker background.

I need some more work here. Also, I can't use linear momentum for Robertson-Walker; I need pseudo-translation (= rotation about axis 90 degrees from body). Finally, once I get it right, it will be in principle different from the geodesic equation. But since the geodesic equation can be derived from the field equations, this will be inconsistent with the field equations. Therefore, I need to modify the field equations.

Chapter 64

Gravitation is fundamentally not geometry[1]

Mach's idea was that inertia was a gravitational interaction with the rest of matter in the universe. If we take the equivalence principle qualitatively, rather than quantitatively, then it really says no more than that inertia is a gravitational force.

That gravitation can be expressed in terms of geometry rests partly on the fact that both inertial force and gravitational force are proportional to the mass m. In the absence of other forces, the motion of a mass is then independent of m, since it cancels out.

If, however, the universe were charged, then there might be an electromagnetic component to inertia. In that case, gravitation could not be expressed in terms of geometry, because the mass would no longer cancel. Therefore, expressing gravitation as geometry is limited to the case where all of the inertia comes from gravitation. In the more general case, we would need to express gravitation in terms of something other than geometry.

[1]Draft: 17 September 1998

Chapter 65

Quantum Selection of a Classical Cosmology or Why our inertial frame seems not to rotate relative to the stars[1]

abstract

An explanation for why our inertial frame seems not to rotate relative to the stars is found in a straightforward application of semiclassical approximations to quantum cosmology. When a saddlepoint approximation is valid, only those classical geometries whose action I_{cl} satisfies $|I_{cl} - I_{saddlepoint}| < \hbar$ contribute significantly to the integration to give the wave function now. Using estimates for the amount of matter in our universe, this implies that only those classical geometries for which the present relative rotation rate of inertial frames and matter are less than about 10^{-71} radians per year contribute significantly to the integration. This is well below the limit set by experiment. The small value is due mainly to the ratio of the Planck length to the Hubble distance, which in turn depends on the amount of matter in the universe. The results do not depend significantly on the details of the theory of quantum gravity.

65.1 Introduction

Although Newton recognized that inertial frames seem not to rotate relative to the stars, he seems to have taken that as evidence for the existence of absolute space. Ernst Mach [120, 122] was probably the first physicist to recognize that such an apparent coincidence requires an explanation. He based his arguments on the observation that only relative positions of bodies are observable. Mach further suggested that matter might cause inertia.

Einstein tried to include what he called "Mach's Principle" in General Relativity, and although it is generally agreed that matter is a source of inertia in his theory, it is not the only source, because there are solutions (such as empty Minkowski space) which have inertia without matter. Further, solutions of Einstein's field equations include cosmologies where inertial frames rotate relative to the bulk of matter in the universe, so that an explanation for why our inertial frame does not rotate relative to the stars is still needed.

[1]Revised version (24 November 1998 or March 1999) of the paper that was presented at (and submitted to the Proceedings of) the conference on Mach's Principle: "From Newton's Bucket to Quantum Gravity", held at Tübingen, Germany, July 26-30, 1993, File: q-cos3.tex

It has been suggested (e.g., [115]) that Mach's principle be used as a boundary condition to eliminate those solutions of the field equations that have inertia from sources other than from matter. Similar suggestions (e.g., [154, 155, 16, 156, 109]) are that such selection might take place automatically if a satisfactory integral formulation of Einstein's field equations might be found, eliminating the need for boundary conditions. Other suggestions in which gravitation (including inertia) is represented by a theory analogous to Newtonian gravitation or Maxwell theory (e.g. [11, 5]) are also intriguing, and have the similar property that the gravitational field would be determined solely by the matter distribution.

Even if we could successfully find a theory along the lines of those mentioned above, in which matter somehow determined the gravitational field, it might not be satisfactory. Although that would explain why our inertial frame seems not to rotate relative to the stars, it takes away all degrees of freedom from the gravitational field. We are not so restrictive with the electromagnetic field. We allow arbitrary initial values on the electromagnetic field that are consistent with Maxwell's equations. We don't require that the field be completely determined by charges and currents. The gravitational field should be as fundamental as matter [157, 158, 159] and others at the Mach's Principle conference.

It is easy to see that allowing arbitrary initial conditions for the gravitational field (consistent with the field equations) is inconsistent with trying to find an explanation for why inertial frames seem not to rotate relative to the stars, at least on the classical level.

On the quantum level, however, we might imagine that somehow the selection takes place automatically through wave interference, and this turns out to be the case.

65.2 An example from ordinary wave mechanics

To help explain the ideas that follow, we first consider elementary wave mechanics. If we have an initial single-particle state specified by an initial wave function $< x_1, t_1|\psi >$ at time t_1 then the wave function $< x_2, t_2|\psi >$ at time t_2 is [112, p. 57]

$$< x_2, t_2|\psi >= \int_{-\infty}^{\infty} < x_2, t_2|x_1, t_1 >< x_1, t_1|\psi > dx_1, \tag{65.1}$$

where $< x_2, t_2|x_1, t_1 >$ is the propagator for the particle to go from (x_1, t_1) to (x_2, t_2). We consider the case where the semiclassical approximation for the propagator is valid. That is, [112, p. 60]

$$< x_2, t_2|x_1, t_1 >\approx f(t_1, t_2)e^{\frac{i}{\hbar}I_{cl}[x_2,t_2;x_1,t_1]}, \tag{65.2}$$

where $I_{cl}[x_2, t_2; x_1, t_1]$ is the action calculated along the classical path from (x_1, t_1) to (x_2, t_2). Thus, (65.1) becomes

$$< x_2, t_2|\psi >\approx f(t_1, t_2) \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}I_{cl}[x_2,t_2;x_1,t_1]} < x_1, t_1|\psi > dx_1. \tag{65.3}$$

Notice that because of the initial wave function we have an infinite number of classical paths contributing to each value of the final wave function.

There are two cases to consider. In the first, I_{cl} is not a sharply peaked function of x_1. In that case, there will be contributions to the wave function at t_2 from classical paths that differ greatly from each other.

In the second case, which we now consider, I_{cl} is sharply peaked about some value of x_1, say x_{sp}. That is, we have

$$\frac{\partial}{\partial x_1} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}} = 0. \tag{65.4}$$

Thus, x_{sp} is a saddlepoint of the integral (65.3), and significant contributions to the integral are limited to values of x_1 such that

$$|x_1 - x_{sp}|^2 < \left| \frac{2\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}}} \right|. \tag{65.5}$$

If $< x_1, t_1|\psi >$ is nearly constant over that range, then we can take it outside of the integral. A saddlepoint evaluation of the integral then gives

$$\begin{aligned} < x_2, t_2|\psi > \quad \approx \quad & f(t_1, t_2) < x_{sp}, t_1|\psi > \\ & \left[\frac{2\pi i\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}}} \right]^{1/2} e^{\frac{i}{\hbar} I_{cl}[x_2, t_2; x_{sp}, t_1]}. \end{aligned} \tag{65.6}$$

We notice from (65.4) that the momentum at t_1 at the saddlepoint is zero. That is,

$$p_1|_{x_1 = x_{sp}} = 0. \tag{65.7}$$

However, for the paths that contribute significantly to the integral in (65.3), there is a range of values of the momentum, namely

$$|p_1^2| < 2\hbar \left| \frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}} \right|, \tag{65.8}$$

consistent with (65.5) and the uncertainty relation.

As a check, using a special case, we consider the free-particle propagator [112, p. 42]

$$< x_2, t_2|x_1, t_1 >= \left[\frac{m}{2\pi i\hbar(t_2 - t_1)} \right]^{1/2} exp \left[\frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)} \right], \tag{65.9}$$

and we choose

$$< x_1, t_1|\psi >=< A, t_1|\psi > exp[-B(x_1 - A)^2] \tag{65.10}$$

to represent a broad initial wave function. For this case, the integral in (65.1) or (65.3) can be evaluated exactly to give

$$< x_2, t_2|\psi >=< A, t_1|\psi > \left[1 + \frac{2B\hbar(t_2 - t_1)}{im} \right]^{-1/2} exp \left[\frac{-B(x_2 - A)^2}{1 - \frac{2B\hbar(t_2 - t_1)}{im}} \right]. \tag{65.11}$$

The condition that the initial wave function is slowly varying is now

$$|B| \ll \left| \frac{m}{2\hbar(t_2 - t_1)} \right|, \tag{65.12}$$

so that (65.11) is approximately

$$< x_2, t_2|\psi >=< x_2, t_1|\psi > \tag{65.13}$$

in agreement with (65.6), since

$$I[x_2, t_2; x_1, t_1] = I_{cl}[x_2, t_2; x_1, t_1] = \frac{m}{2} \frac{(x_2 - x_1)^2}{t_2 - t_1}, \tag{65.14}$$

and

$$x_{sp} = x_2. \tag{65.15}$$

Notice that it is the sharply peaked action that determines which classical paths in (65.3) dominate the integral in this case, not the maximum of the initial wave function.

The calculations can clearly be generalized to two or three dimensions. The main point is that whenever the initial wave function is broad and the action of the classical propagator is not a sharply peaked function of x_1, many paths may contribute to the wave function in the final state, even when a semiclassical approximation is valid for the propagator. In that case, many classical paths may contribute significantly to the wave function in the final state.

When the classical action is sharply peaked as a function of the coordinates of the initial state, however, only a narrow range of classical paths contribute significantly to the wave function in the final state. This is thus a mechanism for selecting classical paths in wave mechanics. As we shall argue in the next sections, this principle has broader application.

65.3 Quantum cosmology

In the case of quantum cosmology, we have a formula analogous to (65.1) to give the wave function over 3-geometries g_2 and matter fields ϕ_2 on a 3-dimensional hypersurface S_2.

$$< g_2, \phi_2, S_2|\psi >= \int < g_2, \phi_2, S_2|g_1, \phi_1, S_1 >< g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1), \tag{65.16}$$

where $< g_1, \phi_1, S_1|\psi >$ is the wave function over 3-geometries g_1 and matter fields ϕ_1 on a 3-dimensional hypersurface S_1, and $< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >$ is the amplitude to go from a state with 3-geometry g_1 and matter fields ϕ_1 on a surface S_1 to a state with 3-geometry g_2 and matter fields ϕ_2 on a surface S_2 [123]. $D(g_1)$ and $D(\phi_1)$ are the measures on the 3-geometry and matter fields. The integration is over all initial 3-geometries g_1 and matter fields ϕ_1 for which the integral is defined.

We assume that the initial wave function is not sharply peaked. This is the most reasonable assumption without any knowledge of why it might be peaked.

65.4 Semiclassical approximation

As in section 2, we want to consider the case where the semiclassical approximation for the propagator is valid. That is, [160]

$$< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >\approx f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{cl}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]}, \tag{65.17}$$

where the function outside of the exponential is a slowly varying function. Substituting (65.17) into (65.16) gives

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > \;\; = \;\; & \int f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{cl}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]} \\ & < g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1). \end{aligned} \tag{65.18}$$

Each value of the integrand in (65.18) corresponds to one classical 4-geometry. As in (65.3), because the initial wave function can be broad, there will be an infinite number of classical 4-geometries that contribute to each value of the final wave function. Here, however, we do not have only one single integration, but an infinite number of integrations, because the integration is carried out over all possible 3-geometries and all matter fields on the initial surface.

In the simple example in Section 2, there were two cases to consider for the single integration being carried out. In the first case, the classical action was not a sharply peaked function. In the second case, the classical action was a sharply peaked function so that a saddlepoint approximation could be applied to the integration. Following that strategy, we would need to consider those two cases for each of the infinite number of integrations in (65.18).

Here, however, we consider two cases. In the first, I_{cl} is not a sharply peaked function of the 3-geometry g_1 for at least one of the infinite number of integrations in (65.18). In that case, there will be contributions to the wave function on S_2 from classical 4-geometries that differ significantly from each other. We consider this case in a later section.

In the second case, I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (65.18). We consider this case in the following section.

65.5 Saddlepoint approximation for the integral over initial states

We consider the case here where I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (65.18). In that case, we can formally make the saddlepoint approximation for each of the integrations in (65.18). In analogy with (65.4), we have the saddlepoint condition

$$\frac{\partial}{\partial g_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\Big|_{g_1=g_{sp}} = 0 \tag{65.19}$$

and

$$\frac{\partial}{\partial \phi_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\Big|_{\phi_1=\phi_{sp}} = 0, \tag{65.20}$$

where the derivatives in (65.19) and (65.20) are with respect to each parameter that defines the 3-geometry g_1 and matter fields ϕ_1. We consider the case where there is only one solution to the saddlepoint conditions (65.19) and (65.20). In that case, (65.19) selects a single classical 4-geometry. However, there will be a range of classical 4-geometries in the neighborhood that contribute significantly to the integral in (65.18). These are determined by

$$|I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] - I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]| < \hbar. \tag{65.21}$$

We can formally write the saddlepoint approximation to the integration in (65.18) as

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > &= f(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) < g_{sp}, \phi_{sp}, S_1|\psi > \\ & f_1(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) e^{\frac{i}{\hbar} I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]} \end{aligned} \tag{65.22}$$

where the exponential is assumed to dominate the behavior in (65.22). The main result, however, is that classical 4-geometries that contribute significantly to (65.22) lie within a narrow range specified by (65.21).

Equation (65.19) requires that the momentum canonical to the initial 3-geometry for the classical 4-geometry at the saddlepoint be zero. That is

$$\pi^{ij}\Big|_{g_1=g_{sp}} = 0. \tag{65.23}$$

(The extrinsic curvature on S_1 will therefore also be zero at the saddlepoint.) However, there will be a range of initial canonical momenta and a range of initial 3-geometries corresponding to the range of classical 4-geometries that satisfy (65.21), so that the uncertainty relations between initial 3-geometries and their canonical momenta are satisfied.

Whether there is a narrow or broad range of classical 4-geometries that satisfy (65.21) depends on the second derivative of the action with respect to the initial 3-geometry.

65.6 Spatially homogeneous spacetimes

The integration in (65.18) is an integration over functions g_1 and ϕ_1 defined on S_1. In that sense, it is similar to a path integral. For example, there are six independent functions that define g_1. As in the integration for a path integral, there are approximations that can be made to reduce the number of integrations that must be performed.

Here, we want to consider matter distributions similar to that observed, at least for the large scale in our universe. Thus, we want to restrict the integration in (65.18) to classical spatially homogeneous 4-geometries that have a homogeneous matter distribution in calculating the classical action in the exponential. The integration in (65.18) would then be over the 3-geometries that form the boundary of those 4-geometries on S_1.

As an example, we shall use Einstein's General Relativity for the classical 4-geometries, but the same calculations could be done for other classical gravitational theories, in case it turns out that General Relativity is not the correct theory of gravity. Thus, we want to consider the integration in (65.18) in which the classical 4-geometries used to calculate the action in the exponential are restricted to Bianchi cosmologies.

The appropriate calculation would be to consider the most general Bianchi model, with all of the parameters that describe that model, and carry out the integration over all of those parameters. We notice that the Bianchi parameters (which are time independent) define the initial three geometry, and therefore are valid integration variables in (65.18). On the other hand, if it is suspected that the saddlepoint for the integration will correspond to the Friedmann-Robertson-Walker (FRW) model, then one can restrict consideration to only those Bianchi models that include the FRW model as a special case, and consider integration in (65.18) for only one Bianchi parameter at a time, holding the others fixed at the FRW value. Here, we do that for only one of the Bianchi models for illustration.

In choosing which Bianchi model to use, we would like one that has a parameter that can be varied continuously to give the FRW model. In addition, we would like to choose a parameter that represents rotation of inertial frames relative to the matter distribution. In that way, we could directly test the ability of quantum selection to implement Mach's ideas about inertia.

So far, I have not been able to find a completely satisfactory example. The Bianchi VI_h model seems to partially satisfy these criteria, since it has a parameter that represents an angular velocity of inertial frames relative to matter, and setting that parameter to zero seems to give the FRW metric. However, there seem to be some difficulties with the Bianchi VI_h model being able to change continuously into the FRW model, and also a possible problem with the topology. Until I find a better example, however, I shall use this one.

We can take the action to be

$$I = \int (-g^{(4)})^{1/2}(L_{geom} + L_{matter})d^4x + \frac{1}{8\pi}\int (g^{(3)})^{1/2} K d^3x, \tag{65.24}$$

where [183, 123] show the importance of the surface term. [123] also points out a potential problem in that the action can be changed by conformal transformations, but suggests a solution.

$$K = g^{(3)ij}K_{ij} \tag{65.25}$$

is the trace of the extrinsic curvature. Although the extrinsic curvature is zero on S_1 at the saddlepoint, it will be nonzero in a region around the saddlepoint. The extrinsic curvature is given by

$$K_{ij} = -\frac{1}{2}\frac{\partial g^{(3)}_{ij}}{\partial t}, \tag{65.26}$$

where $g_{ij}^{(3)}$ is the 3-metric. In this example, we take the Lagrangian for the geometry as

$$L_{geom} = \frac{R}{16\pi}, \tag{65.27}$$

where R is the scalar curvature, but we realize that a different Lagrangian might eventually be shown to be more appropriate in a correct theory of quantum gravity.

For a perfect fluid, the energy momentum tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{65.28}$$

where p is the pressure, ρ is the density, and u is the 4-velocity. For solutions to Einstein's field equations for a perfect fluid, (65.27) becomes

$$L_{geom} = \frac{1}{2}\rho - \frac{3}{2}p, \tag{65.29}$$

and we can take the Lagrangian for the matter as [161]

$$L_{matter} = \rho \tag{65.30}$$

Thus, the classical action for perfect fluids is

$$I_{cl} = \frac{3}{2}\int (-g^{(4)})^{1/2}(\rho - p)d^4x - \frac{1}{16\pi}\int (g^{(3)})^{1/2} g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t}d^3x. \tag{65.31}$$

We can take

$$p = (\gamma - 1)\rho \tag{65.32}$$

for the equation of state, where $1 \leq \gamma < 2$. Then (65.31) becomes

$$I_{cl} = \frac{3}{2}\int (-g^{(4)})^{1/2}(2 - \gamma)\rho d^4x - \frac{1}{16\pi}\int (g^{(3)})^{1/2} g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t}d^3x. \tag{65.33}$$

Equation (65.33) diverges for a spatially open universe. The significance of that might be that only spatially closed universes make sense. On the other hand, it might be that the calculation of the action for the correct theory of quantum gravity will give a finite value for the action, even for a spatially open universe, but here, we shall restrict our calculation to the case of a spatially closed universe.

We use the solution for the Bianchi VI_h model from [163] with $h = -1/9$. After some algebra, we have

$$I_{cl} = \frac{3\pi^2}{4a_0}\int_{T^*}^{t} \frac{Y(t)Z(t)}{X(t)}dt, \tag{65.34}$$

where the spatial part of the 4-volume integration has already been carried out, a_0 is a parameter of the model, T^* is the Planck time, and X(t), Y(t), and Z(t) are functions of the model that must be determined by differential equations given by [163]. As expected, the surface term in (65.33) has canceled.

This cosmological model is relevant here because it has a relative rotation of inertial frames with respect to the matter. Specifically,

$$\Omega(t) = \frac{b}{Y^2(t)Z(t)} \tag{65.35}$$

is the angular velocity in the rest frame of an observer moving with the fluid, of a set of Fermi-propagated axes with respect to a particular inertial triad. The parameter b is an arbitrary constant of the model, and is zero if and only if there is no rotation of inertial frames relative to matter. Thus, we are interested to know the dependence of the classical action on b.

If we define

$$r^3(t) = X(t)Y(t)Z(t)\left(\frac{-3k}{3a_0^2+q_0^2}\right)^{3/2} \tag{65.36}$$

and

$$1+\alpha(t) = \frac{Y(t)^{2/3}Z(t)^{2/3}}{X(t)^{4/3}} \tag{65.37}$$

then the classical action in (65.34) becomes

$$I_{cl} = \frac{3\pi^2}{4}\left(\frac{3+q_0^2/a_0^2}{-3k}\right)^{1/2}\int_{L^*}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{65.38}$$

where L^* is the Planck distance and $k=+1$ for a closed universe. We choose the lower limit to be the boundary where quantum effects would be important. We expect that the value of the action would not depend significantly on the value of the lower limit as long as it is small, and this turns out to be the case.

For the $h=-1/9$ case, we have

$$q_0 = -3a_0. \tag{65.39}$$

if and only if $b \neq 0$. However, when integrating over b, the behavior for small b dominates over the exactly $b=0$ point. Therefore, we shall use (65.39) in any case. Substituting (65.39) into (65.38) gives

$$I_{cl} = \frac{3\pi^2}{2}\left(\frac{-1}{k}\right)^{1/2}\int_{L^*}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{65.40}$$

The form of the equation of state in (65.32) allows one of the differential equations for the model to be integrated in closed form to give

$$8\pi\rho = 3r_m^{3\gamma-2}r^{-3\gamma} \tag{65.41}$$

where r_m is a constant of integration that depends on the amount of matter in the universe and the speed of expansion relative to the gravitational attraction. Equation (65.41) shows that r_m is a measure of the amount of matter in the universe for a given value of r. Therefore, we might expect Machian effects (inertial induction) to increase for larger values of r_m.

Using (65.41), we have

$$\dot{r}^2 = \left(\frac{r_m}{r}\right)^{3\gamma-2} - k - k\alpha - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{r^2}{12}\left(\frac{\dot{\alpha}}{1+\alpha}\right)^2. \tag{65.42}$$

For the isotropic case, only the first two terms on the right hand side of (65.42) are nonzero. r_m is the value of r where those two terms are equal. For a closed universe for the isotropic case, r_m is the maximum value of r.

Equation (65.42) can be written

$$\dot{r} = \sqrt{\left(\frac{r}{r_m}\right)^{2-3\gamma} - k - k\alpha - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{V^2}{3r^4}}, \tag{65.43}$$

and the remaining differential equations to solve are

$$\dot{V} = 3k(1+\alpha)r - \frac{k}{(2a_0)^6}\frac{2b^2}{(1+\alpha)^2r^3} \tag{65.44}$$

and

$$\frac{\dot{\alpha}}{1+\alpha} = \frac{2V}{r^3}, \tag{65.45}$$

where (65.45) is the definition of $V(t)$, and $r(t)$ represents the size of the universe. The angular rotation (65.35) of an inertial frame relative to local matter is given by

$$\Omega(t) = \left(\frac{-k}{4a_0^2}\right)^{3/2} \frac{b}{[1+\alpha(t)]r^3(t)} \tag{65.46}$$

It is not possible to solve the differential equations exactly in closed form, but we can find approximate solutions assuming that α is less than one and neglecting all but the first term under the radical in (65.43). This gives

$$\alpha = \frac{12k}{(3\gamma+2)(3\gamma-2)}\left(\frac{r}{r_m}\right)^{3\gamma-2} - \frac{k}{72a_0^6 r_m^4}\frac{b^2}{(\gamma-2)^2}\left(\frac{r}{r_m}\right)^{3\gamma-6}. \tag{65.47}$$

This solution is valid for small r if b is small enough.

We then perform the integration in (65.40). The result is

$$\begin{aligned}
\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m1}}{L^*}\right)^2 \left\{\frac{2}{3\gamma_1+2}\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1} - \left(\frac{L^*}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]\right. \\
&+\frac{2}{3\gamma_2+2}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1} - \left(\frac{r_1}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}\right]\left(\frac{r_{m2}}{r_{m1}}\right)^2 \\
&+\frac{24k\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1} - \left(\frac{L^*}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}\right]}{(3\gamma_1+2)(3\gamma_1-2)(9\gamma_1-2)} \\
&+\frac{24k\left[\left(\frac{r}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-1} - \left(\frac{r_1}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-1}\right]}{(3\gamma_2+2)(3\gamma_2-2)(9\gamma_2-2)}\left(\frac{r_{m2}}{r_{m1}}\right)^2 \\
&-\frac{kb^2\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5} - \left(\frac{L^*}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}\right]}{36a_0^6 r_{m1}^4(\gamma_1-2)^2(9\gamma_1-10)} \\
&\left.-\frac{kb^2\left[\left(\frac{r}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-5} - \left(\frac{r_1}{r_{m2}}\right)^{\frac{9}{2}\gamma_2-5}\right]}{36a_0^6 r_{m2}^4(\gamma_2-2)^2(9\gamma_2-10)}\left(\frac{r_{m2}}{r_{m1}}\right)^2\right\},
\end{aligned} \tag{65.48}$$

where I have assumed that γ changes from an early-universe value of γ_1 to its late-universe value of γ_2 at $r = r_1$. To satisfy continuity of ρ at $r = r_1$, we must also have r_m change from r_{m1} to r_{m2} at $r = r_1$, where

$$\frac{r_{m2}}{r_{m1}} = \left(\frac{r_1}{r_{m1}}\right)^{\frac{3\gamma_2-3\gamma_1}{3\gamma_2-2}}. \tag{65.49}$$

Substituting (65.49) into (65.48) gives

$$\begin{aligned}
\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m1}}{L^*}\right)^2 \left\{\frac{2}{3\gamma_1+2}\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1} - \left(\frac{L^*}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]\right. \\
&+\frac{2}{3\gamma_2+2}\left[\left(\frac{r}{r_{m1}}\right)^{\frac{3}{2}\gamma_2+1} - \left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_2+1}\right]\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1-\frac{3}{2}\gamma_2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{24k\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}-\left(\frac{L^*}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}\right]}{(3\gamma_1+2)(3\gamma_1-2)(9\gamma_1-2)} \\
&+\frac{24k\left[\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_2-1}-\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_2-1}\right]}{(3\gamma_2+2)(3\gamma_2-2)(9\gamma_2-2)}\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-\frac{9}{2}\gamma_2} \\
&-\frac{kb^2\left[1-\left(\frac{L^*}{r_1}\right)^{\frac{9}{2}\gamma_1-5}\right]\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}\left(\frac{r_1}{r}\right)^{\frac{9}{2}\gamma_1-5}}{36a_0^6 r_{m1}^4(\gamma_1-2)^2(9\gamma_1-10)} \\
&\left.-\frac{kb^2\left[1-\left(\frac{r}{r_1}\right)^{5-\frac{9}{2}\gamma_2}\right]}{36a_0^6 r_{m1}^4(\gamma_2-2)^2(9\gamma_2-10)}\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}\left(\frac{r_1}{r}\right)^{\frac{9}{2}\gamma_1-\frac{9}{2}\gamma_2}\right\},
\end{aligned}
\tag{65.50}
$$

To get a rough estimate, we take

$$\gamma_1 = 4/3 \tag{65.51}$$

to represent a relativistic early universe,

$$\gamma_2 = 1 \tag{65.52}$$

to represent a matter-dominated late universe, and

$$r_1 = \frac{r}{100} \tag{65.53}$$

as an estimate that the universe changed from radiation-dominated to matter-dominated when the universe was about one-hundredth of its present size [13, Section 15.3, p. 481]. Substituting (65.51), (65.52), and (65.53) into (65.48) gives

$$
\begin{aligned}
\frac{I_{cl}}{\hbar} = {} & \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m1}}{L^*}\right)^2\left\{\frac{1}{3}\left[10^{-6}\left(\frac{r}{r_{m1}}\right)^3-\left(\frac{L^*}{r_{m1}}\right)^3\right]\right. \\
&+\frac{2}{5}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}-10^{-5}\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}\right]\left(\frac{r_{m1}}{r_1}\right)^2 \\
&+\frac{k}{5}\left[10^{-10}\left(\frac{r}{r_{m1}}\right)^5-\left(\frac{L^*}{r_{m1}}\right)^5\right] \\
&+\frac{24k}{35}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{7}{2}}-10^{-7}\left(\frac{r}{r_{m2}}\right)^{\frac{7}{2}}\right]\left(\frac{r_{m1}}{r_1}\right)^2 \\
&\left.-\frac{kb^2}{4a_0^6 r_{m1}^4}\left[\frac{1}{800}\left(\frac{r}{r_{m1}}\right)-\frac{1}{8}\left(\frac{L^*}{r_{m1}}\right)+\left(\frac{r_1}{r_{m1}}\right)^2\left(\frac{r_{m2}}{r}\right)^{\frac{1}{2}}\right]\right\}.
\end{aligned}
\tag{65.54}
$$

Neglecting some small terms, letting $k = +1$ for a closed universe, and using (65.49) gives

$$
\begin{aligned}
\frac{I_{cl}}{\hbar} = {} & \frac{3i\pi^2}{2}\left(\frac{r_{m1}}{L^*}\right)^2\left\{\frac{10^{-6}}{3}\left(\frac{r}{r_{m1}}\right)^3\right. \\
&+\frac{2}{5}\left(\frac{r_{m1}}{r_1}\right)^2\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}+\frac{10^{-10}}{5}\left(\frac{r}{r_{m1}}\right)^5 \\
&+\frac{24}{35}\left(\frac{r_{m1}}{r_1}\right)^2\left(\frac{r}{r_{m2}}\right)^{\frac{7}{2}} \\
&\left.-\frac{9b^2}{16000a_0^6 r_{m1}^4}\frac{r}{r_{m1}}\right\}.
\end{aligned}
\tag{65.55}
$$

Because the parameter b is an initial value for the cosmology, it is one of the variables of integration in (65.18). In making the saddlepoint approximation for that integration, we need to locate the saddlepoint (that is, the value of b that makes the action in (65.50) stationary. We see that the action is stationary with respect to variation of b at the isotropic case of $b = 0$, as expected. The range of values of b that contribute significantly to the integral in (65.18) is given by (65.21). That is

$$\left|\frac{I_{cl}(b)}{\hbar} - \frac{I_{cl}(b=0)}{\hbar}\right| < 1. \tag{65.56}$$

Thus, substituting (65.50) into (65.56) gives

$$\begin{aligned}&\frac{3\pi^2}{2}\left(\frac{r_{m1}}{L^*}\right)^2 \frac{b^2}{36a_0^6 r_{m1}^4}\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-5}\\ &\left\{\frac{\left[1-\left(\frac{L^*}{r_1}\right)^{\frac{9}{2}\gamma_1-5}\right]}{(\gamma_1-2)^2(9\gamma_1-10)}\right.\\ &\left.+\frac{\left[1-\left(\frac{r}{r_1}\right)^{5-\frac{9}{2}\gamma_2}\right]}{(\gamma_2-2)^2(9\gamma_2-10)}\left(\frac{r_1}{r}\right)^{\frac{9}{2}\gamma_1-\frac{9}{2}\gamma_2}\right\} < 1.\end{aligned} \tag{65.57}$$

The restriction on the size of b in (65.57) guarantees that α in (65.47) is less than one for r less than r_m and also that the terms neglected in (65.43) are small enough.

Substituting (65.51), (65.52), and (65.53) into (65.57) and neglecting a small term gives

$$\frac{27\pi^2}{32000a_0^6}\left(\frac{r_{m1}}{L^*}\right)^2 \frac{r}{r_{m1}}\frac{b^2}{r_{m1}^4} < 1. \tag{65.58}$$

This gives

$$b < \frac{20\sqrt{5}(2a_0)^3}{3\pi\sqrt{3}}\left(\frac{L^*}{r_{m1}}\right)\left(\frac{r_{m1}}{r}\right)^{\frac{1}{2}} r_{m1}^2. \tag{65.59}$$

Thus, from (65.46), the rotation rate of inertial frames is

$$|\Omega(t)| < \frac{20\sqrt{5}}{3\pi\sqrt{3}}\left(\frac{L^*}{r_{m1}}\right)\left(\frac{r_{m1}}{r}\right)^{\frac{1}{2}} \frac{r_{m1}^2}{[1+\alpha(t)]r(t)^3}. \tag{65.60}$$

If we now take the Planck distance L^* to be 1.6×10^-33 cm, use the Hubble distance of 1.7×10^{28} cm for r and r_{m1}, and neglect α compared to 1, then we get

$$|\Omega| < 1.4\times 10^{-71} \text{ radians per year}, \tag{65.61}$$

which is much less than the bound set by experiment [164]. The rotation rate in (65.61) is so small because the Planck distance is so much smaller than the Hubble distance. The ratio of the Planck distance to the Hubble distance entered in (65.59) to give a small value for the ratio of b to r_{m1}^2. The small value for b in (65.59) comes in turn from using the action (65.50) in (65.56).

That the estimate of the action in (65.48) is nearly independent of the lower limit in the integral in (65.40) suggests that the semiclassical approximation for the action is valid. That is, there is no significant contribution to the action from the time when the universe was a size comparable to the Planck distance. Specifically, it suggests that the small allowed value for b (relative to r_{m1}^2) in (65.59) when L^* is much less than r_{m1} is probably valid.

That the small value of allowed rotation rate depends only on the universe being much larger than a Planck distance rather than on details of the model suggests that the result has more generality.

We notice also, that the selection criterion in (65.56) is so sharp that the initial wave function in the integration in (65.18) would have to be very sharply peaked to overcome it.

65.7 Empty and sparse universes

We use the action in (65.24) and General Relativity as an example, but the results obviously have more generality. For an empty universe, $T^{\alpha\beta}$ is zero and therefore L_{matter} is also zero. In the semiclassical calculation, the integration in (65.18) is restricted to actions evaluated for classical cosmologies. Therefore, all cosmologies in the integrations in (65.18) satisfy Einstein's equations, so that $T^{\alpha\beta}$ equal zero implies that $R^{\alpha\beta}$ also equal zero, and therefore $R = 0$, and therefore L_{geom} is also zero. If we further consider only static universes, then the surface term in the action in (65.24) is also zero. Under those conditions, the action in (65.24) is zero.

Therefore, the action is zero for all empty static universes. (It may be possible to show that the action for all empty universes is zero.) Thus, the integration in (65.18) will weight all empty universes equally (except for the initial wave function). Therefore, for empty universes, there are no saddlepoints for the integrations in (65.18), and therefore no selection of solutions as we had for the solutions with matter in the previous section. Therefore, the only way to have a single matter free static solution would be for it to be selected by the initial wave function. That is, the initial wave function would have to be sharply peaked.

We would thus not expect to find an empty universe, in agreement with our concept that such solutions are non Machian. Similar arguments apply to asymptotically flat solutions, which many agree are also not Machian.

Sparse universes (that is, universes with some, but not much, matter) are perhaps more interesting. We should expect that the action in (65.24) is a continuous function of the amount of matter in the universe, even as the amount of matter approaches zero. Therefore, we would expect the action to be arbitrarily small for an arbitrarily small amount of matter.

We would therefore expect the integrations in (65.18) to not be sharply peaked in such cases, but to be broad, even for the cases where a saddlepoint exists. We would therefore not expect the saddlepoint approximation to be valid, and many classical cosmologies would contribute significantly to the wave function in the final state.

Thus, for sparse universes also, there would be no selection among classical cosmologies that contribute to the wave function in (65.18). To test this idea quantitatively, we consider the Bianchi model VI_h in the previous section, and let the amount of matter approach zero. Specifically, let us consider the density of matter given by (65.41) for r equal to some small value r_0.

$$8\pi\rho = 3r_{m1}^{3\gamma_1-2}r_0^{-3\gamma_1} \tag{65.62}$$

For a fixed value of r_0, decreasing the amount of matter in the universe requires a decrease in ρ, which in turn requires a decrease in r_{m1}. If the amount of matter decreases enough, r_{m1} will eventually no longer be much larger than the Planck distance. In that case, the action will no longer be sharply peaked at the saddlepoint $b = 0$, and the allowed values for b in (65.59) will no longer be restricted to be so small, and in turn, the allowed rotation rates in (65.60) will be much larger.

65.8 Discussion

We see that considerations of quantum cosmology show how a range of classical cosmologies can be selected that contribute significantly to the wave function in the final state. The effect enters through the action. Using semiclassical calculations gives results that should not depend on particular features of the theory of quantum gravity.

For our universe (which is much larger than the Planck distance) the selection is very sharp. The wave function over 3-geometries would have to be extremely sharp (not a probable occurrence) to dominate over the effect of the action.

The selection process seems to occur very soon in the development of a cosmology. That is, for a broad wave function over 3-geometries in the initial state, the wave function becomes sharply peaked after the universe has become a few orders of magnitude larger than the Planck distance.

The quantum selection process seems to agree with previous ideas about Machian cosmologies. Thus, cosmologies with enough matter will behave in a Machian way by the quantum selection process, but not empty universes, asymptotically flat universes, or sparse (nearly empty) universes.

If we were to use the consistent histories (or decoherent histories) approach to quantum cosmology [165], then the results presented here could possibly be extended to include boundary conditions in addition to initial conditions.

A different choice that (65.30) [162] is

$$L_{matter} = p \tag{65.63}$$

The latter choice gives a third of (65.31) for the total actin. A correct theory of quantum gravity will choose which (if either) of these two choices is correct, but for this illustration, a factor of three in the action makes little difference.

It appears likely now, however, that there is not enough matter to keep our universe from expanding forever. To accommodate that with a spatially closed universe within General Relativity would require a positive cosmological constant.

65.9 Acknowledgments

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Chapter 66

Why our inertial frame seems not to rotate relative to the stars[1]

abstract

Our inertial frame seems not to rotate relative to the stars because cosmological models with relative rotation cancel each other by phase interference. A straightforward application of semiclassical approximations plus a saddlepoint approximation to estimate the wave function in quantum cosmology requires that only those classical geometries whose action $I_{classical}$ satisfies $|I_{classical} - I_{saddlepoint}| < \hbar$ contribute significantly to the integration to give the wave function. For our universe, this implies that only those classical geometries for which the present relative rotation rate of inertial frames and matter is less than about 10^{-130} radians per year contribute significantly to the integration. The results do not depend on the details of the theory of quantum gravity.

66.1 Introduction

That our inertial frame seems not to rotate relative to the stars has been known since Newton, but Ernst Mach [120, 122] was the first physicist to recognize that such an apparent coincidence requires an explanation. Einstein [152] tried to include what he called "Mach's Principle" in General Relativity, but because solutions of Einstein's field equations include cosmologies where the average inertial frame rotates relative to the bulk of matter in the universe, we still need to explain why our inertial frame does not rotate relative to the stars.

At first glance, suggestions that Mach's principle be used as a boundary condition to select appropriate solutions of the field equations [115, 16, 156, 109] or proposed formulations in which gravitation (including inertia) should be determined solely by the matter distribution [11, 154, 155, 5, 184] seem to solve the problem. However, such solutions would eliminate independent degrees of freedom for the gravitational field, and would simply enforce the desired effect rather than explain the observation.

My purpose here is to explain without proposing any new theories the observation that our inertial frame seems not to rotate relative to the stars while still allowing independent degrees of freedom (through arbitrary initial conditions) for the gravitational field. I argue that, in principle, the gravitational field has independent degrees of freedom [157, 158, 159], but that a selection of cosmological solutions takes place automatically through phase interference [185].

[1] submitted to "essays on gravitation", March 1999, File: q-cos4.tex

66.2 Quantum cosmology

In quantum cosmology (in analogy with the path-integral formulation of quantum mechanics [112]), the wave function over 3-geometries g_2 and matter fields ϕ_2 on a 3-dimensional hypersurface S_2 is given by [123]

$$< g_2, \phi_2, S_2|\psi >= \int < g_2, \phi_2, S_2|g_1, \phi_1, S_1 >< g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1), \tag{66.1}$$

where $< g_1, \phi_1, S_1|\psi >$ is the wave function over 3-geometries g_1 and matter fields ϕ_1 on a 3-dimensional hypersurface S_1, and $< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >$ is the amplitude to go from a state with 3-geometry g_1 and matter fields ϕ_1 on a surface S_1 to a state with 3-geometry g_2 and matter fields ϕ_2 on a surface S_2. $D(g_1)$ and $D(\phi_1)$ are the measures on the 3-geometry and matter fields. The integration is over all initial 3-geometries g_1 and matter fields ϕ_1 for which the integral is defined.[2]

66.3 Semiclassical approximation

The semiclassical approximation[3] for the propagator [160] is

$$< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >\approx f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{classical}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]}, \tag{66.2}$$

where the function outside of the exponential is a slowly varying function and $I_{classical}$ is the action for a classical 4-geometry bounded by $[g_1, \phi_1, S_1]$ and $[g_2, \phi_2, S_2]$. Substituting (66.2) into (66.1) gives

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > \;\; &= \int f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{classical}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]} \\ & < g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1). \end{aligned} \tag{66.3}$$

66.4 Saddlepoint approximation

If $I_{classical}$ is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (66.3), then we can formally make the saddlepoint approximation for each of the integrations in (66.3). We have the saddlepoint conditions

$$\frac{\partial}{\partial g_1} I_{classical}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\Big|_{g_1=g_{sp}} = 0 \tag{66.4}$$

and

$$\frac{\partial}{\partial \phi_1} I_{classical}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\Big|_{\phi_1=\phi_{sp}} = 0, \tag{66.5}$$

where the derivatives in (66.4) and (66.5) are with respect to each parameter that defines the 3-geometry g_1 and matter fields ϕ_1. Even if there is only one solution to the saddlepoint conditions (66.4) and (66.5), there will be a range of classical 4-geometries in the neighborhood of the saddlepoint that contribute significantly to the integral in (66.3). These are determined by [186]

$$|I_{classical}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] - I_{saddlepoint}| < \hbar, \tag{66.6}$$

[2]Because a matter distribution cannot be specified independent of geometry, we take the above integration to mean that we first specify a 3-geometry, then on that 3-geometry, we specify a matter distribution.

[3]I consider only cases where the semiclassical approximation is valid, since the goal is to show why some classical cosmological models contribute significantly to the integration while others do not. This requires choosing S_1 to be at least a few Planck lengths away from the initial singularity.

where

$$I_{saddlepoint} = I_{classical}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]. \tag{66.7}$$

We can formally write the saddlepoint approximation to the integration in (66.3) as

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > &= f(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) < g_{sp}, \phi_{sp}, S_1|\psi > \\ & f_1(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) e^{\frac{i}{\hbar} I_{classical}[g_2,\phi_2,S_2;g_{sp},\phi_{sp},S_1]}, \end{aligned} \tag{66.8}$$

where classical 4-geometries that contribute significantly to (66.8) (through the function f_1) lie within the range specified by (66.6).

We can take the action to be

$$I = \int (-g^{(4)})^{1/2} (L_{geom} + L_{matter}) d^4x + \frac{1}{8\pi} \int (g^{(3)})^{1/2} K d^3x, \tag{66.9}$$

where York [183] and Hawking [123] explain the importance of the surface term. K is the trace of the extrinsic curvature and $L_{geom} = R/16\pi$, where R is the scalar curvature.

66.5 Perfect fluid models

To approximate the large-scale structure of our universe, we consider perfect fluids in (66.9). Taking $L_{matter} = \rho$ [161] gives the classical action (for solutions to Einstein's field equations) as

$$I_{classical} = \frac{3}{2} \int (-g^{(4)})^{1/2} (\rho - p) d^4x - \frac{1}{16\pi} \int (g^{(3)})^{1/2} g^{(3)ij} \frac{\partial g_{ij}^{(3)}}{\partial t} d^3x, \tag{66.10}$$

where p is pressure and ρ is density.[4] We can take [163]

$$p = (\gamma - 1)\rho \tag{66.11}$$

for the equation of state, where $1 \leq \gamma < 2$.

66.6 Spatially homogeneous spacetimes

Spatial homogeneity approximates the large-scale behavior of our universe and simplifies the calculations. Although the Bianchi IX Cosmology is often used to represent anisotropy, it seems inappropriate for the present purpose because it is simply a superposition of gravitational waves on a Friedman-Robertson-Walker cosmology [187].

Although the Bianchi VI_h cosmological model [163] has some problems, it seems to be a better homogeneous model to illustrate a relative rotation of inertial frames with respect to matter. Specifically,

$$\Omega(t) = \frac{b}{Y^2(t)Z(t)} \tag{66.12}$$

is the angular velocity in the rest frame of an observer moving with the fluid, of a set of Fermi-propagated axes with respect to a particular inertial triad. $Y(t)$ and $Z(t)$ [and $X(t)$ below] are functions of time in the model that must be determined by solutions of differential equations given by [163]. The constant b is an initial condition for the model, and is zero if and only if there is no rotation of inertial frames relative to matter. Thus, we are interested to know the dependence of the classical action on b.

[4] A different choice for the matter Lagrangian for a perfect fluid [162] would give a third of (66.10) for the total action, but would have insignificant effect on the final result.

The form of the equation of state in (66.11) allows one of the differential equations for the model to be integrated in closed form to give

$$8\pi\rho = 3r_m^{3\gamma-2}r^{-3\gamma}, \tag{66.13}$$

where

$$r^3(t) \equiv X(t)Y(t)Z(t)\left(\frac{-3k}{3a_0^2+q_0^2}\right)^{3/2} \tag{66.14}$$

gives the size of the universe and r_m is a constant of integration. a_0 and q_0 are parameters of the model, $q_0 = -3a_0$ for the $h = -1/9$ case, and $k = +1$ for a closed universe.

After some algebra, we have from (66.10)

$$I_{classical} = \frac{3\pi^2}{4a_0}\int_{t_0}^{t}\frac{Y(t)Z(t)}{X(t)}dt = \frac{3\pi^2}{2}\left(\frac{-1}{k}\right)^{1/2}\int_{r_0}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{66.15}$$

where the spatial part of the 4-volume integration has been carried out, $\dot{r} = dr/dt$,

$$\dot{r}^2 = \left(\frac{r_m}{r}\right)^{3\gamma-2} - k(1+\alpha) - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{r^2}{12}\left(\frac{\dot{\alpha}}{1+\alpha}\right)^2, \tag{66.16}$$

and

$$\alpha(t) \equiv \frac{Y(t)^{2/3}Z(t)^{2/3}}{X(t)^{4/3}} - 1. \tag{66.17}$$

As expected, the surface term in (66.10) has canceled.[5] For $k = +1$, b is imaginary for real Ω, and the path of integration is along the imaginary b axis.

Choosing isotropic values for some initial conditions gives a smaller minisuperspace and leads to an approximate solution for X, Y, and Z for small values of b, which yields

$$\begin{aligned}
1+\alpha \;&\approx\; (1+\alpha_0)exp\left\{\frac{2r_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3} - r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma} + \right.\\
&12k\frac{(3\gamma-6)r^{3\gamma-2} - (6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3} + (3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\\
&\left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6} - 2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3} + r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\right\}\\
&\approx\; (1+\alpha_0)\left\{1 + \frac{2r_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3} - r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma} + \right.\\
&12k\frac{(3\gamma-6)r^{3\gamma-2} - (6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3} + (3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\\
&\left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6} - 2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3} + r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\right\},
\end{aligned} \tag{66.18}$$

where α_0 and $\dot{\alpha}_0$ are constants of integration.

[5] The lower and upper limits in (66.15) correspond to the surfaces S_1 and S_2 in (66.1) and (66.3). We take the lower limit to be just enough larger than the Planck time and Planck length T^* and L^* that the semiclassical approximation for the action is valid.

Assuming that γ changes from $\gamma_1 = 4/3$ (to represent a relativistic early universe) to $\gamma_2 = 1$ (to represent a matter-dominated late universe) at $r = r_1$, continuity of ρ at $r = r_1$ requires r_m to change from r_{m1} to r_{m2} at $r = r_1$, where

$$\frac{r_{m1}}{r_{m2}} = \left(\frac{r_1}{r_{m2}}\right)^{\frac{3\gamma_1-3\gamma_2}{3\gamma_1-2}} = \left(\frac{r_1}{r_{m2}}\right)^{1/2}. \tag{66.19}$$

We choose the constants of integration in (66.18) to insure continuity of α and $\dot{\alpha}$ at $r = r_1$.

Using the approximation (66.18) and neglecting all but the first term in (66.16) leads to an approximation for (66.15). Keeping only the dominant terms, letting $k = +1$ for a closed universe, and using (66.19) gives

$$\frac{I_{classical}}{\hbar} \approx \frac{I_{saddlepoint}}{\hbar} - i\frac{6}{5}(1+\alpha_0)\left(\frac{r(t)}{r_1}\right)\left(\frac{r(t)}{r_{m2}}\right)^{3/2}\left(\frac{L^*}{r_0}\right)^2\left(\frac{\pi b}{(2a_0)^3 L^{*2}}\right)^2, \tag{66.20}$$

where [188] gives the details of the calculation.

Because the parameter b is an initial value for the cosmology, it is one of the variables of integration in (66.3). In making the saddlepoint approximation for that integration, we see that the action is stationary with respect to variation of b at the isotropic case of $b = 0$, as expected. The steepest descent path is along the imaginary b axis. The range of values of b that contribute significantly to the integral in (66.3) is given by (66.6). Thus,

$$\left|\frac{6}{5}(1+\alpha_0)\frac{\pi^2}{(2a_0)^6}\left(\frac{r(t)}{r_1}\right)\left(\frac{r(t)}{r_{m2}}\right)^{3/2}\left(\frac{L^*}{r_0}\right)^2\frac{b^2}{L^{*4}}\right| < 1. \tag{66.21}$$

For values of b restricted by (66.21), all of the terms neglected in (66.16) are smaller than the first term.

Thus, from (66.12), (66.14), and (66.18), the rotation rates of inertial frames that contribute significantly to (66.3) are limited by

$$|\Omega(t)| < \frac{L^{*2}(1+\alpha_0)^{-3/2}}{\pi r(t)^3}\left(\frac{5r_1}{6r(t)}\right)^{\frac{1}{2}}\left(\frac{r_{m2}}{r(t)}\right)^{\frac{3}{4}}\left(\frac{r_0}{L*}\right). \tag{66.22}$$

Taking the Planck length L^* to be 1.6×10^-33 cm, twice that for r_0, using the Hubble distance of 1.7×10^{28} cm for r_{m2}, a tenth of that for r, a hundredth of r for r_1,[6] and zero (the isotropic value) for α_0 to get a rough estimate,[7] we see that the present relative rotation rate is limited by

$$|\Omega| < 1.6 \times 10^{-130} \text{ radians per year}, \tag{66.23}$$

which is much less than the bound set by experiment of 10^{-14} to 7×10^{-17} radians per year if the universe is spatially closed [164].

66.7 Discussion

There are no new theories here, simply a calculation based on general properties of what we would most likely expect for quantum cosmology. The integrals in (66.1) or (66.3) (or something very similar) are implicit in the amplitude for any observation or experiment that we might imagine.

The argument can be made more general by including the cosmological constant and extending the calculation to the spatially open case (in case that turns out to be the correct situation [189, 190]).

[6] as an estimate that the universe changed from radiation-dominated to matter-dominated when the universe was about one-hundredth of its present size [13].

[7] Notice that choosing slightly different values would not give significantly different results.

66.8 Acknowledgment

I would like to thank Douglas Gough for first bringing the paper [11] by Dennis Sciama to my attention in 1967.

Chapter 67

The classical action for a Bianchi VI_h model[1]

abstract

An estimate for the classical action $I_{classical}$ for a Bianchi VI_h homogeneous spatially closed model with $h = -1/9$ is given by

$$\begin{aligned} \frac{I_{classical}}{\hbar} &\approx \frac{3}{5} i\pi^2 (1+\alpha_0) \left[\left(\frac{r(t)}{L^*}\right)^2 \left(\frac{r(t)}{r_{m2}}\right)^{1/2} \right. \\ &- \left. 2 \left(\frac{r(t)}{r_1}\right) \left(\frac{r(t)}{r_{m2}}\right)^{3/2} \left(\frac{L^*}{r_0}\right)^2 \left(\frac{b}{8a_0^3 L^{*2}}\right)^2 \right], \end{aligned}$$

where b and a_0 are parameters of the model, b is zero if and only if the relative rotation of inertial frames and matter is zero, α_0 is an initial value of an anisotropy of the expansion rate at $r = r_0$, L^* is the Planck length, and r_{m2} is a constant of integration that gives the maximum size of the universe for the isotropic ($b = 0$, $\alpha_0 = 0$) case.

It is assumed that the equation of state is $p = (\gamma - 1)\rho$, where p is pressure and ρ is density. It is assumed that γ has a constant value of $\gamma = 4/3$ (to represent a relativistic early universe) for $r < r_1$ and a constant value of $\gamma = 1$ (to represent a matter-dominated late universe) for $r > r_1$. The approximation is valid for b small enough that $|I_{classical}[b] - I_{classical}[b = 0]| < \hbar$.

An explanation for why our inertial frame seems not to rotate relative to the stars is found in a straightforward application of semiclassical approximations to quantum cosmology. Application of a saddlepoint approximation leads to the result that only those classical geometries whose action $I_{classical}$ satisfies $|I_{classical} - I_{saddlepoint}| < \hbar$ contribute significantly to the integration to give the present value of the wave function. Using estimates for our universe implies that only those classical geometries for which the present relative rotation rate of inertial frames and matter are less than about 10^{-130} radians per year contribute significantly to the integration. This is well below the limit set by experiment. The result depends on the Hubble distance being much larger than the Planck length, but does not depend on the details of the theory of quantum gravity.

67.1 Introduction

Although Newton recognized that inertial frames seem not to rotate relative to the stars, he seems to have taken that as evidence for the existence of absolute space. Ernst Mach [120, 122] was

[1] 18 March 1999, File: q-cos5.tex, available at <http://arxiv.org/abs/gr-qc/9903073>

probably the first physicist to recognize that such an apparent coincidence requires an explanation. He based his arguments on the observation that only relative positions of bodies are observable. Mach further suggested that matter might cause inertia.

Einstein [152] tried to include what he called "Mach's Principle" in General Relativity, and although it is generally agreed that matter is a source of inertia in his theory, it is not the only source, because there are solutions (such as empty Minkowski space) that have inertia without matter. Further, solutions of Einstein's field equations include cosmologies where inertial frames rotate relative to the bulk of matter in the universe, so that an explanation for why our inertial frame does not rotate relative to the stars is still needed.

It has been suggested (e.g. [115]) that Mach's principle be used as a boundary condition to eliminate those solutions of the field equations that have inertia from sources other than from matter. Similar suggestions (e.g. [154, 155, 16, 156, 109]) that such selection might take place automatically if a satisfactory integral formulation of Einstein's field equations might be found would eliminate the need for explicit boundary conditions. Other suggestions in which gravitation (including inertia) is represented by a theory analogous to Newtonian gravitation or Maxwell theory (e.g. [11, 5]) are also intriguing, and have the similar property that the gravitational field would be determined solely by the matter distribution.

Even if we could successfully find a theory along the lines of those mentioned above, in which matter somehow determined the gravitational field, it might not be satisfactory. Rather than explain why our inertial frame seems not to rotate relative to the stars, such a theory would simply impose that condition, and would take away all degrees of freedom from the gravitational field. We are not so restrictive with the electromagnetic field, for example. We allow arbitrary initial values on the electromagnetic field that are consistent with Maxwell's equations. We don't require that the electromagnetic field be completely determined by charges and currents. Just as the electromagnetic field should be as fundamental as charges and currents, the gravitational field should be as fundamental as matter [157, 158, 159].

Allowing arbitrary initial conditions for the gravitational field (consistent with the field equations) is inconsistent with trying to explain why our inertial frame seems not to rotate relative to the stars, at least on the classical level.

On the quantum level, however, we might imagine that somehow the selection takes place automatically through wave interference, and this turns out to be the case. To show that requires calculating the action for a classical cosmology as a function of the appropriate parameters of the model, which is the goal here.

67.2 An example from ordinary wave mechanics

To help explain the ideas that follow, we first consider elementary wave mechanics as an example. If we have an initial single-particle state specified by an initial wave function $< x_1, t_1|\psi >$ at time t_1 then the wave function $< x_2, t_2|\psi >$ at time t_2 is [112, p. 57]

$$< x_2, t_2|\psi >= \int_{-\infty}^{\infty} < x_2, t_2|x_1, t_1 >< x_1, t_1|\psi > dx_1, \tag{67.1}$$

where $< x_2, t_2|x_1, t_1 >$ is the propagator for the particle to go from (x_1, t_1) to (x_2, t_2). We consider the case where the semiclassical approximation for the propagator is valid. That is, [112, p. 60]

$$< x_2, t_2|x_1, t_1 >\approx f(t_1, t_2)e^{\frac{i}{\hbar}I_{cl}[x_2,t_2;x_1,t_1]}, \tag{67.2}$$

where $I_{cl}[x_2, t_2; x_1, t_1]$ is the action calculated along the classical path from (x_1, t_1) to (x_2, t_2). Thus, (67.1) becomes

$$< x_2, t_2|\psi >\approx f(t_1, t_2) \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}I_{cl}[x_2,t_2;x_1,t_1]} < x_1, t_1|\psi > dx_1. \tag{67.3}$$

Notice that because of the initial wave function we have an infinite number of classical paths contributing to each value of the final wave function.

There are two cases to consider. In the first, I_{cl} is not a sharply peaked function of x_1. In that case, there will be contributions to the wave function at t_2 from classical paths that differ greatly from each other.

In the second case, which we now consider, I_{cl} is sharply peaked about some value of x_1, say x_{sp}. That is, we have

$$\frac{\partial}{\partial x_1} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}} = 0. \tag{67.4}$$

Thus, x_{sp} is a saddlepoint of the integral (67.3), and significant contributions to the integral are limited to values of x_1 such that

$$|x_1 - x_{sp}|^2 < \left| \frac{2\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}}} \right|. \tag{67.5}$$

If $< x_1, t_1 | \psi >$ is nearly constant over that range, then we can take it outside of the integral. A saddlepoint evaluation of the integral then gives

$$\begin{aligned} < x_2, t_2 | \psi > \quad \approx \quad & f(t_1, t_2) < x_{sp}, t_1 | \psi > \\ & \left[\frac{2\pi i \hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}}} \right]^{1/2} e^{\frac{i}{\hbar} I_{cl}[x_2, t_2; x_{sp}, t_1]}. \end{aligned} \tag{67.6}$$

We notice from (67.4) that the momentum at t_1 at the saddlepoint is zero. That is,

$$p_1|_{x_1 = x_{sp}} = 0. \tag{67.7}$$

However, for the paths that contribute significantly to the integral in (67.3), there is a range of momenta, namely

$$|p_1^2| < 2\hbar \left| \frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}} \right|, \tag{67.8}$$

consistent with (67.5) and the uncertainty relation.

As a check, using a special case, we consider the free-particle propagator [112, p. 42]

$$< x_2, t_2 | x_1, t_1 >= \left[\frac{m}{2\pi i \hbar (t_2 - t_1)} \right]^{1/2} exp \left[\frac{im(x_2 - x_1)^2}{2\hbar (t_2 - t_1)} \right], \tag{67.9}$$

and we choose

$$< x_1, t_1 | \psi >=< A, t_1 | \psi > exp[-B(x_1 - A)^2] \tag{67.10}$$

to represent a broad initial wave function. For this case, the integral in (67.1) or (67.3) can be evaluated exactly to give

$$< x_2, t_2 | \psi >=< A, t_1 | \psi > \left[1 + \frac{2B\hbar (t_2 - t_1)}{im} \right]^{-1/2} exp \left[\frac{-B(x_2 - A)^2}{1 - \frac{2B\hbar (t_2 - t_1)}{im}} \right]. \tag{67.11}$$

The condition that the initial wave function is slowly varying is now

$$|B| \ll \left| \frac{m}{2\hbar (t_2 - t_1)} \right|, \tag{67.12}$$

so that (67.11) is approximately

$$< x_2, t_2|\psi >=< x_2, t_1|\psi > \tag{67.13}$$

in agreement with (67.6), since

$$I[x_2, t_2; x_1, t_1] = I_{cl}[x_2, t_2; x_1, t_1] = \frac{m}{2}\frac{(x_2 - x_1)^2}{t_2 - t_1}, \tag{67.14}$$

and

$$x_{sp} = x_2. \tag{67.15}$$

Notice that it is the sharply peaked action that determines which classical paths in (67.3) dominate the integral in this case, not the maximum of the initial wave function.

The calculations can clearly be generalized to two or three dimensions. The main point is that whenever the initial wave function is broad and the action of the classical propagator is not a sharply peaked function of x_1, some very different classical paths may contribute to the wave function in the final state, even when a semiclassical approximation is valid for the propagator.

When the classical action is sharply peaked as a function of the coordinates of the initial state, however, only a narrow range of classical paths contribute significantly to the wave function in the final state. This is thus a mechanism for selecting classical paths in wave mechanics. As we shall argue in the next sections, this principle has broader application.

67.3 Quantum cosmology

In the case of quantum cosmology, we have a formula analogous to (67.1) to give the wave function over 3-geometries g_2 and matter fields ϕ_2 on a 3-dimensional hypersurface S_2.

$$< g_2, \phi_2, S_2|\psi >= \int < g_2, \phi_2, S_2|g_1, \phi_1, S_1 >< g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1), \tag{67.16}$$

where $< g_1, \phi_1, S_1|\psi >$ is the wave function over 3-geometries g_1 and matter fields ϕ_1 on a 3-dimensional hypersurface S_1, and $< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >$ is the amplitude to go from a state with 3-geometry g_1 and matter fields ϕ_1 on a surface S_1 to a state with 3-geometry g_2 and matter fields ϕ_2 on a surface S_2 [123]. $D(g_1)$ and $D(\phi_1)$ are the measures on the 3-geometry and matter fields. The integration is over all initial 3-geometries g_1 and matter fields ϕ_1 for which the integral is defined.

67.4 Semiclassical approximation

As in section 2, we want to consider the case where the semiclassical approximation for the propagator is valid. That is, [160]

$$< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >\approx f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{cl}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]}, \tag{67.17}$$

where the function outside of the exponential is a slowly varying function and I_{cl} is the action for a classical 4-geometry. Substituting (67.17) into (67.16) gives

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > \;\; = \;\; & \int f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{cl}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]} \\ & < g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1). \end{aligned} \tag{67.18}$$

Each value of the integrand in (67.18) corresponds to one classical 4-geometry. As in (67.3), there will be an infinite number of classical 4-geometries that contribute to each value of the final wave function. Here, however, we do not have only one single integration, but an infinite number of integrations, because the integration is carried out over all possible 3-geometries and all matter fields on the initial surface.

In the simple example in Section 2, there were two cases to consider for the single integration being carried out. In the first case, the classical action was not a sharply peaked function. In the second case, the classical action was a sharply peaked function so that a saddlepoint approximation could be applied to the integration. Following that strategy, we would need to consider those two cases for each of the infinite number of integrations in (67.18).

Here, however, we consider only the case where I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (67.18). We consider this case in the following section.

67.5 Saddlepoint approximation for the integral over initial states

We consider the case here where I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (67.18). In that case, we can formally make the saddlepoint approximation for each of the integrations in (67.18). In analogy with (67.4), we have the saddlepoint condition

$$\frac{\partial}{\partial g_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\bigg|_{g_1 = g_{sp}} = 0 \tag{67.19}$$

and

$$\frac{\partial}{\partial \phi_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\bigg|_{\phi_1 = \phi_{sp}} = 0, \tag{67.20}$$

where the derivatives in (67.19) and (67.20) are with respect to each parameter that defines the 3-geometry g_1 and matter fields ϕ_1. We consider the case where there is only one solution to the saddlepoint conditions (67.19) and (67.20). In that case, (67.19) selects a single classical 4-geometry. However, there will be a range of classical 4-geometries in the neighborhood that contribute significantly to the integral in (67.18). These are determined by (e.g. [186])

$$|I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] - I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]| < \hbar. \tag{67.21}$$

We can formally write the saddlepoint approximation to the integration in (67.18) as

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > \quad &= \quad f(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) < g_{sp}, \phi_{sp}, S_1|\psi > \\ & \quad\quad f_1(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) e^{\frac{i}{\hbar} I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]} \end{aligned} \tag{67.22}$$

where classical 4-geometries that contribute significantly to (67.22) (through the function f_1) lie within a narrow range specified by (67.21).

Equation (67.19) requires that the momentum canonical to the initial 3-geometry for the classical 4-geometry at the saddlepoint be zero. That is

$$\pi^{ij}\Big|_{g_1 = g_{sp}} = 0. \tag{67.23}$$

(The extrinsic curvature on S_1 will therefore also be zero at the saddlepoint.) However, there will be a range of initial canonical momenta and a range of initial 3-geometries corresponding to the range of classical 4-geometries that satisfy (67.21), so that the uncertainty relations between initial 3-geometries and their canonical momenta are satisfied.

Whether there is a narrow or broad range of classical 4-geometries that satisfy (67.21) depends on the second derivative of the action with respect to the initial 3-geometry.

We can take the action to be

$$I = \int (-g^{(4)})^{1/2}(L_{geom} + L_{matter})d^4x + \frac{1}{8\pi}\int (g^{(3)})^{1/2} K d^3x, \tag{67.24}$$

where [183, 123] show the importance of the surface term. [123] also points out a potential problem in that the action can be changed by conformal transformations, but suggests a solution.

$$K = g^{(3)ij}K_{ij} \tag{67.25}$$

is the trace of the extrinsic curvature. Although the extrinsic curvature is zero on S_1 at the saddlepoint, it will be nonzero in a region around the saddlepoint. The extrinsic curvature is given by

$$K_{ij} = -\frac{1}{2}\frac{\partial g_{ij}^{(3)}}{\partial t}, \tag{67.26}$$

where $g_{ij}^{(3)}$ is the 3-metric. In this example, we take the Lagrangian for the geometry as

$$L_{geom} = \frac{R}{16\pi}, \tag{67.27}$$

where R is the scalar curvature, but we realize that a different Lagrangian might eventually be shown to be more appropriate in a correct theory of quantum gravity.

67.6 Perfect fluid models

For a perfect fluid, the energy momentum tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{67.28}$$

where p is the pressure, ρ is the density, and u is the 4-velocity. For solutions to Einstein's field equations for a perfect fluid, (67.27) becomes

$$L_{geom} = \frac{1}{2}\rho - \frac{3}{2}p, \tag{67.29}$$

and we can take the Lagrangian for the matter as [161]

$$L_{matter} = \rho \tag{67.30}$$

Thus, the classical action for perfect fluids is

$$I_{cl} = \frac{3}{2}\int (-g^{(4)})^{1/2}(\rho - p)d^4x - \frac{1}{16\pi}\int (g^{(3)})^{1/2} g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t} d^3x. \tag{67.31}$$

We can take

$$p = (\gamma - 1)\rho \tag{67.32}$$

for the equation of state, where $1 \leq \gamma < 2$. Then (67.31) becomes

$$I_{cl} = \frac{3}{2}\int (-g^{(4)})^{1/2}(2-\gamma)\rho d^4x - \frac{1}{16\pi}\int (g^{(3)})^{1/2} g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t} d^3x. \tag{67.33}$$

Equation (67.33) diverges for a spatially open universe. The significance of that might be that only spatially closed universes make sense. On the other hand, it might be that the calculation of the action for the correct theory of quantum gravity will give a finite value for the action, even for a spatially open universe, but here, we shall restrict our calculation to the case of a spatially closed universe.

67.7 Spatially homogeneous spacetimes

The integration in (67.18) is an integration over functions g_1 and ϕ_1 defined on S_1. In that sense, it is similar to a path integral. For example, there are six independent functions that define g_1. As in the integration for a path integral, there are approximations that can be made to reduce the number of integrations that must be performed.

Here, we want to consider matter distributions similar to that observed, at least for the large scale in our universe. Thus, we want to restrict the integration in (67.18) to classical spatially homogeneous 4-geometries that have a homogeneous matter distribution in calculating the classical action in the exponential. The integration in (67.18) would then be over the 3-geometries that form the boundary of those 4-geometries on S_1.

As an example, we shall use Einstein's General Relativity for the classical 4-geometries, but the same calculations could be done for other classical gravitational theories, in case it turns out that General Relativity is not the correct theory of gravity. Thus, we want to consider the integration in (67.18) in which the classical 4-geometries used to calculate the action in the exponential are restricted to Bianchi cosmologies.

The appropriate calculation would be to consider the most general Bianchi model, with all of the parameters that describe that model, and carry out the integration over all of those parameters. We notice that the Bianchi parameters (which are time independent) define the initial three geometry, and therefore are valid integration variables in (67.18). On the other hand, if it is suspected that the saddlepoint for the integration will correspond to the Friedmann-Robertson-Walker (FRW) model, then one can restrict consideration to only those Bianchi models that include the FRW model as a special case, and consider integration in (67.18) for only one Bianchi parameter at a time, holding the others fixed at the FRW value. Here, we do that for only one of the Bianchi models for illustration.

In choosing which Bianchi model to use, we would like one that has a parameter that can be varied continuously to give the FRW model. In addition, we would like to choose a parameter that represents rotation of inertial frames relative to the matter distribution. In that way, we could directly test the ability of quantum selection to implement Mach's ideas about inertia.

So far, I have not been able to find a completely satisfactory example. Although the Bianchi IX cosmology is often used to represent anisotropy, it seems inappropriate for the present case because it is only a superposition of gravitational waves on a Friedman-Robertson-Walker background.

The Bianchi VI_h model seems to be a better homogeneous model that has a parameter that represents an angular velocity of inertial frames relative to matter, and setting that parameter to zero seems to give the FRW metric. However, there seem to be some difficulties with the Bianchi VI_h model being able to change continuously into the FRW model, and also a possible problem with the topology. Until I find a better example, however, I shall use this one.

We use the solution for the Bianchi VI_h model from [163] with $h = -1/9$. This cosmological model is relevant here because it has a relative rotation of inertial frames with respect to the matter. Specifically,

$$\Omega(t) = \frac{b}{Y^2(t)Z(t)} \tag{67.34}$$

is the angular velocity in the rest frame of an observer moving with the fluid, of a set of Fermi-propagated axes with respect to a particular inertial triad. The parameter b is an arbitrary constant of the model, and is zero if and only if there is no rotation of inertial frames relative to matter. Thus, we are interested to know the dependence of the classical action on b.

After some algebra, we have

$$I_{cl} = \frac{3\pi^2}{4a_0}\int_{t_0}^{t}\frac{Y(t)Z(t)}{X(t)}dt, \tag{67.35}$$

where the spatial part of the 4-volume integration has already been carried out, a_0 is a parameter of the model, t_0 corresponds to the surface S_1 in (67.16) and (67.18) and is enough larger than the Planck time T^* that the semiclassical approximation is valid, the upper limit in (67.35) corresponds to the surface S_2 in (67.16) and (67.18), and $X(t)$, $Y(t)$, and $Z(t)$ are functions of the model that must be determined by differential equations given by [163]. As expected, the surface term in (67.33) has canceled.

If we define

$$r^3(t) = X(t)Y(t)Z(t)\left(\frac{-3k}{3a_0^2+q_0^2}\right)^{3/2} \tag{67.36}$$

and

$$1+\alpha(t) = \frac{Y(t)^{2/3}Z(t)^{2/3}}{X(t)^{4/3}} \tag{67.37}$$

then the classical action in (67.35) becomes

$$I_{cl} = \frac{3\pi^2}{4}\left(\frac{3+q_0^2/a_0^2}{-3k}\right)^{1/2}\int_{r_0}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{67.38}$$

where $\dot{r} = dr/dt$, r_0 is enough larger than the Planck length L^* that quantum effects can be neglected, and $k = +1$ for a closed universe.

For the $h = -1/9$ case, we have

$$q_0 = -3a_0 \tag{67.39}$$

if and only if $b \neq 0$. Substituting (67.39) into (67.38) gives

$$I_{cl} = \frac{3\pi^2}{2}\left(\frac{-1}{k}\right)^{1/2}\int_{r_0}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{67.40}$$

The form of the equation of state in (67.32) allows one of the differential equations for the model to be integrated in closed form to give

$$8\pi\rho = 3r_m^{3\gamma-2}r^{-3\gamma} \tag{67.41}$$

where r_m is a constant of integration that depends on the amount of matter in the universe and the speed of expansion relative to the gravitational attraction. Equation (67.41) shows that r_m is a measure of the amount of matter in the universe for a given value of r. Therefore, we might expect Machian effects (inertial induction) to increase for larger values of r_m.

Using (67.41), we have

$$\dot{r}^2 = \left(\frac{r_m}{r}\right)^{3\gamma-2} - k - k\alpha - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{r^2}{12}\left(\frac{\dot{\alpha}}{1+\alpha}\right)^2. \tag{67.42}$$

For the isotropic case, only the first two terms on the right hand side of (67.42) are nonzero. r_m is the value of r where those two terms are equal. For a closed universe for the isotropic case, r_m is the maximum value of r.

Equation (67.42) can be written

$$\dot{r} = \sqrt{\left(\frac{r}{r_m}\right)^{2-3\gamma} - k(1+\alpha) - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{V^2}{3r^4}}, \tag{67.43}$$

and the remaining differential equations to solve are

$$\dot{V} = 3k(1+\alpha)r - \frac{k}{(2a_0)^6}\frac{2b^2}{(1+\alpha)^2r^3} \tag{67.44}$$

and

$$\frac{\dot{\alpha}}{1+\alpha} = \frac{2V}{r^3}, \tag{67.45}$$

where (67.45) is the definition of $V(t)$, and $r(t)$ represents the size of the universe. The angular rotation (67.34) of an inertial frame relative to local matter is given by

$$\Omega(t) = \left(\frac{-k}{4a_0^2}\right)^{3/2} \frac{b}{[1+\alpha(t)]r^3(t)} \tag{67.46}$$

It is not possible to solve the differential equations exactly in closed form, but we can find approximate solutions. In the limit as b approaches zero, it is valid to neglect all but the first term under the radical in (67.43). The appendix then gives

$$\begin{aligned}
1+\alpha \;&\approx\; (1+\alpha_0) exp\left\{ \frac{2r_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3} - r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma} + \right.\\
&\quad 12k\frac{(3\gamma-6)r^{3\gamma-2} - (6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3} + (3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\\
&\quad \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6} - 2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3} + r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}} \right\}\\
&\approx\; (1+\alpha_0)\left\{1 + \frac{2r_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3} - r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma} + \right.\\
&\quad 12k\frac{(3\gamma-6)r^{3\gamma-2} - (6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3} + (3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\\
&\quad \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6} - 2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3} + r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}} \right\}\\
&\approx\; (1+\alpha_0)\left\{1 + \right.\\
&\quad 12k\frac{r^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)r_m^{3\gamma-2}}\\
&\quad \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}} \right\}.
\end{aligned} \tag{67.47}$$

This solution is valid for r smaller than r_m if b is small enough.

We assume that γ changes from γ_1 to γ_2 at $r = r_1$. To satisfy continuity of ρ at $r = r_1$, we must have r_m change from r_{m1} to r_{m2} at $r = r_1$, where

$$\frac{r_{m1}}{r_{m2}} = \left(\frac{r_1}{r_{m2}}\right)^{\frac{3\gamma_1-3\gamma_2}{3\gamma_1-2}}. \tag{67.48}$$

Equation (67.47) applies for $r \le r_1$ with $\gamma = \gamma_1$ and $r_m = r_{m1}$.

For $r > r_1$, (67.47) gives

$$\begin{aligned}
1+\alpha \;&\approx\; (1+\alpha_1) exp\left\{ \frac{2r_1^3}{3\gamma_2-6}\frac{\dot{\alpha}_1}{1+\alpha_1}(r^{\frac{3}{2}\gamma_2-3} - r_1^{\frac{3}{2}\gamma_2-3})r_{m2}^{1-\frac{3}{2}\gamma_2} + \right.\\
&\quad 12k\frac{(3\gamma_2-6)r^{3\gamma_2-2} - (6\gamma_2-4)r_1^{\frac{3}{2}\gamma_2+1}r^{\frac{3}{2}\gamma_2-3} + (3\gamma_2+2)r_1^{3\gamma_2-2}}{(3\gamma_2+2)(3\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}}
\end{aligned}$$

$$
\begin{aligned}
& \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma_2-6}-2r_1^{\frac{3}{2}\gamma_2-3}r^{\frac{3}{2}\gamma_2-3}+r_1^{3\gamma_2-6}}{(\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}}\right\} \\
\approx\ & (1+\alpha_1)\left\{1+\frac{2r_1^3}{3\gamma_2-6}\frac{\dot{\alpha}_1}{1+\alpha_1}(r^{\frac{3}{2}\gamma_2-3}-r_1^{\frac{3}{2}\gamma_2-3})r_{m2}^{1-\frac{3}{2}\gamma_2}+\right. \\
& 12k\frac{(3\gamma_2-6)r^{3\gamma_2-2}-(6\gamma_2-4)r_1^{\frac{3}{2}\gamma_2+1}r^{\frac{3}{2}\gamma_2-3}+(3\gamma_2+2)r_1^{3\gamma_2-2}}{(3\gamma_2+2)(3\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}} \\
& \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma_2-6}-2r_1^{\frac{3}{2}\gamma_2-3}r^{\frac{3}{2}\gamma_2-3}+r_1^{3\gamma_2-6}}{(\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}}\right\} \\
\approx\ & (1+\alpha_1),
\end{aligned}
\tag{67.49}
$$

where continuity of α and $\dot{\alpha}$ at $r=r_1$ requires that

$$
\begin{aligned}
1+\alpha_1 \approx\ & (1+\alpha_0)exp\left\{\frac{2r_0^3}{3\gamma_1-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r_1^{\frac{3}{2}\gamma_1-3}-r_0^{\frac{3}{2}\gamma_1-3})r_{m1}^{1-\frac{3}{2}\gamma_1}+\right. \\
& 12k\frac{(3\gamma_1-6)r_1^{3\gamma_1-2}-(6\gamma_1-4)r_0^{\frac{3}{2}\gamma_1+1}r_1^{\frac{3}{2}\gamma_1-3}+(3\gamma_1+2)r_0^{3\gamma_1-2}}{(3\gamma_1+2)(3\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}} \\
& \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_1^{3\gamma_1-6}-2r_0^{\frac{3}{2}\gamma_1-3}r_1^{\frac{3}{2}\gamma_1-3}+r_0^{3\gamma_1-6}}{(\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}}\right\} \\
\approx\ & (1+\alpha_0)\left\{1+\frac{2r_0^3}{3\gamma_1-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r_1^{\frac{3}{2}\gamma_1-3}-r_0^{\frac{3}{2}\gamma_1-3})r_{m1}^{1-\frac{3}{2}\gamma_1}+\right. \\
& 12k\frac{(3\gamma_1-6)r_1^{3\gamma_1-2}-(6\gamma_1-4)r_0^{\frac{3}{2}\gamma_1+1}r_1^{\frac{3}{2}\gamma_1-3}+(3\gamma_1+2)r_0^{3\gamma_1-2}}{(3\gamma_1+2)(3\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}} \\
& \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_1^{3\gamma_1-6}-2r_0^{\frac{3}{2}\gamma_1-3}r_1^{\frac{3}{2}\gamma_1-3}+r_0^{3\gamma_1-6}}{(\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}}\right\} \\
\approx\ & (1+\alpha_0)\{1+ \\
& 12k\frac{r_1^{3\gamma_1-2}}{(3\gamma_1+2)(3\gamma_1-2)r_{m1}^{3\gamma_1-2}} \\
& \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}}{(\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}}\right\}.
\end{aligned}
\tag{67.50}
$$

We then perform the integration in (67.40). The result is

$$
\begin{aligned}
\frac{I_{cl}}{\hbar} =\ & \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\left\{\frac{2}{3\gamma_1+2}\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]\left(\frac{r_{m1}}{r_{m2}}\right)^2\right. \\
& +\frac{24k\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}\right]}{(3\gamma_1+2)(3\gamma_1-2)(9\gamma_1-2)}\left(\frac{r_{m1}}{r_{m2}}\right)^2 \\
& -\frac{8k}{3}\frac{b^2r_{m1}^{-4}}{(2a_0)^6}\frac{\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]}{(\gamma_1-2)^2 3(3\gamma_1+2)}2\left(\frac{r_0}{r_{m1}}\right)^{3\gamma_1-6}\left(\frac{r_{m1}}{r_{m2}}\right)^2 \\
& +\left[1+12k\frac{r_1^{3\gamma_1-2}r_{m1}^{2-3\gamma_1}}{(3\gamma_1+2)(3\gamma_1-2)}-\frac{8k}{3}\frac{b^2r_{m1}^{-4}}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}r_{m1}^{6-3\gamma_1}}{(\gamma_1-2)(3\gamma_1-6)}\right]
\end{aligned}
$$

$$\frac{2}{3\gamma_2+2}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}-\left(\frac{r_1}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}\right]\Bigg\}. \tag{67.51}$$

Equation (67.51) neglects all but the first term under the radical in (67.43). In making the calculation, I actually included the other terms under the radical to first order, but then determined after the integration that they could be neglected.

Neglecting some small terms gives

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\frac{2}{3\gamma_2+2}\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}\\ &\quad\left[1-\frac{8k}{3}\frac{b^2r_{m1}^{-4}}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}r_{m1}^{6-3\gamma_1}}{(\gamma_1-2)(3\gamma_1-6)}\right].\end{aligned} \tag{67.52}$$

Substituting (67.48) into (67.52) gives

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\frac{2}{3\gamma_2+2}\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}\\ &\quad\left[1-\frac{8k}{3}\left(\frac{r_1}{r_{m2}}\right)^{3\gamma_2-3\gamma_1}\frac{b^2r_{m2}^{-4}}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}r_{m2}^{6-3\gamma_1}}{(\gamma_1-2)(3\gamma_1-6)}\right].\end{aligned} \tag{67.53}$$

To get a rough estimate, we take

$$\gamma_1 = 4/3 \tag{67.54}$$

to represent a relativistic early universe and

$$\gamma_2 = 1 \tag{67.55}$$

to represent a matter-dominated late universe. Substituting (67.54) and (67.55) into (67.51) gives

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\left\{\frac{1}{3}\left[\left(\frac{r_1}{r}\right)^3\left(\frac{r}{r_{m1}}\right)^3-\left(\frac{r_0}{r_{m1}}\right)^3\right]\left(\frac{r_{m1}}{r_{m2}}\right)^2\right.\\ &\quad+\frac{2}{5}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}-\left(\frac{r_1}{r}\right)^{5/2}\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}\right]\left[1+\left(\frac{r_1}{r}\right)^2k\left(\frac{r}{r_{m1}}\right)^2\right]\\ &\quad+\frac{k}{5}\left[\left(\frac{r_1}{r}\right)^5\left(\frac{r}{r_{m1}}\right)^5-\left(\frac{r_0}{r_{m1}}\right)^5\right]\left(\frac{r_{m1}}{r_{m2}}\right)^2\\ &\quad-\frac{2kb^2}{(2a_0)^6r_{m1}^2r_0^2}\\ &\quad\left[\frac{1}{3}\left(\frac{r_{m1}}{r_{m2}}\right)^2\left(\left(\frac{r_1}{r}\right)^3\left(\frac{r}{r_{m1}}\right)^3-\left(\frac{r_0}{r_{m1}}\right)^3\right)\right.\\ &\quad\left.\left.+\frac{2}{5}\left(\frac{r}{r_{m2}}\right)^{5/2}\left(1-\left(\frac{r_1}{r}\right)^{5/2}\right)\right]\right\}.\end{aligned} \tag{67.56}$$

Neglecting some small terms, letting $k=+1$ for a closed universe, and using (67.48) gives

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3i\pi^2}{2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\left\{\frac{1}{3}\left(\frac{r_1}{r}\right)^3\left(\frac{r}{r_{m2}}\right)^{5/2}\left(\frac{r}{r_1}\right)^{1/2}\right.\\ &\quad+\frac{2}{5}\left[1+\left(\frac{r_1}{r}\right)^2\left(\frac{r}{r_{m2}}\right)^2\frac{r_{m2}}{r_1}\right]\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}+\frac{1}{5}\left(\frac{r_1}{r}\right)^5\left(\frac{r}{r_{m2}}\right)^{7/2}\left(\frac{r}{r_1}\right)^{3/2}\end{aligned}$$

$$
\begin{aligned}
& \left. - \frac{2}{5} \frac{2b^2}{(2a_0)^6 r_{m2}^2 r_0^2} \left(\frac{r}{r_1}\right) \left(\frac{r}{r_{m2}}\right)^{3/2} \right\} \\
\approx \; & \frac{3i\pi^2}{2} \left(\frac{r_{m2}}{L^*}\right)^2 (1+\alpha_0) \\
& \left\{ \frac{2}{5} \left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}} - \frac{2}{5} \frac{2b^2}{(2a_0)^6 r_{m2}^2 r_0^2} \left(\frac{r}{r_1}\right) \left(\frac{r}{r_{m2}}\right)^{3/2} \right\}. \qquad (67.57)
\end{aligned}
$$

Because the parameter b is an initial value for the cosmology, it is one of the variables of integration in (67.18). In making the saddlepoint approximation for that integration, we need to locate the saddlepoint (that is, the value of b that makes the action in (67.52), (67.53), or (67.57) stationary. We see that the action is stationary with respect to variation of b at the isotropic case of $b = 0$, as expected. The range of values of b that contribute significantly to the integral in (67.18) is given by (67.21). That is

$$\left| \frac{I_{cl}(b)}{\hbar} - \frac{I_{cl}(b=0)}{\hbar} \right| < 1. \qquad (67.58)$$

Thus, substituting (67.53) into (67.58) gives

$$
\begin{aligned}
& \frac{3\pi^2}{2} \left(\frac{r_{m2}}{L^*}\right)^2 \frac{(1+\alpha_0)b^2}{(2a_0)^6 r_{m2}^4} \left(\frac{r}{r_{m2}}\right)^{\frac{9}{2}\gamma_2 - 3\gamma_1 + 1} \\
& \frac{16}{(\gamma_1-2)^2 9(3\gamma_2+2)} \left(\frac{r}{r_1}\right)^{3(\gamma_1-\gamma_2)} \left(\frac{r_{m2}}{r_0}\right)^{6-3\gamma_1} < 1. \qquad (67.59)
\end{aligned}
$$

The approximations made so far are valid whenever (67.59) holds.

Substituting (67.54) and (67.55) into (67.59) gives

$$\frac{6\pi^2}{5}(1+\alpha_0) \left(\frac{r}{r_1}\right) \left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}} \left(\frac{L^*}{r_0}\right)^2 \frac{b^2}{(2a_0)^6 L^{*4}} < 1. \qquad (67.60)$$

This gives

$$b < \frac{\sqrt{5}(2a_0)^3}{\pi\sqrt{6}} \left(\frac{r_1}{r}\right)^{1/2} \frac{L^{*2}}{(1+\alpha_0)^{1/2}} \left(\frac{r_{m2}}{r}\right)^{\frac{3}{4}} \left(\frac{r_0}{L^*}\right). \qquad (67.61)$$

Thus, from (67.46), the rotation rate of inertial frames is

$$|\Omega(t)| < \frac{\sqrt{5}}{\pi\sqrt{6}} \left(\frac{L^*}{r_{m2}}\right)^2 \frac{1}{(1+\alpha_0)^{1/2}[1+\alpha(t)]r_{m2}} \left(\frac{r_{m2}}{r(t)}\right)^{\frac{15}{4}} \left(\frac{r_1}{r}\right)^{1/2} \left(\frac{r_0}{L^*}\right). \qquad (67.62)$$

If we now take the Planck length L^* to be $1.6 \times 10^{-}33$ cm, use the Hubble distance of 1.7×10^{28} cm for r_{m2}, a tenth of that for r, neglect α and α_0 compared to 1, and take

$$r_1 = \frac{r}{100} \qquad (67.63)$$

as an estimate that the universe changed from radiation-dominated to matter-dominated when the universe was about one-hundredth of its present size [13, Section 15.3, p. 481], then we get

$$|\Omega| < 1.6 \times 10^{-130} \text{ radians per year,} \qquad (67.64)$$

which is much less than the bound set by experiment of 10^{-14} to 7×10^{-17} radians per year if the universe is spatially closed [164].

The rotation rate in (67.62) and (67.64) is so small because the Planck length is so much smaller than the Hubble distance.

That the small value of allowed rotation rate depends mostly on the universe being much larger than a Planck length rather than on details of the model suggests that the result has some generality.

We notice also, that the selection criterion in (67.58) is so sharp that the initial wave function in the integration in (67.18) would have to be very sharply peaked to overcome it.

67.8 Discussion

We see that considerations of quantum cosmology show how a range of classical cosmologies can be selected that contribute significantly to the wave function in the final state. The effect enters through the action. Using semiclassical calculations gives results that should not depend on particular details of the theory of quantum gravity.

For our universe (which is much larger than the Planck length) the selection is very sharp. The initial wave function over 3-geometries would have to be extremely sharp (not a probable occurrence) to dominate over the effect of the action.

The selection process seems to occur very soon in the development of a cosmology. That is, for a broad wave function over 3-geometries in the initial state, the wave function becomes sharply peaked after the universe has become a few orders of magnitude larger than the Planck length.

A different choice than (67.30) [162] is

$$L_{matter} = p. \tag{67.65}$$

The choice in (67.65) gives a third of (67.31) for the total action. A correct theory of quantum gravity will determine which (if either) of these two choices is correct, but for this illustration, a factor of three in the action makes little difference.

It appears likely now, however, that there is not enough matter to keep our universe from expanding forever (e.g. [189]). To accommodate that with a spatially closed universe within General Relativity would require a positive cosmological constant (e.g. [190]). I shall try to include the cosmological constant in a future calculation.

67.9 Acknowledgment

I would like to thank Douglas Gough for first bringing the paper by [11] to my attention in 1967.

67.10 Appendix: Bianchi VI_h Models

We start with Equations (6.7) of Ellis and MacCallum (1969). For the case of zero cosmological constant, Λ, these can be written as

$$\frac{\dot{\rho}}{\rho + p} = -\frac{3\dot{R}}{R} \tag{67.66}$$

$$4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot{X}}{X}\right)^{\cdot} - \frac{2(a_0^2 + q_0^2)}{X^2} + \frac{2b^2}{Y^4Z^2} \tag{67.67}$$

$$4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot{Y}}{Y}\right)^{\cdot} - \frac{2(a_0^2 + a_0q_0)}{X^2} - \frac{2b^2}{Y^4Z^2} \tag{67.68}$$

$$4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot{Z}}{Z}\right)^{\cdot} - \frac{2(a_0^2 - a_0q_0)}{X^2} \tag{67.69}$$

$$8\pi\rho + \frac{3a_0^2 + q_0^2}{X^2} = \frac{1}{2}\left[9\left(\frac{\dot{R}}{R}\right)^2 - \left(\frac{\dot{X}}{X}\right)^2 - \left(\frac{\dot{Y}}{Y}\right)^2 - \left(\frac{\dot{Z}}{Z}\right)^2\right] + \frac{2b^2}{Y^4Z^2}, \tag{67.70}$$

where

$$R(t)^3 = X(t)Y(t)Z(t), \tag{67.71}$$

b, a_0, and q_0 are constants that are parameters of the model, and the variation of ρ, X, Y, and Z with time is determined by (67.66) through (67.70).

According to Ellis and MacCallum (1969), (67.70) is a first integral of the other equations. If I understand that correctly, then taking the derivative of (67.70) should be a combination of the other equations. When I try that, the result is close, but not quite correct. I have not been able to find a similar equation that does work, although I have found one that works for the special case ($q_0 + 3a_0 = 0$ and $XY = Z^2$) that I use later. This is[2]

$$8\pi\rho + \frac{3a_0^2 + q_0^2}{X^2} = \frac{1}{2}\left[9\left(\frac{\dot{R}}{R}\right)^2 - \left(\frac{\dot{X}}{X}\right)^2 - \left(\frac{\dot{Y}}{Y}\right)^2 - \left(\frac{\dot{Z}}{Z}\right)^2\right] - \frac{b^2}{Y^4Z^2}. \tag{67.72}$$

We can add (67.67), (67.68), and (67.69) with coefficients A, B, and C to give

$$(A+B+C)\left[4\pi(\rho - p) + \frac{2a_0^2}{X^2}\right] = R^{-3}\left(R^3\frac{\dot{U}}{U}\right)^{\cdot} - (B-C)\frac{2a_0q_0}{X^2} - 2A\frac{q_0^2}{X^2} + (A-B)\frac{2b^2}{Y^4Z^2}, \tag{67.73}$$

where

$$U = X^AY^BZ^C. \tag{67.74}$$

We can take special cases of A, B, and C. Taking $A = B = C = 1/3$ in (67.73) gives

$$4\pi(\rho - p) = R^{-3}(R^2\dot{R})^{\cdot} - \frac{2}{3}\frac{3a_0^2 + q_0^2}{X^2}, \tag{67.75}$$

where I have used (67.71). For another special case, we take $A = B = 1$ and $C = -2$ in (67.73) to give

$$R^{-3}\left(R^3\frac{\dot{U}}{U}\right)^{\cdot} = -2q_0\frac{3a_0 + q_0}{X^2}, \tag{67.76}$$

where

$$U = \frac{XY}{Z^2}. \tag{67.77}$$

For a third special case, we take $A = -4/3$ and $B = C = 2/3$ in (67.73) to give

$$R^{-3}\left(R^3\frac{\dot{\alpha}}{1+\alpha}\right)^{\cdot} = \frac{4b^2}{Y^4Z^2} - \frac{8q_0^2}{3X^2}, \tag{67.78}$$

where

$$1 + \alpha = \frac{Y^{2/3}Z^{2/3}}{X^{4/3}}. \tag{67.79}$$

We can use (67.71), (67.77), and (67.79) to determine X, Y, and Z in terms of R, U, and α. This gives

$$X = \frac{R}{(1+\alpha)^{1/2}}, \tag{67.80}$$

$$Y = U^{1/3}(1+\alpha)^{1/2}R, \tag{67.81}$$

and

$$Z = \frac{R}{U^{1/3}}. \tag{67.82}$$

[2]Note added 16 May 2012: MacCallum[191, Note 3, page 60] points out that the last term in (67.70) should be multiplied by -1/2. This would make it the same as the last term in (67.72).

Using (67.80), (67.81), and (67.82) in (67.75), (67.76), (67.78), and (67.72) gives

$$R^{-3}\left(R^2\dot{R}\right)^{\cdot} = 4\pi(\rho - p) + \frac{2(3a_0^2+q_0^2)(1+\alpha)}{3R^2}, \tag{67.83}$$

$$R^{-3}\left(R^3\frac{\dot{U}}{U}\right)^{\cdot} = -2q_0\frac{(3a_0+q_0)(1+\alpha)}{R^2}, \tag{67.84}$$

$$R^{-3}\left(R^3\frac{\dot{\alpha}}{1+\alpha}\right)^{\cdot} = \frac{4b^2}{(1+\alpha)^2U^{2/3}R^6} - \frac{8q_0^2(1+\alpha)}{3R^2}, \tag{67.85}$$

and

$$\begin{aligned}\left(\frac{\dot{R}}{R}\right)^2 &= \frac{1}{3}\left[\left(\frac{1}{2}\frac{\dot{\alpha}}{1+\alpha}\right)^2 + \left(\frac{1}{2}\frac{\dot{\alpha}}{1+\alpha}\right)\left(\frac{1}{3}\frac{\dot{U}}{U}\right) + \left(\frac{1}{3}\frac{\dot{U}}{U}\right)^2\right] \\ &\quad + \frac{8\pi\rho}{3} + \frac{(3a_0^2+q_0^2)(1+\alpha)}{3R^2} + \frac{b^2}{3(1+\alpha)^2U^{2/3}R^6}.\end{aligned} \tag{67.86}$$

If we change variables from R to r, where

$$r = \sqrt{\frac{-3k}{3a_0^2+q_0^2}}R, \tag{67.87}$$

(and where k is +1 for a closed universe and -1 for an open universe), then (67.83) through (67.86) become

$$r^{-3}(r^2\dot{r})^{\cdot} = 4\pi(\rho - p) - \frac{2k(1+\alpha)}{r^2}, \tag{67.88}$$

$$r^{-3}\left(r^3\frac{\dot{U}}{U}\right)^{\cdot} = \frac{6kq_0(3a_0+q_0)(1+\alpha)}{(3a_0^2+q_0^2)r^2}, \tag{67.89}$$

$$r^{-3}\left(r^3\frac{\dot{\alpha}}{1+\alpha}\right)^{\cdot} = \left(\frac{-3k}{3a_0^2+q_0^2}\right)^3\frac{4b^2}{(1+\alpha)^2U^{2/3}r^6} + \frac{8kq_0^2(1+\alpha)}{(3a_0^2+q_0^2)r^2}, \tag{67.90}$$

and

$$\begin{aligned}\left(\frac{\dot{r}}{r}\right)^2 &= \frac{1}{3}\left[\left(\frac{1}{2}\frac{\dot{\alpha}}{1+\alpha}\right)^2 + \left(\frac{1}{2}\frac{\dot{\alpha}}{1+\alpha}\right)\left(\frac{1}{3}\frac{\dot{U}}{U}\right) + \left(\frac{1}{3}\frac{\dot{U}}{U}\right)^2\right] \\ &\quad + \frac{8\pi\rho}{3} - \frac{k(1+\alpha)}{r^2} + \left(\frac{-3k}{3a_0^2+q_0^2}\right)^3\frac{b^2}{3(1+\alpha)^2U^{2/3}r^6}.\end{aligned} \tag{67.91}$$

If we define W, V, and K by

$$r^2\dot{r} = W, \tag{67.92}$$

$$\frac{\dot{\alpha}}{1+\alpha} = \frac{2V}{r^3} \tag{67.93}$$

and

$$\frac{\dot{U}}{U} = \frac{3K}{r^3}, \tag{67.94}$$

then (67.88) through (67.91) become

$$\dot{W} = 4\pi(\rho - p)r^3 - 2k(1+\alpha)r, \tag{67.95}$$

$$\dot{K} = \frac{2kq_0(3a_0 + q_0)(1+\alpha)r}{3a_0^2 + q_0^2}, \tag{67.96}$$

$$\dot{V} = \left(\frac{-3k}{3a_0^2 + q_0^2}\right)^3 \frac{2b^2}{(1+\alpha)^2 U^{2/3} r^3} + \frac{4kq_0^2(1+\alpha)r}{3a_0^2 + q_0^2}, \tag{67.97}$$

$$\begin{aligned} W^2 &= \frac{V^2 + VK + K^2}{3} + \frac{8\pi\rho r^6}{3} \\ &\quad - k(1+\alpha)r^4 + \left(\frac{-3k}{3a_0^2 + q_0^2}\right)^3 \frac{b^2}{3(1+\alpha)^2 U^{2/3}}. \end{aligned} \tag{67.98}$$

If we take

$$p = (\gamma - 1)\rho \tag{67.99}$$

for the equation of state, where

$$1 \le \gamma < 2 \tag{67.100}$$

is a constant, then (67.66) can be integrated to give

$$8\pi\rho = 3r_m^{3\gamma-2} r^{-3\gamma} \tag{67.101}$$

where r_m is a constant of integration. Substituting (67.99) and (67.101) into (67.95) and (67.98) gives

$$\dot{W} = 3(1 - \gamma/2) r_m^{3\gamma-2} r^{3-3\gamma} - 2k(1+\alpha)r, \tag{67.102}$$

$$\begin{aligned} W^2 &= \frac{V^2 + VK + K^2}{3} + r_m^{3\gamma-2} r^{6-3\gamma} \\ &\quad - k(1+\alpha)r^4 + \left(\frac{-3k}{3a_0^2 + q_0^2}\right)^3 \frac{b^2}{3(1+\alpha)^2 U^{2/3}}. \end{aligned} \tag{67.103}$$

Let us now convert to a set of dimensionless variables defined by

$$\tau = t/r_m \tag{67.104}$$

$$x = r/r_m \tag{67.105}$$

$$y = V/r_m^2 \tag{67.106}$$

$$z = W/r_m^2 \tag{67.107}$$

$$C = K/r_m^2 \tag{67.108}$$

$$D = b/r_m^2 \tag{67.109}$$

Let us also use $'$ for $d/d\tau$. Then (67.92) through (67.97), (67.102), and (67.103) become

$$x^2 x' = z, \tag{67.110}$$

$$\frac{\alpha'}{1+\alpha} = \frac{2y}{x^3} \tag{67.111}$$

$$\frac{U'}{U} = \frac{3C}{x^3}, \tag{67.112}$$

$$C' = \frac{2kq_0(3a_0 + q_0)(1+\alpha)x}{3a_0^2 + q_0^2}, \tag{67.113}$$

$$y' = \left(\frac{-3k}{3a_0^2+q_0^2}\right)^3 \frac{2D^2}{(1+\alpha)^2U^{2/3}x^3} + \frac{4kq_0^2(1+\alpha)x}{3a_0^2+q_0^2}, \tag{67.114}$$

$$z' = 3(1-\gamma/2)x^{3-3\gamma} - 2k(1+\alpha)x, \tag{67.115}$$

$$z^2 = x^{6-3\gamma} - k(1+\alpha)x^4 + \left(\frac{-3k}{3a_0^2+q_0^2}\right)^3 \frac{D^2}{3(1+\alpha)^2U^{2/3}} + \frac{y^2+yC+C^2}{3}. \tag{67.116}$$

For the $h=-1/9$ case, we have

$$q_0 = -3a_0. \tag{67.117}$$

if and only if $b \neq 0$. However, the $b=0$ case is only a single point. When integrating over b, the behavior for small b is more important than exactly at $b=0$. Therefore, we shall use (67.117) in any case. Therefore, from (67.113), we have that C is a constant.

So both b and C are constants that are determined by initial conditions. There are two possibilities. Either they are connected by a relation, or they are independent. There seems to be no obvious connection between them from the equations, so we shall assume they are independent unless we shall find it necessary to make a connection to get a consistent solution to the equations. Therefore, we can assume that both b and C must be integrated over on the initial hypersurface. For this demonstration, however, it is sufficient to integrate over only one initial constant, b. Therefore, we shall fix C at the value we would guess for the FRW case, which is zero. Therefore, we take C to be zero.

That means from (67.112) that U is constant. Again, we guess that U is independent from b, and choose the isotropic value. In the isotropic case, we would have $X=Y=Z$, and therefore, from (67.77) we have $U=1$ as the isotropic value.

Therefore, from (67.114) we have

$$y' = 3k(1+\alpha)x - \frac{k}{(2a_0)^6}\frac{2D^2}{(1+\alpha)^2x^3}, \tag{67.118}$$

and from (67.116) we have

$$z^2 = x^{6-3\gamma} - k(1+\alpha)x^4 - \frac{k}{(2a_0)^6}\frac{D^2}{3(1+\alpha)^2} + \frac{y^2}{3}. \tag{67.119}$$

We notice at this point that if we take the derivative of (67.119) and substitute from (67.110), (67.111), (67.118), and (67.115), that we get an identity, confirming the consistency of the equations at this point.

Combining (67.110) with (67.119) gives

$$x' = \sqrt{x^{2-3\gamma} - k(1+\alpha) - \frac{k}{(2a_0)^6}\frac{D^2}{3(1+\alpha)^2x^4} + \frac{y^2}{3x^4}}. \tag{67.120}$$

Combining (67.118) with (67.120) gives

$$\frac{dy}{dx} = \frac{3k(1+\alpha)x - \frac{k}{(2a_0)^6}\frac{2D^2}{(1+\alpha)^2x^3}}{\sqrt{x^{2-3\gamma} - k(1+\alpha) - \frac{k}{(2a_0)^6}\frac{D^2}{3(1+\alpha)^2x^4} + \frac{y^2}{3x^4}}}. \tag{67.121}$$

Both b and a_0 are constants that are determined by initial conditions on the initial hypersurface. They are either related or independent. We assume first that they are independent. In that case, the third term under the radical in (67.120) and (67.121) will get smaller as D gets smaller. We shall assume that we can neglect that term relative to the first term under the radical for all values of x. We can check that assumption later. There is also the possibility of iterating later by assuming that this term is small instead of zero.

We shall also neglect the last term under the radical in (67.120) and (67.121). We can check that approximation later. There is also the possibility of iterating by substituting an approximate solution for y into (67.121). We shall see that it it permissible to neglect the fourth term for all values of x within the integration range if b is small enough.

Neglecting the third and fourth terms under the radical in (67.120) and (67.121) gives

$$x' = \sqrt{x^{2-3\gamma} - k(1+\alpha)}. \tag{67.122}$$

and

$$\frac{dy}{dx} = \frac{3k(1+\alpha)x - \frac{k}{(2a_0)^6}\frac{2D^2}{(1+\alpha)^2 x^3}}{\sqrt{x^{2-3\gamma} - k(1+\alpha)}}. \tag{67.123}$$

We shall assume that

$$\alpha \ll 1. \tag{67.124}$$

We can test that approximation from the solution later and iterate if necessary. In that case, (67.120) and (67.121) become

$$x' = \sqrt{x^{2-3\gamma} - k}. \tag{67.125}$$

and

$$\frac{dy}{dx} = \frac{3kx - \frac{k}{(2a_0)^6}\frac{2D^2}{x^3}}{\sqrt{x^{2-3\gamma} - k}}. \tag{67.126}$$

We notice that (67.126) could be integrated numerically to obtain an approximate solution for $y(x)$. For a closed universe (which is the case we are considering), we can get an estimate of that solution, at least for small x. The second term under the radical in (67.125) and (67.126) is smaller than the first term under the radical except at the point of maximum expansion, when they are equal (when x is one). Thus, before the universe gets too close to the point of maximum expansion, we can neglect the second term under the radical in (67.125) and (67.126) to give

$$x' = x^{1-\frac{3}{2}\gamma}. \tag{67.127}$$

and

$$\frac{dy}{dx} = 3kx^{\frac{3}{2}\gamma} - 2kE^2 x^{\frac{3}{2}\gamma - 4}, \tag{67.128}$$

where

$$E \equiv D/(2a_0)^3 = \frac{b/r_{m1}^2}{(2a_0)^3}. \tag{67.129}$$

We can integrate (67.128) to give

$$y = y_0 + \frac{6k}{3\gamma+2}(x^{\frac{3}{2}\gamma+1} - x_0^{\frac{3}{2}\gamma+1}) - \frac{4k}{3}\frac{E^2}{\gamma-2}(x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3}), \tag{67.130}$$

where, without loss of generality, the constant of integration has been chosen such that $y = y_0$ when $x = x_0$.

We notice that on substituting (67.130) into (67.120) and (67.121) the terms it gives that do not involve b have a higher power of x than the dominant term. Therefore, for small b and small x, it was justified to neglect the fourth term under the radical in (67.120) and (67.121), since the terms that involve b are proportional to a positive power of b.

Using (67.130) in (67.111) gives

$$\frac{\alpha'}{1+\alpha} = 2y_0 x^{-3} + \frac{12k}{3\gamma+2}(x^{\frac{3}{2}\gamma-2} - x_0^{\frac{3}{2}\gamma+1}x^{-3}) - \frac{8k}{3}\frac{E^2}{\gamma-2}(x^{\frac{3}{2}\gamma-6} - x_0^{\frac{3}{2}\gamma-3}x^{-3}). \tag{67.131}$$

Combining (67.131) with (67.127) gives

$$\frac{d\ln(1+\alpha)}{dx} = 2y_0 x^{\frac{3}{2}\gamma-4} + \frac{12k}{3\gamma+2}(x^{3\gamma-3} - x_0^{\frac{3}{2}\gamma+1}x^{\frac{3}{2}\gamma-4}) - \frac{8k}{3}\frac{E^2}{\gamma-2}(x^{3\gamma-7} - x_0^{\frac{3}{2}\gamma-3}x^{\frac{3}{2}\gamma-4}). \tag{67.132}$$

We can integrate (67.132) to give

$$\begin{aligned}\ln\frac{1+\alpha}{1+\alpha_0} &= \frac{4y_0}{3\gamma-6}(x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3}) + \\ &\quad \frac{12k}{3\gamma+2}\frac{(3\gamma-6)x^{3\gamma-2} - (6\gamma-4)x_0^{\frac{3}{2}\gamma+1}x^{\frac{3}{2}\gamma-3} + (3\gamma+2)x_0^{3\gamma-2}}{(3\gamma-2)(3\gamma-6)} \\ &\quad -\frac{8k}{3}\frac{E^2}{\gamma-2}\frac{x^{3\gamma-6} - 2x_0^{\frac{3}{2}\gamma-3}x^{\frac{3}{2}\gamma-3} + x_0^{3\gamma-6}}{3\gamma-6},\end{aligned} \tag{67.133}$$

where, without loss of generality, the constant of integration has been chosen such that $\alpha = \alpha_0$ when $x = x_0$.

Writing (67.130) and (67.133) in original variables using (67.105), (67.106), (67.129), and (67.93) gives

$$\begin{aligned}V &\equiv \frac{r^3}{2}\frac{\dot\alpha}{1+\alpha} = \frac{r_0^3}{2}\frac{\dot\alpha_0}{1+\alpha_0} + \\ &\quad [6k\frac{r^{\frac{3}{2}\gamma+1} - r_0^{\frac{3}{2}\gamma+1}}{3\gamma+2} - \frac{4k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{\frac{3}{2}\gamma-3} - r_0^{\frac{3}{2}\gamma-3}}{\gamma-2}]r_m^{1-\frac{3}{2}\gamma},\end{aligned} \tag{67.134}$$

$$\begin{aligned}1+\alpha &\approx (1+\alpha_0)exp\left\{\frac{2r_0^3}{3\gamma-6}\frac{\dot\alpha_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3} - r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma} + \right. \\ &\quad 12k\frac{(3\gamma-6)r^{3\gamma-2} - (6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3} + (3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}} \\ &\quad \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6} - 2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3} + r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\right\},\end{aligned} \tag{67.135}$$

Chapter 68

Derivation of Special Relativity[1]

68.1 Introduction

We now realize that since Special Relativity already contains inertia that it is necessary to derive Special Relativity from some sort of interaction with the matter in the universe. This means also deriving all of the light-cone properties and Maxwell's equations.

The simplest generalization I can think of for Maxwell's equations is to allow ϵ_0 and μ_0 to depend somehow on the matter distribution. This may mean that the speed of light in vacuum may also depend on the matter distribution.

One possible mechanism I can think of is the one that slows photons getting out of stars. The photons are generated by nuclear reactions in the center of stars, but it takes about ten million years for a photon to get to the surface because the photons keep bumping into things.

So let us imagine that in the absence of a lot of matter, a photon has infinite speed. (This corresponds to taking the bare action for the photon to be zero.) Let us assume further that each time a photon hits something it takes a specific amount of time, say T_0. Then if a photon has N interactions to go a distance L, it would take N times T_0 seconds with an average speed of $L/(NT_0)$.

Let us assume that each time the photon has a collision it moves off in a random direction with equal probability. A random-walk calculation will give a distance of lambda times square root of N for the root-mean-square distance it will have gone after N collisions if λ is the mean free path between collisions. The effective speed will be $\lambda\sqrt{N}/(NT_0)$, or $\lambda/(T_0\sqrt{N})$.

[1] written before April 1999

Chapter 69

Symmetries in relative coordinates[1]

Symmetry means different things for a relative position theory. Usually, symmetries have to do with coordinate systems and the position, motion, or movement of bodies in that coordinate system. In the case of a relative position or relative distance theory, the symmetries may depend on the relative positions.

Originally, Donald Lynden-Bell defined relative positions between two bodies as the difference of the positions of the two bodies in some coordinate system. However, once we have the Lagrangian or whatever defined in terms of those relative positions, then we can forget the original definitions, and consider the relative positions as fundamental in themselves. That way, we can work with those Lagrangians as fundamental.

We may want to generalize the Lagrangians, however to be more general functions of the relative positions and velocities, rather than just the $1/r$ etc. functions that Lynden-Bell had.

So, how do we define symmetries in terms of relative positions? In some sense, we already have some symmetries simply because we have relative coordinates. However, some symmetries will depend on the distribution of matter. Let us assume a distribution of matter that is on the average spherically symmetric. That should give us an O(3) symmetry. The Lie algebra is the same as for SU(2), which is that of the angular momentum algebra. The Casimir operator is J^2, which has the eigenvalue J(J+1).

We might guess, however, a larger symmetry, namely O(4), which is the same as SU(2) cross SU(2). There will be 2 Casimir operators, for two kinds of spin. (We can take any two axes for each spin, I think.)

So, what do these symmetries mean for relative-position theories?

So, here is the point of the symmetries. Lynden-Bell gave a Lagrangian that was based on $\frac{1}{2}mv^2$ for the kinetic energy. We know that is a small-velocity approximation. However, going to 4-vectors is based on the Poincare group, but we have a different group, in which mass is not a Casimir operator. So, what is the group? What are the Casimir operators? What is the relationship between energy and momentum (or the corresponding parameters) for this group?

It at least seems clear that the system will not be invariant under translation, but instead, under rotation to give a quasi-translation. In that case, the Casimir operator replacing mass will be a rotation operator, in which the axis is 90 degrees away. There should be 3 of these. It is starting to sound like O(4) again. So, the Casimir operator is J^2, with eigenvalue J(J+1). It is quantized because the sphere is finite if the universe is closed. The levels are probably spaced very close together, however.

So, how do I use this information to get a formula for energy? Let's start with SU(2). We have $L_x^2 + L_y^2 + L_z^2 = l(l+1)$. So, we need to express kinetic energy in terms of L or J. Let us assume O(4) on a 4-sphere of radius R. Then the kinetic energy equals the sum of several terms.

[1] 4 November 1999

(1) the expansion rate of the sphere. (This means things moving away from each other.)

(2) rotation of the sphere. I think we need to think only of relative rotations.

We could start with . . .

On the other hand, we don't really want symmetry built into the theory. We want to get some symmetries depending on the matter distribution. But we definitely don't want to build in Poincare symmetry. So, back to our problem. How do we express the energy in general?

Related to this, we also have a problem for potential energy. In the non-relativistic case, it was $1/r_{ij}$. In the relativistic case, it came from $g_{\mu\nu}$, which in the Lagrangian was represented by R. So, maybe we need something like $g_{\mu\nu}$ and R for kinetic energy also. Except we don't really want potentials that are due to other bodies and apply to another bodies. We want something that depends on relative coordinates and relative motion.

I think I have it figured out. Special Relativity applies to relative, not absolute velocities. I simply have to replace any relative velocity by the corresponding replacement that would be used to replace absolute velocities in the usual way of going from Newtonian dynamics to special relativity. I still have to try this out to make sure it works. The rotations might be a problem, but we shall see.

It doesn't work. It doesn't give the kind of inertial effect I was expecting that Sciama got in 1953. The problems are (1) the contribution of each term to inertia has a $1/M$ in it and (2) the contribution of each term to inertia does not have a $1/r$ in it.

Chapter 70

The Essence of Ernst Mach's ideas[1]

The essence of Mach's ideas about inertia is not that inertia is caused by distant matter rather than an intrinsic property of a body, but that physical law must be expressed in terms of relative rather than absolute coordinates. The idea (often referred to a Mach's Principle) that inertia is caused by distant matter, is only a derived effect (principle?, law?). Einstein's principle of general relativity, that all coordinate systems (including accelerated coordinate systems) are valid for expressing physical law, is not equivalent to Mach's idea that physical law must be expressed in terms of relative coordinates.

[1] 30 December 1999

Chapter 71

Physics in a sparse universe[1]

abstract

I refer to a universe in which there is so little matter that the connection between the matter distribution and inertial frames (if there is any) could be discovered by experiment. Although we do not have the ability to make such experiments, speculating on the outcome of such experiments may shed light on the connection between inertia and matter distribution, and in so doing, may also shed light on the nature of quantum theory, electromagnetic theory, and quantum gravity. We want to know what the laws of physics would be in such a universe. Clearly, we cannot create such a universe to test our theories. Our test will be consistency, including being able to make a consistent theory of quantum gravity.

71.1 Introduction: absolute space versus Mach's principle

We know that local inertial frames (after subtracting out the gravitational attraction of known bodies in the solar system) do not rotate or accelerate relative to distant matter to a very high accuracy. If that relationship has a cause rather than being an amazing coincidence, then there seem to be two possibilities:

1. The inertial frame is given, and the matter in the universe is constrained to have zero acceleration (on the average) in that frame. This situation is normally referred to as "absolute space."

2. The inertial frame is to be at rest with some average in the distribution of matter, and may even arise from the presence of the matter. This situation is usually referred to as "Mach's principle."

I don't know which of the two situations (if either) is the correct description, but finding out may have great consequences for the future of physics. However, knowing about frame dragging suggests that the latter is the situation.

71.2 Mach's principle

I shall assume that the latter is the situation, because it is more interesting than the first, and may have deep consequences for quantum physics, electromagnetic theory, and quantum gravity. Why is this? All local physics as we know it, depends implicitly on the existence of an inertial frame. We

[1] 18 July 2000, revised 26 January 2001 and 23 May 2014, files sparse2.tex, sparse.tex, and sparse4.tex

could not formulate quantum theory without an inertial frame. Electromagnetic theory assumes an inertial background.

71.3 A sparse universe

Thus, knowing what happens to our local inertial frame as the matter distribution in the universe changes, or as we get more or less matter is of crucial importance. Suppose there were half as much matter in the universe, or twice as much. Would we notice? What would an inertial frame look like if there were only a few stars?

71.4 Empty and sparse universes, 24 July 2001

We use the action in (49.24)[166, equation (24)] and General Relativity as an example, but the results obviously have more generality. For an empty universe, $T^{\alpha\beta}$ is zero and therefore L_{matter} is also zero. In the semiclassical calculation, the integration in (49.18)[166, equation (18)] is restricted to actions evaluated for classical cosmologies. Therefore, all cosmologies in the integrations in (49.18)[166, equation (18)] satisfy Einstein's equations, so that $T^{\alpha\beta}$ equal zero implies that $R^{\alpha\beta}$ also equal zero, and therefore $R = 0$, and therefore L_{geom} is also zero. If we further consider only static universes, then the surface term in the action in (49.24)[166, equation (24)] is also zero. Under those conditions, the action in (49.24)[166, equation (24)] is zero.

Therefore, the action is zero for all empty static universes. (It may be possible to show that the action for all empty universes is zero.) Thus, the integration in (49.18)[166, equation (18)] will weight all empty universes equally (except for the initial wave function). Therefore, for empty universes, there are no saddlepoints for the integrations in (49.18)[166, equation (18)], and therefore no selection of solutions as we had for the solutions with matter in the previous section. Therefore, the only way to have a single matter free static solution would be for it to be selected by the initial wave function. That is, the initial wave function would have to be sharply peaked.

We would thus not expect to find an empty universe, in agreement with our concept that such solutions are non Machian. Similar arguments apply to asymptotically flat solutions, which many agree are also not Machian.

Sparse universes (that is, universes with some, but not much, matter) are perhaps more interesting. We should expect that the action in (49.24)[166, equation (24)] is a continuous function of the amount of matter in the universe, even as the amount of matter approaches zero. Therefore, we would expect the action to be arbitrarily small for an arbitrarily small amount of matter.

We would therefore expect the integrations in (49.18)[166, equation (18)] to not be sharply peaked in such cases, but to be broad, even for the cases where a saddlepoint exists. We would therefore not expect the saddlepoint approximation to be valid, and many classical cosmologies would contribute significantly to the wave function in the final state.

Thus, for sparse universes also, there would be no selection among classical cosmologies that contribute to the wave function in (49.18)[166, equation (18)]. To test this idea quantitatively, we consider the Bianchi model VI_h in the previous section, and let the amount of matter approach zero. Specifically, let us consider the density of matter given by (49.41)[166, equation (41)] for r equal to some small value r_0.

$$8\pi\rho = 3r_{m1}^{3\gamma_1-2}r_0^{-3\gamma_1} \tag{71.1}$$

For a fixed value of r_0, decreasing the amount of matter in the universe requires a decrease in ρ, which in turn requires a decrease in r_{m1}. If the amount of matter decreases enough, r_{m1} will eventually no longer be much larger than the Planck distance. In that case, the action will no longer be sharply peaked at the saddlepoint $b = 0$, and the allowed values for b in (49.59)[166, equation (61)] will no longer be restricted to be so small, and in turn, the allowed rotation rates in (49.60)[166, equation (62)] will be much larger.

71.5 Addition, 16 August 2009, with changes, January 2010

First, a short review of the importance of considering physics in a sparse universe.[2] Quantum theory is based on the existence of local inertial frames. If inertial frames result from a gravitational interaction with the rest of matter in the universe, and if (as observed) inertial frames are aligned with the matter in the universe, then we need to determine the form for that interaction.

In a sparse universe, we would not observe (as we do in our universe) that matter plus gravitational theory leads to a cosmological model that has inertial frames that gives a background for quantum theory. Instead, we would have matter plus "theory to be determined" leads to interactions which leads to something like quantum theory.[3]

The goal is to discover the "theory to be determined." To do that, we first try to imagine what physics would be like in a sparse universe. There would be no (or very little) inertia. There would be no inertial frames, because there would not be enough matter to fix them. There would be no geometry. There would be no distances. No positions. It might be like Wheeler's "bucket of dust" pregeometry.

It might be like something we expect for distances smaller than a Planck length, because we expect there to be no geometry for such short distances. (See, for example, the work of Ronald Adler [193, 194].) Maybe we can get some insight into physics in a sparse universe by considering the Planck length, L^*,

$$L^{*2} = \frac{\hbar G}{c^3} \ , \tag{71.2}$$

where $\hbar = h/(2\pi)$, h is Planck's constant, G is the gravitational constant, and c is the free-space speed of light. It is useful to consider the ratio of the Planck length to $c\tau$, the radius of the universe, where τ is the Hubble time,

$$\left(\frac{L^*}{c\tau}\right)^2 = \frac{\hbar G}{c^5 \tau^2} \ , \tag{71.3}$$

because if the Planck length is ever comparable to the size of the universe, then we might have something similar to a "sparse universe," which might give us insight into the physics in such a sparse universe.

Dennis Sciama wrote a paper[12], "On the origin of inertia," in 1953 in which he arrived at the formula

$$G\rho\tau^2 = 1 \ , \tag{71.4}$$

where ρ is the mean density of matter in the universe (now). He got this formula by deriving inertia as an induction force in analogy to electrodynamics, and comparing to Newtonian gravitation and dynamics. He argued that the gravitational constant G is really an inertial constant, and will depend on the amount of matter in the universe and the size $c\tau$ of the universe. The formula (71.4) is approximately correct for our universe (now), especially if we include dark matter and dark energy.

Although ordinary baryonic matter is apparently only about 4% of the matter in the universe, we do not yet know what makes up dark matter, and we do not understand what dark energy is. For the present purposes, I will assume that matter is made up of mostly protons and neutrons

[2] I thank David L. Peterson for pointing out some errors in a previous version.

[3] Another feature of a sparse universe would be large inhomogeneities, which may be important to consider in our universe as well. [192]

because that is all we really know about and I am doing only an order-of-magnitude calculation anyway. Then, we can write

$$m_p c = \hbar k_p = \frac{\hbar}{\lambda_p^{\text{bar}}} \ , \tag{71.5}$$

where m_p is the mass of the proton, and λ_p^{bar} is the reduced compton wavelength of the proton. Since protons and neutrons have about the same mass, they also have about the same Compton wavelength.

We can now solve for G and $\hbar$ in (71.4) and (71.5) and substitute into (71.3) to give

$$\frac{\lambda_p^{\text{bar}}}{c\tau} = \frac{N_p}{4\pi/3}\left(\frac{L^*}{c\tau}\right)^2 \ , \tag{71.6}$$

where

$$N_p \equiv \frac{4}{3}\pi(c\tau)^3\rho/m_p \tag{71.7}$$

is the number of protons and neutrons in the universe. I should use a more appropriate factor than $4\pi/3$ for the volume for a closed sphere, but for order-of-magnitude calculations, this is good enough.

To simplify the formulas, we take the radius of the universe as our unit length. That is, we set $c\tau$ equal to one. In addition, we define $n_p \equiv N_p/(4\pi/3)$. Then (71.6) becomes

$$\lambda_p^{\text{bar}} = n_p L^{*2} \ . \tag{71.8}$$

For our universe, in these units,

$$\lambda_p^{\text{bar}} \approx 10^{-39} \ , \tag{71.9}$$

$$n_p \approx 10^{80} \to 10^{83} \ , \tag{71.10}$$

and

$$L^* \approx 10^{-61} \ . \tag{71.11}$$

which approximately satisfies (71.8).

We do not know if this equation would be satisfied for universes whose initial conditions were such that n_p were much smaller than it is for our universe, but it seems useful to explore that possibility. As n_p becomes smaller, λ_p^{bar} or L^* or both must change to continue to satisfy (71.8). We could try to hold λ_p^{bar} fixed while lowering n_p. Eventually, when n_p got down to about 10^{39}, L^* would equal λ_p^{bar}. If we were to lower n_p further, L^* would become larger than λ_p^{bar}. This would mean that quantum gravity effects would be more important than normal quantum effects. It is possible that that is what would happen in a sparse universe, but I suspect not.

Instead, we could require that

$$L^* < \lambda_p^{\text{bar}} \ . \tag{71.12}$$

In that case, (71.8) shows that

$$L^* > 1/n_p \tag{71.13}$$

and

$$\lambda_p^{\text{bar}} > 1/n_p \ . \tag{71.14}$$

Then, as n_p approaches a few thousand or so, both L^* and λ_p^{bar} would become significant relative to the size of the universe. We would describe such a universe as very quantum, not classical at all. Of course, any atoms that might exist would be very different from our universe, and it is doubtful if life could develop.

That is why we need to investigate physics in such a sparse universe theoretically. I am hoping that looking at (71.8) can shed some light on physics in a sparse universe.

We do not have a choice in these scenarios. Actual physics will decide. It is up to us to guess what would happen. However, we have no way of testing our guess except through consistency and seeing what solves other problems in physics, like reconciling gravitation and quantum theory. A more likely scenario would be both L^* and λ_p^{bar} varying by keeping some relation between them. We need to guess that relation.

Another thing to consider is what happens to the fine structure constant α as λ_p^{bar} increases. My guess is that α also depends on the matter in the universe, so that α would change also.

Let us now consider the significance of (71.8). Equation (71.2) was gotten by dimensional analysis. However, it is generally accepted that L^* gives the length below which geometry (and therefore, classical gravitation) ceases to exist.

Equation (71.4) was derived by assuming that inertia was an inductive force. Although the derivation was not rigorous, frame dragging effects in General Relativity suggest that it has some approximate validity. It is valid only when geometry (and therefore, classical gravitation) exist. There is some question about whether it is valid when the wave properties of the matter that is a source of the inertia are significant. Probably a semi-classical gravitation will still be valid in that case, in which the effective gravitational matter is some form of smeared-out expectation value of the quantum matter fields.

Equation (71.5) gives the reduced Compton wavelength λ_p^{bar} of the proton. Generally, this gives the length below which wave properties of a proton are important, although there are more valid tests in specific cases. Generally, the WKB approximation is valid when wave properties are not significant. Using the Compton wavelength is just a rule of thumb. It is not valid for massless particles, such as photons. Generally a better measure is the size of the first Fresnel zone. Still, the Compton wavelength is some indication of the length where wave properties of particles become significant. Of course, the Compton wavelength is different for particles of different mass. Equation (71.5) is valid only when geometry exists, and therefore only for distances larger that L^*. That gives some support for the requirement that L^* be smaller than λ_p^{bar}. λ_p^{bar} has no validity otherwise.

In summary, we can say that (71.8) is valid for lengths larger than L^*, but not for lengths smaller than that. That is, (71.8) can be used to determine when a universe can be considered a "sparse universe," but cannot be used to tell us anything about the physics in a sparse universe. Still, that is a first step.

Notice that a correct theory of quantum gravitation will determine a relation between λ_p^{bar} and L^*. From that knowledge, if we can make a correct guess about what that relation is, that will give us a clue about what the correct theory of quantum gravitation should be.

71.6 Pregeometry

Just a reminder of the difficult problem we are up against. We need a mechanism for how a collection of particles can generate an effective geometry in the presence of no background geometry. In this collection of particles, we have no distances, no directions, no time, no clocks, and no dimensions. We have to generate those things by interactions.

Suppose we try it by path integrals. (Actually a path sum in this discrete case) We can define a path in the following way. A particle interacts with particle A, then with particle B, etc. We have to define an action that depends on path. We have to have a way of describing paths that are close to each other. We have to look for least-action paths. We have to be able to calculate Fresnel zones. This is difficult!

The usual way of trying to do quantum gravity is by assuming that we have wave functions over 3-geometries. That is, we have an amplitude for each 3-geometry. Then, in the classical limit, one 3-geometry (surrounded by a Fresnel zone of 3-geometries) will dominate. Geometry arises because it is put into the theory. Is it likely this is how nature really works? I doubt it. It is simply that our imagination is limited.

Returning to pregeometry, I was mistaken above when I was thinking of particles. When geometry breaks down, we are already in a quantum world. We don't have N_p protons and neutrons as particles or billiard balls, we have wave functions. So we don't have separate bodies or particles; we have superimposed waves. Now, what happens if the Planck length is not negligible compared with the wavelength of these waves? Maybe it is like digitizing sound. Or, maybe like sampling. Maybe the waves have values only on a grid.

Another possibility is that turbulence sets in. Maybe we get nonlinear effects. Maybe there is wavebreaking, or clipping of the waves.

Maybe it is more like filtering. Maybe the waves start falling off in amplitude for wavelengths equal to the Planck length.

Maybe it is all about degrees of freedom. It isn't that we have 10^{80} particles; we have 10^{80} degrees of freedom. Actually, we probably have more degrees of freedom than that because there are electrons, neutrinos, and other fields. It is just that protons and neutrons make up the bulk of mass for ordinary matter, so that they are important for gravitation. There may be different kinds of degrees of freedom for each kind of field. It may be that (71.8) shows how the degrees of freedom are allocated between gravitation and particle waves.

So, maybe our model should be degrees of freedom in the volume of the universe.[4] In a 3-dimensional universe, each particle (wave) has at least three degrees of freedom. So, in a cube of length $c\tau$ on a side, $3N_p$ degrees of freedom would allow N_p Fourier components in each direction. The smallest allowed wavelength would be $c\tau/N_p$. That agrees with (71.14) when we remember what units we were using for λ_p^{bar}.

So, I think we basically have it now in terms of degrees of freedom. I just have to refine it. To do that, define

$$N^* \equiv c\tau/L^* \ , \tag{71.15}$$

which is equal to the number of gravitational degrees of freedom per dimension. We also define

$$N \equiv c\tau/\lambda_p^{\mathrm{bar}} \ . \tag{71.16}$$

Then (71.8) can be written

$$Nn_p = N^{*2} \ , \tag{71.17}$$

which might give more insight into the correct interpretation of the situation. On the other hand, we can also write (71.17) as

$$\frac{n_p}{N^*} = \frac{\lambda_p^{\mathrm{bar}}}{L^*} > 1 \ , \tag{71.18}$$

which might also give more insight into the correct interpretation of the situation, where I have used (71.12). On the left of (71.18), we have the ratio of roughly the number of protons and neutrons in the universe (which is approximately equal to the number of degrees of freedom in the particles that give inertia to the universe) to N^* (which is roughly equal to the number of gravitational degrees of freedom). On the right, we have the ratio of the Compton wavelength of the proton and

[4]Of course, by "volume of the universe" we mean that only for a closed universe. For an open universe, we mean "volume of the part of the universe accessible to us."

neutron (which are the particles that give inertia to the universe) to the Planck length. Equation (71.18) can also be written

$$\frac{n_p}{\lambda_p^{\text{bar}}} = \frac{N^*}{L^*} \; . \tag{71.19}$$

Equation (71.19) can also be written

$$n_p \frac{c\tau}{\lambda_p^{\text{bar}}} = N^{*2} \; . \tag{71.20}$$

The right side of (71.20) seems to follow the holographic principle that information is proportional to area rather than volume. That may help us to understand the significance of (71.20).

There are two goals at this point. The first is to try to develop a theory the predicts (71.20). The second is to use that theory to develop a second relation among those variables such that the two together will tell us how λ_p^{bar} and N^* depend on n_p. At that point, we should have a theory of physics in a sparse universe, so that we can see how present theory develops from a more basic theory.

Combining (71.15) through (71.20) gives[5]

$$N \equiv \frac{c\tau}{\lambda_p^{\text{bar}}} = \frac{N^{*2}}{n_p} = \frac{(c\tau)^2}{n_p L^{*2}} < N^* \equiv \frac{c\tau}{L^*} < n_p = \frac{N^{*2}}{N} \; . \tag{71.21}$$

For our universe (now), this is

$$N = 10^{39} = \frac{N^{*2}}{n_p} = \frac{10^{122}}{10^{83}} < N^* = 10^{61} < n_p = 10^{83} \; . \tag{71.22}$$

In words, this is

proton wavelengths in universe < gravitational degrees of freedom[6] < number of protons & neutrons . (71.23)

Stripped to bare bones, (71.21) is

$$N = \frac{N^{*2}}{n_p} < N^* < n_p = \frac{N^{*2}}{N} \; . \tag{71.24}$$

The question is, "As n_p (which is presumably determined by some initial condition for the universe) decreases, (that is, as we consider universes whose initial condition is such that n_p is smaller) what happens to the values of the other numbers in (71.24)? More specifically, is there some functional relation of either N or N^* on n_p? And if so, what is that relationship? If we could guess the correct relationship, we might go a long way to determining the correct form of quantum gravity.

We could start by testing some particular functional relationships, and see where those lead. If they lead to nonsense, we could reject those hypotheses. One interesting hypothesis, which approximately satisfies (71.22) is

$$N = \sqrt{n_p} \; . \tag{71.25}$$

Substituting into (71.24) gives

$$N^* = n_p^{3/4} \tag{71.26}$$

[5] I thank Gerd Hartmann from Bilshausen, Germany for the discussion in May 2014 that led to the following additions.

[6] per dimension (equals number of Planck lengths in universe)

and

$$n_p^{1/2} < n_p^{3/4} < n_p \ , \tag{71.27}$$

or

$$N < N^{3/2} < N^2 \ , \tag{71.28}$$

or

$$(N^*)^{2/3} < N^* < (N^*)^{4/3} \ . \tag{71.29}$$

The inequality in (71.27) would be satisfied as long as n_p is greater than one. Of course, the hypothesis could be slightly modified by multiplying by some numerical constant to make it fit the data for our universe. The point now is, what kind of a theory of quantum gravity would we get from this hypothesis? Does it lead to nonsense, or something reasonable? That remains to be seen.

Chapter 72

Separating gravitation from geometry in physics[1]

New forms for the geodesic equation and Dirac equation are derived in which gravitation appears on the same footing as the electromagnetic interaction. This seems to be a first step to try to separate gravitation from geometry.

72.1 Introduction

Energy and momentum play important roles in Newtonian mechanics. Their role in Einsteinian mechanics is less clear.

On the one hand, they seem to play the role of sources of the gravitational field, analogous to charge and current in electromagnetism. But the separation of role is not so clear cut as in electromagnetism. We don't have something like $J \to A$ and A plus $J \to$ force as in electromagnetism.

On the other hand, momentum in Einsteinian mechanics seems to play more the role of a vector potential (dP/dt acts like $\partial A/\partial t$) as demonstrated by Sciama in 1953.[12] In Newtonian mechanics, Energy and momentum played only the latter role. Active gravitational mass played the former role. Einstein showed the equivalence of mass and energy, and that mixed the roles.

So, is this dual-role aspect a confusion that will be later cleared up by a better theory, or is this dual role a nature of gravitation, related to its nonlinearity?

Gravitation is an odd interaction for other reasons. Because gravitation is represented as geometry in General Relativity, it seems to act as the arena for all other physics. But we do not know if that is intrinsically true or just happens because of circumstances, such as a lot of mass in the universe. In fact, it is very difficult to separate the gravitational role of the metric tensor from its geometrical role of raising and lowering indices, for example.

Here, I consider two examples, the force equation in General Relativity and the Dirac equation on a curved background, to illustrate the problem.

Consider the force equation,[195]

$$-m(g_{\mu\nu}\dot{U}^{\nu} + \Gamma_{\mu\alpha\beta}U^{\alpha}U^{\beta}) + qF_{\mu\alpha}U^{\alpha} = 0, \tag{72.1}$$

for a body of mass m and charge q, including both the geodesic gravitational force and the Lorentz force, where $U^{\alpha} = \dot{x}^{\alpha}$ is the 4-velocity,

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}) \tag{72.2}$$

[1] Written probably ca 6 March 2001, with corrections, 31 August 2006, based on the gravitational vector potential, first draft 18 December 1974

is the affine connection,[196]

$$F_{\mu\alpha} = A_{\alpha,\mu} - A_{\mu,\alpha} \tag{72.3}$$

is the electromagnetic field tensor,[197] and A is the electromagnetic 4-vector potential.

There are several odd things about (72.1). Equation (72.1) includes both gravitational interaction (the part with the m) and the electromagnetic interaction (the part with the q). But although both gravitational and electromagnetic interactions are fields, the two parts are very different, especially the first gravitational term, which arises even in the absence of curvature.

The qU^ν is a current, so maybe $mU^\mu U^\nu$ is a gravitational current. Maybe we have twice the U^μ factor because it is a tensor rather than a vector interaction. But how about the $m\dot{U}^\mu$ term? Is that also a gravitational current? If so, it should be a source of gravitation, but nothing like it appears in $T_{\mu\nu}$, the source of gravitation in Einstein's theory. Actually, in Einstein's theory, $T_{\mu\nu}$ is a source of curvature, not gravitation. So there is something wrong with considering $T_{\mu\nu}$ as a source of gravitation.

To state this a little better: In electromagnetism, it seems to work well. J_μ is the 4-current that produces the electromagnetic field $F_{\mu\nu}$ (through the 4-vector potential A_μ). Then $F_{\mu\nu}$ acts on the 4-current J_μ to produce a force.

In the gravitational case, it is not so simple. Supposedly, we have $T_{\mu\nu}$ as the source of a gravitational potential $g_{\mu\nu}$. There is nothing that corresponds to a gravitational field, unless it is $\Gamma^\mu_{\alpha\beta}$, but that is not a tensor, and does not give the whole force. There is still the $\dot{U}^\mu$ term, which seems to have no source. At best, $T_{\mu\nu}$ is a source of curvature, not gravitation.

Now consider the Dirac equation[198]

$$\gamma^\mu(i\hbar\nabla_\mu - qA_\mu)\psi = m\psi, \tag{72.4}$$

where the γ matrices are defined by

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{72.5}$$

$$\nabla_\mu = \frac{\partial}{\partial x^\mu} - \Gamma_\mu \tag{72.6}$$

is the covariant derivative, and Γ_μ is the spinor connection.

The electromagnetic term qA_μ is clear, but where is gravitation? It certainly does not appear in an obvious way. It seems that it appears only as an arena on which quantum mechanics and electromagnetism can interact.

The above two examples illustrate some of the problems that occur when gravitation is not separated from geometry.

As long as gravitation is equated with geometry and is an arena for other interactions, it will be difficult to quantize it. Here is a partial attempt to express physics in a way that gravitation appears as a field in the same way as other interactions such as electromagnetism.

In section 72.2, we consider how to write the geodesic equation in a way that puts the gravitational and electromagnetic interactions on more equal footings. In section 72.3, we do the same for the Dirac equation. In section 72.4, we consider the interpretation.

72.2 Geodesic Equation

In the rest frame of the body, we have $\dot{U}^\nu = 0$, $U^0 = 1$, $U^i = 0$ for $i = 1, 2, 3$, so that (72.1) becomes

$$-m(g_{\mu 0,0} - \frac{1}{2}g_{00,\mu}) + q(A_{0,\mu} - A_{\mu,0}) = 0. \tag{72.7}$$

The body moves so that in the frame of the body, the gravitational force (the part with the m) just balances the EM part (the part with the q). In this frame, gravitation and EM appear on equal footing. For the case of many bodies, we can write the equation of motion for each body in its own frame.

Doing this would partially solve the problem of putting gravitation and electromagnetism on the same footing, because in the frame of the body, only the explicit part of the gravitational interaction appears.

We could require that the force law for any body in the frame of that body is $F = 0$, and that the force law on any body cannot be written in any other frame.

Another possibility is to define a covariant vector g_μ which in the rest frame of the body has the components

$$(g_0, g_1, g_2, g_3) = (g_{00}/2, g_{10}, g_{20}, g_{30}). \tag{72.8}$$

Then, (72.7) is

$$-mg_{00,0} = 0 \tag{72.9}$$

for $\mu = 0$ and

$$m(g_{0,\mu} - g_{\mu,0}) + q(A_{0,\mu} - A_{\mu,0}) = 0 \tag{72.10}$$

for $\mu = 1$, 2, or 3. In this notation, gravitation and electromagnetism appear with the same form.

We then require that g_μ transform as a vector under an arbitrary coordinate transformation. This gives in an arbitrary frame

$$g_\mu = U_\mu - \frac{1}{2} g^{\alpha\beta} U_\alpha U_\beta \frac{\partial \tau}{\partial x^\mu} = g_{\mu\alpha} U^\alpha - \frac{1}{2} g_{\alpha\beta} U^\alpha U^\beta \tau_{,\mu}, \tag{72.11}$$

where U_μ and $U^\mu = g^{\mu\nu} U_\nu$ are the covariant and contravariant components of a vector field that satisfies

$$(g^{\alpha\beta} U_\alpha U_\beta)_{,\nu} g^{\mu\nu} U_\mu = (g_{\alpha\beta} U^\alpha U^\beta)_{,\nu} U^\nu = 0 \tag{72.12}$$

and τ is a scalar field that satisfies

$$\tau_{,\beta} g^{\alpha\beta} U_\alpha = \tau_{,\beta} U^\beta = 1. \tag{72.13}$$

The force equation (72.1) can now be written

$$m(g_{\nu,\mu} - g_{\mu,\nu}) U^\nu + q(A_{\nu,\mu} - A_{\mu,\nu}) U^\nu = 0. \tag{72.14}$$

The equivalence of (72.14) with (72.1) is verified in the Appendix to this chapter.

The interpretation of (72.14) is slightly different from (72.1) in that here, U^ν is a vector field, which happens to have the same value as the 4-velocity on the trajectory of the body.

Thus, we may consider a total vector potential

$$q_\mu = mg_\mu + qA_\mu. \tag{72.15}$$

The interesting thing is that there is no way the body can tell the difference between the vector-potential representation of gravitation and the geometric representation. The only difference is in how the field relates to sources.

We can also define a Lagrangian

$$L = mU^\mu g_\mu + qU^\mu A_\mu = mU^\mu U_\mu - \frac{1}{2} m g^{\alpha\beta} U_\alpha U_\beta \frac{\partial \tau}{\partial x^\mu} U^\mu + qU^\mu A_\mu. \tag{72.16}$$

Considering (72.12) and (72.13), we can add zero to (72.16) to give

$$L = mU^\mu g_\mu + qU^\mu A_\mu = mU^\mu U_\mu - \frac{m}{2} g^{\alpha\beta} U_\alpha U_\beta \frac{\partial \tau}{\partial x^\mu} U^\mu + qU^\mu A_\mu + \lambda_1 (g_{\alpha\beta} U^\alpha U^\beta)_{,\nu} U^\nu + \lambda_2 (\tau_{,\beta} U^\beta - 1), \tag{72.17}$$

where λ_1 and λ_2 are arbitrary multipliers.

72.3 Dirac Equation

Brill and Wheeler[198] consider the energy levels of an electron in a gravitational field by adding the gravitational scalar potential to the A_0 term in the Dirac equation (72.4). A single electron cannot distinguish the effect of the gravitational vector potential (72.14) from the geodesic equation (72.1). Therefore, it seems reasonable to continue that ad hoc change in the Dirac equation (72.4) by substituting the total vector potential (72.15) for the electromagnetic vector potential. This gives

$$\gamma^\mu(i\hbar\nabla_\mu - mg_\mu - qA_\mu)\psi = m\psi, \tag{72.18}$$

Or, more explicitly,

$$\gamma^\mu(i\hbar\nabla_\mu - mU_\mu + m\frac{1}{2}g^{\alpha\beta}U_\alpha U_\beta\frac{\partial\tau}{\partial x^\mu} - qA_\mu)\psi = m\psi, \tag{72.19}$$

This equation has the classical trajectory given by

$$-m(g_{\mu\nu}\dot{U}^\nu + \Gamma_{\mu\alpha\beta}U^\alpha U^\beta) + m(g_{\nu,\mu} - g_{\mu,\nu})U^\nu + q(A_{\nu,\mu} - A_{\mu,\nu})U^\nu = 0. \tag{72.20}$$

This gives twice the gravitational force. So, we have to get rid of part. Thus, we should replace (72.19) with

$$\gamma^\mu(i\hbar\nabla_\mu - mU_\mu + m\frac{1}{2}g^{\alpha\beta}U_\alpha U_\beta\frac{\partial\tau}{\partial x^\mu} - qA_\mu)\psi = 0. \tag{72.21}$$

This equation has a classical trajectory given by (72.14).

We need to further test (72.21). When we think of it, Schroedinger's equation, Dirac's equation, and the Klein-Gordon equation are really just ad-hoc equations that guarantee that[2]

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = m^2 \tag{72.22}$$

for a free particle, where ψ is an eigenvector of $i\hbar\nabla_\mu$ with the eigenvalue p_μ. So, let's see how (72.21) satisfies (72.22). We can rewrite (72.21) as

$$\gamma^\mu(i\hbar\nabla_\mu - qA_\mu)\psi = \gamma^\mu(U_\mu - \frac{1}{2}g^{\alpha\beta}U_\alpha U_\beta\frac{\partial\tau}{\partial x^\mu})m\psi. \tag{72.23}$$

For a momentum eigenstate, we have

$$\gamma^\mu(p_\mu - qA_\mu)\psi = \gamma^\mu(U_\mu - \frac{1}{2}g^{\alpha\beta}U_\alpha U_\beta\frac{\partial\tau}{\partial x^\mu})m\psi. \tag{72.24}$$

This gives

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = g^{\mu\nu}(U_\mu - \frac{1}{2}g^{\alpha\beta}U_\alpha U_\beta\frac{\partial\tau}{\partial x^\mu})(U_\nu - \frac{1}{2}g^{\alpha\beta}U_\alpha U_\beta\frac{\partial\tau}{\partial x^\nu})m^2. \tag{72.25}$$

Using

$$g^{\mu\nu}U_\mu U_\nu = -1 \tag{72.26}$$

gives

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = g^{\mu\nu}(U_\mu + \frac{1}{2}\frac{\partial\tau}{\partial x^\mu})(U_\nu + \frac{1}{2}\frac{\partial\tau}{\partial x^\nu})m^2. \tag{72.27}$$

Or,

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = (g^{\mu\nu}U_\mu U_\nu + g^{\mu\nu}U_\nu\frac{\partial\tau}{\partial x^\mu} + \frac{1}{4}g^{\mu\nu}\frac{\partial\tau}{\partial x^\mu}\frac{\partial\tau}{\partial x^\nu})m^2. \tag{72.28}$$

[2] assuming $g = - + ++$

Using (72.26) and (72.13) gives

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = (-1 + 1 + \frac{1}{4} g^{\mu\nu} \frac{\partial\tau}{\partial x^\mu} \frac{\partial\tau}{\partial x^\nu}) m^2. \tag{72.29}$$

Finally,

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = -\frac{1}{4} m^2. \tag{72.30}$$

As far as I can tell, I have not made an error. So, how do I interpret this? The factor of 4 may be related to the factors of 4 noticed by Peterson[199, 200, 201, 202, 203].

On the other hand, let

$$m^2 = 4M^2. \tag{72.31}$$

Then (72.30) becomes

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = -M^2, \tag{72.32}$$

which looks just like the Dirac equation. Using (72.31) in (72.21) gives

$$\gamma^\mu [i\hbar\nabla_\mu - 2M(U_\mu - \frac{1}{2} g^{\alpha\beta} U_\alpha U_\beta \frac{\partial\tau}{\partial x^\mu}) - qA_\mu]\psi = 0. \tag{72.33}$$

This corresponds to a total vector potential from (72.15) of

$$q_\mu = 2Mg_\mu + qA_\mu, \tag{72.34}$$

which leads to a force equation from (72.1) of

$$-2M(g_{\mu\nu}\dot{U}^\nu + \Gamma_{\mu\alpha\beta}U^\alpha U^\beta) + qF_{\mu\alpha}U^\alpha = 0 \tag{72.35}$$

The factor of 2 may be related to the factor of 2 in the effective gravitomagnetic charge [204].

On the other hand, the problem may be related to spin. The Lorentz force and geodesic equation is not the only way an electron interacts with an electromagnetic and gravitational field. There is also the interaction of magnetic moment with the gradient of magnetic field and angular momentum with a gravitational field.

Because the ratio of magnetic moment to angular momentum of an electron is only half of the ratio of electric charge to mass,[205] the electromagnetic and gravitational interactions cannot be simply combined as I have done.

So, as far as the motion of the electron with regard to only its mass and charge are concerned, the total effective vector potential given by (72.15) is what applies. But, as far as interactions of magnetic moment with the electromagnetic field and spin angular momentum with the gravitational field, the total effective vector potential is instead

$$q_\mu = 2mg_\mu + qA_\mu. \tag{72.36}$$

So, what I have to do is to unwrap the Dirac equation so that it reveals explicitly how it contains both of these total vector potentials. At that point, I will have managed to separate gravitation from geometry. I will have the gravitational interaction explicitly in the Dirac equation.

Here is one idea. Consider a quadratic form of the Dirac equation.[206]

$$[i(\partial/\partial t) - e\phi - \sigma\cdot(-i\nabla - e\mathbf{A})][i(\partial/\partial t) - e\phi + \sigma\cdot(-i\nabla - e\mathbf{A})]\mathbf{\Psi} = m^2\mathbf{\Psi} \tag{72.37}$$

Using the process that went from (72.25) to (72.30), we can substitute the m^2 in (72.37) to give

$$\begin{aligned}[i(\partial/\partial t) - e\phi - \sigma\cdot(-i\nabla - e\mathbf{A})][i(\partial/\partial t) - e\phi + \sigma\cdot(-i\nabla - e\mathbf{A})]\mathbf{\Psi} &= \\ -4g^{\mu\nu}(U_\mu - \frac{1}{2} g^{\alpha\beta} U_\alpha U_\beta \frac{\partial\tau}{\partial x^\mu})(U_\nu - \frac{1}{2} g^{\alpha\beta} U_\alpha U_\beta \frac{\partial\tau}{\partial x^\nu}) m^2\mathbf{\Psi}\end{aligned} \tag{72.38}$$

That substitution is valid if U^μ is actually a geodesic. But, if instead, it is the vector field that satisfies (72.12), then (72.38) is not equivalent to (72.37). However, suppose that (72.38) is really the correct form. In that case, we can rewrite (72.38) as

$$[i(\partial/\partial t) - e\phi - \sigma\cdot(-i\nabla - e\mathbf{A})][i(\partial/\partial t) - e\phi + \sigma\cdot(-i\nabla - e\mathbf{A})]\mathbf{\Psi} = -4g^{\mu\nu}g_\mu g_\nu m^2\mathbf{\Psi} \tag{72.39}$$

where g_μ is the gravitational vector potential.

72.4 Discussion

The covariant vector field U_μ that satisfies (72.26) will also satisfy (72.12). If we choose U_μ in (72.11) to satisfy (72.26) in addition to (72.12), then (72.11) becomes

$$g_\mu = U_\mu + \frac{1}{2}\frac{\partial\tau}{\partial x^\mu}, \tag{72.40}$$

The normal interpretation uses

$$\dot{x}^\mu = g^{\mu\nu}U_\nu, \tag{72.41}$$

which has some usefulness when interpreting a particle in a semiclassical approximation. However, here I don't make that interpretation. Instead, it seems that U_μ is more like a covariant vector field that represents something to do with the gravitational field.

Note added, 28 August 2006: I have added some missing factors of m in (72.16) and (72.17). In addition, my alteration of the Dirac equation was overly simplistic. The usual Dirac equation (72.4) guarantees that (72.12) will be satisfied, but neither of my replacements (72.19) or(72.21) guarantees that.

Thus, the real meaning of the results here is that the Dirac equation ensures that the gravitational interaction will look locally like a vector potential, so that it matches with the electromagnetic vector potential. To see this, we first write

$$\gamma^\mu(mU_\mu)\psi = m\psi\ , \tag{72.42}$$

where, solving for U_μ from (72.11) and (72.15) gives

$$\gamma^\mu(\frac{1}{2}mg^{\alpha\beta}U_\alpha U_\beta\frac{\partial\tau}{\partial x^\mu} + q_\mu - qA_\mu)\psi = m\psi\ . \tag{72.43}$$

We need to compare this with the Dirac equation

$$\gamma^\mu(i\hbar\frac{\partial}{\partial x^\mu} - i\hbar\Gamma_\mu - qA_\mu)\psi = m\psi\ , \tag{72.44}$$

where I have used (72.6). We need q_μ proportional to the spinor connection, but I do not know the significance of that. Also, in the case of the Klein-Gordon equation, there is no spinor connection, so I do not know where q_μ fits in then.

72.5 Appendix. Showing that (72.14) is equivalent to (72.1)

The second form of (72.11) is

$$g_\mu = g_{\mu\alpha}U^\alpha - \frac{1}{2}g_{\alpha\beta}U^\alpha U^\beta\tau_{,\mu} \tag{72.45}$$

Taking the derivative of (72.45) gives

$$\begin{aligned} g_{\mu,\nu} &= g_{\mu\alpha,\nu}U^\alpha + g_{\mu\alpha}U^\alpha_{,\nu} \\ &\quad -\frac{1}{2}(g_{\alpha\beta}U^\alpha U^\beta)_{,\nu}\tau_{,\mu} - \frac{1}{2}g_{\alpha\beta}U^\alpha U^\beta\tau_{,\mu,\nu} \end{aligned} \tag{72.46}$$

Interchanging the indices in (72.46) and expanding gives

$$\begin{aligned} g_{\nu,\mu} &= g_{\nu\alpha,\mu}U^\alpha + g_{\nu\alpha}U^\alpha_{,\mu} - \frac{1}{2}g_{\alpha\beta,\mu}U^\alpha U^\beta\tau_{,\nu} \\ &\quad -g_{\alpha\beta}U^\alpha U^\beta_{,\mu}\tau_{,\nu} - \frac{1}{2}g_{\alpha\beta}U^\alpha U^\beta\tau_{,\nu,\mu} \end{aligned} \tag{72.47}$$

Subtracting (72.47) from (72.46) and using $\tau_{,\mu,\nu} = \tau_{,\nu,\mu}$ gives

$$\begin{aligned} g_{\mu,\nu} - g_{\nu,\mu} &= g_{\mu\alpha,\nu}U^\alpha - g_{\nu\alpha,\mu}U^\alpha + g_{\mu\alpha}U^\alpha_{,\nu} - g_{\nu\alpha}U^\alpha_{,\mu} - \frac{1}{2}(g_{\alpha\beta}U^\alpha U^\beta)_{,\nu}\tau_{,\mu} \\ &\quad +\frac{1}{2}g_{\alpha\beta,\mu}U^\alpha U^\beta\tau_{,\nu} + g_{\alpha\beta}U^\alpha U^\beta_{,\mu}\tau_{,\nu} \end{aligned} \tag{72.48}$$

Multiplying by U^ν gives

$$\begin{aligned} (g_{\mu,\nu} - g_{\nu,\mu})U^\nu &= g_{\mu\alpha,\nu}U^\alpha U^\nu - g_{\nu\alpha,\mu}U^\alpha U^\nu + g_{\mu\alpha}U^\alpha_{,\nu}U^\nu - g_{\nu\alpha}U^\alpha_{,\mu}U^\nu \\ &\quad -\frac{1}{2}(g_{\alpha\beta}U^\alpha U^\beta)_{,\nu}U^\nu\tau_{,\mu} + (\frac{1}{2}g_{\alpha\beta,\mu}U^\beta + g_{\alpha\beta}U^\beta_{,\mu})U^\alpha U^\nu\tau_{,\nu} \end{aligned} \tag{72.49}$$

Or,

$$\begin{aligned} (g_{\mu,\nu} - g_{\nu,\mu})U^\nu &= (g_{\mu\alpha}U^\alpha)_{,\nu}U^\nu - g_{\nu\alpha,\mu}U^\alpha U^\nu + \frac{1}{2}g_{\alpha\beta,\mu}U^\alpha U^\beta U^\nu\tau_{,\nu} \\ &\quad -g_{\nu\alpha}U^\alpha_{,\mu}U^\nu + g_{\alpha\beta}U^\alpha_{,\mu}U^\beta U^\nu\tau_{,\nu} - \frac{1}{2}(g_{\alpha\beta}U^\alpha U^\beta)_{,\nu}U^\nu\tau_{,\mu} \end{aligned} \tag{72.50}$$

Changing dummy indices gives

$$\begin{aligned} (g_{\mu,\nu} - g_{\nu,\mu})U^\nu &= (g_{\mu\alpha}U^\alpha)_{,\nu}U^\nu - g_{\alpha\beta,\mu}U^\alpha U^\beta + \frac{1}{2}g_{\alpha\beta,\mu}U^\alpha U^\beta U^\nu\tau_{,\nu} \\ &\quad -g_{\alpha\beta}U^\alpha_{,\mu}U^\beta + g_{\alpha\beta}U^\alpha_{,\mu}U^\beta U^\nu\tau_{,\nu} - \frac{1}{2}(g_{\alpha\beta}U^\alpha U^\beta)_{,\nu}U^\nu\tau_{,\mu} \end{aligned} \tag{72.51}$$

Or,

$$\begin{aligned} (g_{\mu,\nu} - g_{\nu,\mu})U^\nu &= g_{\mu\alpha}U^\alpha_{,\nu}U^\nu + g_{\mu\alpha,\nu}U^\alpha U^\nu - g_{\alpha\beta,\mu}(1 - \frac{1}{2}\tau_{,\nu}U^\nu)U^\alpha U^\beta - g_{\alpha\beta}U^\alpha_{,\mu}U^\beta(1 - \tau_{,\nu}U^\nu) \\ &\quad -\frac{1}{2}(g_{\alpha\beta}U^\alpha U^\beta)_{,\nu}U^\nu\tau_{,\mu} \end{aligned} \tag{72.52}$$

Or,

$$\begin{aligned} (g_{\mu,\nu} - g_{\nu,\mu})U^\nu &= g_{\mu\alpha}U^\alpha_{,\nu}U^\nu + [g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}(1 - \frac{1}{2}\tau_{,\nu}U^\nu)]U^\alpha U^\beta \\ &\quad -g_{\alpha\beta}U^\alpha_{,\mu}U^\beta(1 - \tau_{,\nu}U^\nu) - \frac{1}{2}(g_{\alpha\beta}U^\alpha U^\beta)_{,\nu}U^\nu\tau_{,\mu} \end{aligned} \tag{72.53}$$

Using (72.12) and (72.13), multiplying by $-m$, and adding something to both sides gives

$$\begin{aligned} (g_{\nu,\mu} - g_{\mu,\nu})U^\nu + q(A_{\nu,\mu} - A_{\mu,\nu})U^\nu &= -m[g_{\mu\alpha}U^\alpha_{,\nu}U^\nu + (g_{\mu\alpha,\beta} - \frac{1}{2}g_{\alpha\beta,\mu})U^\alpha U^\beta] \\ &\quad +q(A_{\nu,\mu} - A_{\mu,\nu})U^\nu \end{aligned} \tag{72.54}$$

We define a coordinate system in which $U^\mu = \dot{x}^\mu$. In that case, $\dot{U}^\alpha = U^\alpha_{,\nu}U^\nu$, so that (72.54) becomes

$$\begin{aligned} m(g_{\nu,\mu} - g_{\mu,\nu})U^\nu + q(A_{\nu,\mu} - A_{\mu,\nu})U^\nu &= -m[g_{\mu\alpha}\dot{U}^\alpha + (g_{\mu\alpha,\beta} - \frac{1}{2}g_{\alpha\beta,\mu})U^\alpha U^\beta] \\ &\quad +q(A_{\nu,\mu} - A_{\mu,\nu})U^\nu \end{aligned} \tag{72.55}$$

72.6 Afterthoughts - 2008

I fixed the sign error in (72.22), but then I had to fix other errors in section 72.3 as a result of that correction. I hope the equations are all correct now.

Substituting the gravitational vector potential back into the Dirac equation was a little superficial. It should not be too surprising that it did not give useful results.

The part about τ becoming a scalar field, however, is correct and also U^μ becoming a vector field is correct with regard to solutions of the Dirac equations. In fact, following Weinberg (1962)[207], we can construct a WKB approximation to the Dirac equation. The WKB approximation for each component of the Dirac equation is proportional to

$$e^{\frac{i}{\hbar}\int p_\mu dx^\mu} \; . \tag{72.56}$$

Using

$$p_\mu - qA_\mu = mU_\mu \tag{72.57}$$

gives

$$e^{\frac{i}{\hbar}\int (mU\mu + qA_\mu)dx^\mu} \; . \tag{72.58}$$

Chapter 73

Putting gravitation on the same footing as the other interactions[1]

We change the role of gravitation from acting as the arena for all other physics to being simply another interaction such as electromagnetism. This should make it easier to discover the correct form for quantum gravity.

73.1 Introduction

We have successfully quantized three of the four interactions, but not the gravitational interaction, although most of us believe that gravitation should also be quantized. One of the reasons for the difficulty is that gravitation (as geometry) is taken to be a background (or arena) upon which the rest of physics takes place.

There is nothing in our common experiences that suggests that gravitation is so different from other interactions. Gravitation acts on (and is produced by) masses. Electromagnetism acts on (and is produced by) charges and currents.

The only thing that suggests that gravitation might be different from other interactions is inertia. Inertia is treated as a gravitational (or geometric) effect by Einstein's theory, but flat Minkowski spacetime seems to suggest that inertia might be a property of the background. I think it is this property of inertia that leads us to associate gravitation with a background arena upon which all other physics takes place.

In fact, one usual way of doing physics is to find a local inertial frame and then ignore curvature and do Newtonian physics in that frame, doing a coordinate transformation to another frame to generalize the results if necessary.

In a sense, this keeps us from getting away from Newtonian physics, and may be making it harder to quantize gravitation. Here, I try to put the gravitational interaction on the same footing as other interactions, using the electromagnetic (EM) interaction as an example. Since the weak interaction has already been combined with EM (the electroweak interaction), and QCD are already treated on the same footing as EM, it is sufficient to put gravitation on the same footing as EM.

[1] 25 February 2002

73.2 Action

It is sufficient to consider the action, because that characterizes the physics, either in the classical or the quantum case. The usual form for the action S is

$$S = \int (-g^{(4)})^{-1/2} L d^4x, \tag{73.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$,

$$L = \underbrace{\frac{R - 2\Lambda}{16\pi}}_{geometry} \underbrace{-\frac{1}{2}\rho g_{\mu\nu} U^\mu U^\nu}_{matter} \underbrace{-\frac{\rho_e}{c} A_\mu U^\mu}_{interaction} \underbrace{-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}}_{EM}, \tag{73.2}$$

R is the Riemann scalar, Λ is the cosmological constant, ρ is the mass density, U^μ is the four-velocity, ρ_e is the electric charge density, A_μ is the electromagnetic 4-vector potential, $F_{\mu\nu}$ is the electromagnetic field tensor, and the usual designation of the four terms is shown.

The "interaction" term is actually a term that deals with interaction of electric charges and currents with the electromagnetic field. To be more explicit, I shall call it the "EM-charge-current interaction" term.

The "matter" term is actually the term that deals with the interaction of mass with the gravitational field, including inertia. I shall call that the "mass-gravitational interaction" term.

The "EM" term deals with the behavior of the EM field in the absence of charges and currents. More specifically, it deals with the behavior of EM fields in the presence of inertial frames provided by the gravitational field. And, because it also gives the contribution of the EM field to the stress-energy tensor (that is the source of gravitational fields), I shall call it the "EM-gravitational interaction" term.

The "geometry" term deals with the behavior of the geometry or the gravitational field in the absence of masses, electric charges, or EM fields. I shall call it the "gravitation" term.

Equation (73.2) can then be written

$$L = \underbrace{\frac{R - 2\Lambda}{16\pi}}_{gravitation} \underbrace{-\frac{1}{2}\rho g_{\mu\nu} U^\mu U^\nu}_{m-grav.\, int.} \underbrace{-\frac{\rho_e}{c} A_\mu U^\mu}_{q-EM\, int.} \underbrace{-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}}_{EM-grav.\, int.}, \tag{73.3}$$

The gravitational and EM terms seems much more symmetrical. To make them even more symmetrical, we rewrite (73.3) as

$$L = \underbrace{\frac{R - 2\Lambda}{16\pi}}_{gravitation} \underbrace{-\frac{1}{2} g_{\mu\nu} J^{\mu\nu}}_{m-grav.\, int.} \underbrace{-A_\mu J^\mu}_{q-EM\, int.} \underbrace{-\frac{1}{16\pi} g^{\alpha\mu} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}}_{EM-grav.\, int.}, \tag{73.4}$$

where

$$J^\mu \equiv \frac{\rho_e}{c} U^\mu \tag{73.5}$$

is the electric current density 4-vector (which is the source of EM fields) and

$$J^{\mu\nu} \equiv \rho U^\mu U^\nu \tag{73.6}$$

is the mass tensor (related to the energy-momentum tensor), which is the source of gravitational fields.

On the classical level, variation of L with respect to the appropriate quantities in (73.3) or (73.4) leads to Einstein's field equations, Maxwell's equations, and the geodesic equation, including the

Lorentz force. On the quantum level, (73.3) or (73.4) leads to quantum equations, either through path integrals or through the Klein-Gordon equation.

Considering these four terms, only one of them (the first term) involves only one field (gravitation). That means, if we are to believe this Lagrangian, only the gravitational field can exist by itself in absence of the others. This is why we consider gravitation to be the arena for the other fields. Is this really correct? I do not know, although our universe certainly has all of the other fields, and without other fields, there is no way to detect any such lone gravitational field.

Related to this, is the string-theory result that gravitons are represented by closed strings while other particles or fields are represented by open strings, that must have their ends connected to a brane. This suggests that gravitons could survive alone, but that other particles or fields require the presence of some other field or particle.

Another reason why no other field could stand alone is that all fields have energy so that they will generate gravitational fields. Again, whether this is the correct situation or just a property of present knowledge is still an open question.

Continuing on with strings, the original reason for inventing strings was to avoid some of the infinities associated with the point representation of particles. Related to that problem is the requirement that a field is a function of position. That is, that we can know the field at a point. That goes back to the Green's function representation, where we can consider extended sources to be a sum of points. To avoid that we have to consider a way to represent the Lagrangian not as a function of position, but as something else.

What that something else might be is not at all clear, but it might be something like knowing some sort of average value of the field on the particle (which might be some sort of extended object, but also possibly not determined). For the mass- and charge-interaction terms, that might be fairly straightforward to do, but for the massless particles, like photons, that might be more complicated.

Maybe we can write the gravitational term in the Lagrangian as a graviton-graviton interaction term. If we can, then all terms in the Lagrangian will be interaction terms. Then we have the possibility of considering a "relative-fields" formulation of quantum gravitation, in which the separate fields are not quantized, but only their interactions.

Back to the gravitational field, which apparently can exist alone, Since gravitons are self-interacting, we can imagine the gravitational field evaluated only on gravitons or on other particles if they are present, including virtual particles.

Another fundamental questions concerning particles is whether they exist except in interactions. That is, back to the old wave-particle duality, where they propagate and interfere as waves, but interact as particles. So, how do we represent that property in a Lagrangian? Or, is a Lagrangian even the correct formalism to use? Maybe infinitesimal particle propagators.

This Lagrangian does not apply to electrons, however, because it does not include spin. Although spin could be added to the Lagrangian in an ad-hoc way, it is more instructive to take another route.

73.3 Spinors

To get the Dirac equation instead of the Klein-Gordon equation, we need a linear Lagrangian.

$$L = \underbrace{\frac{R-2\Lambda}{16\pi}}_{gravitation} \underbrace{-m\gamma_\mu U^\mu}_{m-grav.\ int.} \underbrace{-A_\mu J^\mu}_{q-EM\ int.} \underbrace{-\frac{i}{4}\gamma_\mu\gamma_\nu F^{\mu\nu}}_{EM-grav.\ int.}, \tag{73.7}$$

where the γ matrices are defined by

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{73.8}$$

Variation doesn't give an equation of motion for a linear Lagrangian, so we consider the contribution of a particular path to the wave function.

$$\psi = e^{i\hbar S} = e^{i\hbar \int (g^{(4)})^{-1/2} L d^4x} \tag{73.9}$$

Now we make a change that is not equivalent if there are any non-commuting terms in L.

$$\psi = \prod_{i=1}^{i=\infty} K_i \tag{73.10}$$

where K_i is the infinitesimal particle propagator

$$K_i = e^{i\hbar (g^{(4)})^{-1/2} L d^4x} \tag{73.11}$$

Now I think it is possible to vary the path and get something reasonable, but I need to work on this still.

73.4 Interpretation in terms of General Relativity[3]

In Newtonian theory, we had $F = ma$. In General Relativity, inertial is a gravitational force, so we have $F - ma = 0$, or $F_0 = 0$, where $F_0 \equiv F - ma$.

Sciama, in 1953 [12], modeled inertial as an induction force, analogous to electromagnetism. In 1969, Sciama, Waylen, and Gilman [16], showed that it was possible to put the induction calculation on a firm footing by deriving an integral formulation of Einstein's field equations. The resulting equation is

$$g_{\mu\nu}(x) = \int G^{\acute\beta}_{\mu\nu\acute\alpha}(x, \acute x) T^{\acute\alpha}_{\acute\beta}(\acute x) d^4\acute x \text{ plus surface terms,} \tag{73.12}$$

where (and this equation is from memory)[4] we see that the local metric at x is give by an integral of the stress-energy tensor at $\acute x$, and the two-point propagator G, and I have left out the correct normalizing factor. So, to do Sciama's 1953 calculation accurately, we look at the geodesic equation in the frame of the particle.

$$-m(g_{\mu0,0} - \frac{1}{2} g_{00,\mu}) + q(A_{0,\mu} - A_{\mu,0}) = 0. \tag{73.13}$$

If we neglect the surface terms, then

$$g_{\mu0,0}(x) = \int G^{\acute\beta}_{\mu0\acute\alpha,0}(x, \acute x) T^{\acute\alpha}_{\acute\beta}(\acute x) d^4\acute x \tag{73.14}$$

and

$$g_{00,\mu}(x) = \int G^{\acute\beta}_{00\acute\alpha,\mu}(x, \acute x) T^{\acute\alpha}_{\acute\beta}(\acute x) d^4\acute x \tag{73.15}$$

Substituting (73.14) and (73.15) into (73.13), we get

$$-m\left(\int G^{\acute\beta}_{\mu0\acute\alpha,0}(x, \acute x) T^{\acute\alpha}_{\acute\beta}(\acute x) d^4\acute x - \frac{1}{2}\int G^{\acute\beta}_{00\acute\alpha,\mu}(x, \acute x) T^{\acute\alpha}_{\acute\beta}(\acute x) d^4\acute x\right) + q(A_{0,\mu} - A_{\mu,0}) = 0. \tag{73.16}$$

This shows clearly the parallel between electromagnetic and gravitational effect on a particle such as an electron. It seems to put gravitation and EM on the same footing. However, this classical picture is not as fundamental as a wave mechanics picture. Also, notice that this easy parallel between EM and gravitation is apparent only in the frame of the body. In other frames, we have the usual inertial term instead of an induction term.

[3]September 2010

[4]$T^{\acute\alpha}_{\acute\beta}(\acute x)$ should be replace by $[T^{\acute\alpha}_{\acute\beta}(\acute x) - Tg^{\acute\alpha}_{\acute\beta}(\acute x)][-g(\acute x)]^{1/2}$ in all of these equations. See equation (86.32) to include the cosmological constant term. The normalization factor is $2k$.

73.5 Interpretation in terms of wave mechanics

In terms of wave mechanics, the force equation is a refractive equation. That is, we consider the electron to be a wave, and the wave is bent by refraction. In this interpretation, the electromagnetic force on the electron is seen as a refractive index, which bends the wave.

We can use either Lagrange's equations or Hamilton's equations to express the bending of the wave. Either would be equivalent to the classical picture in the previous section. However, here, we recognize that we are simply making a WKB approximation to the wave equation. The wave is the real physics.

The wave picture is easier to see in a frame other than the frame of the particle. Refraction is difficult to see in the frame of the wave.

73.6 Combining the two interpretations

This is the hard part, because in one case, we must be in the frame of the particle, while in the other case, we must not.

Chapter 74

The classical action for a Bianchi VI_h model[1]

abstract

An estimate for the classical action $I_{classical}$ for a Bianchi VI_h homogeneous model with $h = -1/9$ is given by

$$\begin{aligned}\frac{I_{classical}}{\hbar} &\approx \frac{3}{5} i\pi^2 (1+\alpha_0) \left[\left(\frac{r(t)}{L^*}\right)^2 \left(\frac{r(t)}{r_{m2}}\right)^{1/2} \right. \\ &- \left. 2 \left(\frac{r(t)}{r_1}\right) \left(\frac{r(t)}{r_{m2}}\right)^{3/2} \left(\frac{b}{8a_0^3 L^* r_0}\right)^2 \right],\end{aligned}$$

where b and a_0 are parameters of the model, b is zero if and only if the relative rotation of inertial frames and matter is zero, α_0 is an initial value of an anisotropy of the expansion rate at $r = r_0$, L^* is the Planck length, and r_{m2} is a constant of integration that would give the maximum size of the universe for the isotropic ($b = 0$, $\alpha_0 = 0$) case if the universe were spatially closed.

It is assumed that the equation of state is $p = (\gamma - 1)\rho$, where p is pressure and ρ is density. It is assumed that γ has a constant value of $\gamma = 4/3$ (to represent a relativistic early universe) for $r < r_1$ and a constant value of $\gamma = 1$ (to represent a matter-dominated late universe) for $r > r_1$. The approximation is valid for b small enough that $|I_{classical}[b] - I_{classical}[b = 0]| < \hbar$.

An explanation for why our inertial frame seems not to rotate relative to the stars is found in a straightforward application of semiclassical approximations to quantum cosmology. Application of a saddlepoint approximation leads to the result that only those classical geometries whose action $I_{classical}$ satisfies $|I_{classical} - I_{saddlepoint}| < \hbar$ contribute significantly to the integration to give the present value of the wave function. Using estimates for our universe implies that only those classical geometries for which the present relative rotation rate of inertial frames and matter are less than about 10^{-129} radians per year contribute significantly to the integration. This is well below the limit set by experiment. The result depends on the Hubble distance being much larger than the Planck length, but does not depend on the details of the theory of quantum gravity.

74.1 Introduction

To estimate the wave function for the universe in the quasi-classical approximation, it is necessary to calculate the action for a solution of Einstein's equation as a function of the parameters for the model.

[1]23 June 2002, File: q-cos6.tex, adding the cosmological constant Λ

A better example would probably be one that reduces to the FRW model in the isotropic case, such as Bianchi types I, VII$_0$, V, VII$_h$, or IX[208], but the Bianchi VI$_h$ model has some interesting properties. In the future, it would be good to use some of the other models.

If we divide the classical action $I_{classical}$ by Planck's constant $\hbar$, we get the dimensionless action, that is in the exponent of the propagator. We can express this dimensionless action in terms of several length scales. We want to know how the action depends on the various parameters of the Bianchi VI model. In particular, how it depends on the parameter b that gives the anisotropy of the model. The saddlepoint is at $b = 0$, for which the model is isotropic. Because the action is stationary at the saddlepoint for variation of b, for small b, there is no linear term for the variation of the action with respect to b. Therefore, for small b, the next term is a b^2 term. The dimensions of b are that of an angular velocity. Therefore, for $c = 1$, the dimensions of b are inverse length. The inverse $\hbar$ dependence shows that $I_{classical}/\hbar$ depends on the Planck length $L^* \equiv (\hbar G/c^3)^{1/2}$ as $I_{classical}/\hbar \propto (L^*)^{-2}$. Thus, we must have

$$I_{classical}/\hbar = \frac{\ell^2}{L^{*2}} - \frac{b^2\ell^4}{L^{*2}}, \tag{74.1}$$

where ℓ is something having the dimensions of a length. However, the Bianchi VI model also has another parameter a_0, which has the dimensions of inverse length. This enters into the action in always the same combination with b. Thus, we have

$$I_{classical}/\hbar = \frac{\ell^2}{L^{*2}} - \frac{b^2\ell^{-2}}{a_0^6 L^{*2}} \tag{74.2}$$

There are several quantities that have the dimensions of length that can be used in these formulas. These are r_0, the size of the universe at the initial wave function, r_1, the size of the universe when it switches from radiation dominated to matter dominated, r, the size of the universe now, and constants of integration r_{m1} and r_{m2}. r_0 is very small, only a few powers of ten larger than L^*. All of the others are within a few powers of ten of each other, and are many powers of ten larger than L^*.

On the quantum level, however, we might imagine that somehow the selection takes place automatically through wave interference, and this turns out to be the case. To show that requires calculating the action for a classical cosmology as a function of the appropriate parameters of the model, which is the goal here.

74.2 An example from ordinary wave mechanics

To help explain the ideas that follow, we first consider elementary wave mechanics as an example. If we have an initial single-particle state specified by an initial wave function $< x_1, t_1|\psi >$ at time t_1 then the wave function $< x_2, t_2|\psi >$ at time t_2 is [112, p. 57]

$$< x_2, t_2|\psi >= \int_{-\infty}^{\infty} < x_2, t_2|x_1, t_1 >< x_1, t_1|\psi > dx_1, \tag{74.3}$$

where $< x_2, t_2|x_1, t_1 >$ is the propagator for the particle to go from (x_1, t_1) to (x_2, t_2). We consider the case where the semiclassical approximation for the propagator is valid. That is, [112, p. 60]

$$< x_2, t_2|x_1, t_1 >\approx f(t_1, t_2)e^{\frac{i}{\hbar}I_{cl}[x_2,t_2;x_1,t_1]}, \tag{74.4}$$

where $I_{cl}[x_2, t_2; x_1, t_1]$ is the action calculated along the classical path from (x_1, t_1) to (x_2, t_2). Thus, (74.3) becomes

$$< x_2, t_2|\psi >\approx f(t_1, t_2) \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}I_{cl}[x_2,t_2;x_1,t_1]} < x_1, t_1|\psi > dx_1. \tag{74.5}$$

Notice that because of the initial wave function we have an infinite number of classical paths contributing to each value of the final wave function.

There are two cases to consider. In the first, I_{cl} is not a sharply peaked function of x_1. In that case, there will be contributions to the wave function at t_2 from classical paths that differ greatly from each other.

In the second case, which we now consider, I_{cl} is sharply peaked about some value of x_1, say x_{sp}. That is, we have

$$\frac{\partial}{\partial x_1} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}} = 0. \tag{74.6}$$

Thus, x_{sp} is a saddlepoint of the integral (74.5), and significant contributions to the integral are limited to values of x_1 such that

$$|x_1 - x_{sp}|^2 < \left| \frac{2\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}}} \right|. \tag{74.7}$$

If $< x_1, t_1|\psi >$ is nearly constant over that range, then we can take it outside of the integral. A saddlepoint evaluation of the integral then gives

$$\begin{aligned} < x_2, t_2|\psi > \quad \approx \quad & f(t_1, t_2) < x_{sp}, t_1|\psi > \\ & \left[\frac{2\pi i\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}}} \right]^{1/2} e^{\frac{i}{\hbar} I_{cl}[x_2, t_2; x_{sp}, t_1]}. \end{aligned} \tag{74.8}$$

We notice from (74.6) that the momentum at t_1 at the saddlepoint is zero. That is,

$$p_1|_{x_1 = x_{sp}} = 0. \tag{74.9}$$

However, for the paths that contribute significantly to the integral in (74.5), there is a range of momenta, namely

$$|p_1^2| < 2\hbar \left| \frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}} \right|, \tag{74.10}$$

consistent with (74.7) and the uncertainty relation.

As a check, using a special case, we consider the free-particle propagator [112, p. 42]

$$< x_2, t_2|x_1, t_1 >= \left[\frac{m}{2\pi i\hbar(t_2 - t_1)} \right]^{1/2} exp \left[\frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)} \right], \tag{74.11}$$

and we choose

$$< x_1, t_1|\psi >=< A, t_1|\psi > exp[-B(x_1 - A)^2] \tag{74.12}$$

to represent a broad initial wave function. For this case, the integral in (74.3) or (74.5) can be evaluated exactly to give

$$< x_2, t_2|\psi >=< A, t_1|\psi > \left[1 + \frac{2B\hbar(t_2 - t_1)}{im} \right]^{-1/2} exp \left[\frac{-B(x_2 - A)^2}{1 - \frac{2B\hbar(t_2 - t_1)}{im}} \right]. \tag{74.13}$$

The condition that the initial wave function is slowly varying is now

$$|B| \ll \left| \frac{m}{2\hbar(t_2 - t_1)} \right|, \tag{74.14}$$

so that (74.13) is approximately

$$< x_2, t_2|\psi >=< x_2, t_1|\psi > \tag{74.15}$$

in agreement with (74.8), since

$$I[x_2, t_2; x_1, t_1] = I_{cl}[x_2, t_2; x_1, t_1] = \frac{m}{2}\frac{(x_2 - x_1)^2}{t_2 - t_1}, \tag{74.16}$$

and

$$x_{sp} = x_2. \tag{74.17}$$

Notice that it is the sharply peaked action that determines which classical paths in (74.5) dominate the integral in this case, not the maximum of the initial wave function.

The calculations can clearly be generalized to two or three dimensions. The main point is that whenever the initial wave function is broad and the action of the classical propagator is not a sharply peaked function of x_1, some very different classical paths may contribute to the wave function in the final state, even when a semiclassical approximation is valid for the propagator.

When the classical action is sharply peaked as a function of the coordinates of the initial state, however, only a narrow range of classical paths contribute significantly to the wave function in the final state. This is thus a mechanism for selecting classical paths in wave mechanics. As we shall argue in the next sections, this principle has broader application.

74.3 Quantum cosmology

In the case of quantum cosmology, we have a formula analogous to (74.3) to give the wave function over 3-geometries g_2 and matter fields ϕ_2 on a 3-dimensional hypersurface S_2.

$$< g_2, \phi_2, S_2|\psi >= \int < g_2, \phi_2, S_2|g_1, \phi_1, S_1 >< g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1), \tag{74.18}$$

where $< g_1, \phi_1, S_1|\psi >$ is the wave function over 3-geometries g_1 and matter fields ϕ_1 on a 3-dimensional hypersurface S_1, and $< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >$ is the amplitude to go from a state with 3-geometry g_1 and matter fields ϕ_1 on a surface S_1 to a state with 3-geometry g_2 and matter fields ϕ_2 on a surface S_2 [123]. $D(g_1)$ and $D(\phi_1)$ are the measures on the 3-geometry and matter fields. The integration is over all initial 3-geometries g_1 and matter fields ϕ_1 for which the integral is defined.

74.4 Semiclassical approximation

As in section 2, we want to consider the case where the semiclassical approximation for the propagator is valid. That is, [160]

$$< g_2, \phi_2, S_2|g_1, \phi_1, S_1 >\approx f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{cl}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]}, \tag{74.19}$$

where the function outside of the exponential is a slowly varying function and I_{cl} is the action for a classical 4-geometry. Substituting (74.19) into (74.18) gives

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > \;&=\; \int f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}I_{cl}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]} \\ &\quad < g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1). \end{aligned} \tag{74.20}$$

Each value of the integrand in (74.20) corresponds to one classical 4-geometry. As in (74.5), there will be an infinite number of classical 4-geometries that contribute to each value of the final wave function. Here, however, we do not have only one single integration, but an infinite number of integrations, because the integration is carried out over all possible 3-geometries and all matter fields on the initial surface.

In the simple example in Section 2, there were two cases to consider for the single integration being carried out. In the first case, the classical action was not a sharply peaked function. In the second case, the classical action was a sharply peaked function so that a saddlepoint approximation could be applied to the integration. Following that strategy, we would need to consider those two cases for each of the infinite number of integrations in (74.20).

Here, however, we consider only the case where I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (74.20). We consider this case in the following section.

74.5 Saddlepoint approximation for the integral over initial states

We consider the case here where I_{cl} is a sharply peaked function of g_1 and matter fields ϕ_1 for each of the infinite number of integrations in (74.20). In that case, we can formally make the saddlepoint approximation for each of the integrations in (74.20). In analogy with (74.6), we have the saddlepoint condition

$$\frac{\partial}{\partial g_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\bigg|_{g_1=g_{sp}} = 0 \tag{74.21}$$

and

$$\frac{\partial}{\partial \phi_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]\bigg|_{\phi_1=\phi_{sp}} = 0, \tag{74.22}$$

where the derivatives in (74.21) and (74.22) are with respect to each parameter that defines the 3-geometry g_1 and matter fields ϕ_1. We consider the case where there is only one solution to the saddlepoint conditions (74.21) and (74.22). In that case, (74.21) selects a single classical 4-geometry. However, there will be a range of classical 4-geometries in the neighborhood that contribute significantly to the integral in (74.20). These are determined by (e.g. [186])

$$|I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] - I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]| < \hbar. \tag{74.23}$$

We can formally write the saddlepoint approximation to the integration in (74.20) as

$$\begin{aligned} < g_2, \phi_2, S_2|\psi > \quad = \quad & f(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) < g_{sp}, \phi_{sp}, S_1|\psi > \\ & f_1(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) e^{\frac{i}{\hbar} I_{cl}[g_2,\phi_2,S_2;g_{sp},\phi_{sp},S_1]} \end{aligned} \tag{74.24}$$

where classical 4-geometries that contribute significantly to (74.24) (through the function f_1) lie within a narrow range specified by (74.23).

Equation (74.21) requires that the momentum canonical to the initial 3-geometry for the classical 4-geometry at the saddlepoint be zero. That is

$$\pi^{ij}\Big|_{g_1=g_{sp}} = 0. \tag{74.25}$$

(The extrinsic curvature on S_1 will therefore also be zero at the saddlepoint.) However, there will be a range of initial canonical momenta and a range of initial 3-geometries corresponding to the range of classical 4-geometries that satisfy (74.23), so that the uncertainty relations between initial 3-geometries and their canonical momenta are satisfied.

Whether there is a narrow or broad range of classical 4-geometries that satisfy (74.23) depends on the second derivative of the action with respect to the initial 3-geometry.

We can take the action to be

$$I = \int (-g^{(4)})^{1/2}(L_{geom} + L_{matter})d^4x + \frac{1}{8\pi}\int (g^{(3)})^{1/2} K d^3x, \tag{74.26}$$

where [183, 123] show the importance of the surface term. [123] also points out a potential problem in that the action can be changed by conformal transformations, but suggests a solution.

$$K = g^{(3)ij}K_{ij} \tag{74.27}$$

is the trace of the extrinsic curvature. Although the extrinsic curvature is zero on S_1 at the saddlepoint, it will be nonzero in a region around the saddlepoint. The extrinsic curvature is given by

$$K_{ij} = -\frac{1}{2}\frac{\partial g_{ij}^{(3)}}{\partial t}, \tag{74.28}$$

where $g_{ij}^{(3)}$ is the 3-metric. In this example, we take the Lagrangian for the geometry as

$$L_{geom} = \frac{R^{(4)} - 2\Lambda}{16\pi}, \tag{74.29}$$

where $R^{(4)}$ is the four-dimensional scalar curvature, but we realize that a different Lagrangian might eventually be shown to be more appropriate in a correct theory of quantum gravity.

74.6 Perfect fluid models

For a perfect fluid, the energy momentum tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{74.30}$$

where p is the pressure, ρ is the density, and u is the 4-velocity. For solutions to Einstein's field equations for a perfect fluid, (74.29) becomes

$$L_{geom} = \frac{1}{2}\rho - \frac{3}{2}p + \frac{\Lambda}{8\pi}, \tag{74.31}$$

and we can take the Lagrangian for the matter as [161]

$$L_{matter} = \rho \tag{74.32}$$

Thus, the classical action for perfect fluids is

$$I_{cl} = \int (-g^{(4)})^{1/2}(\frac{3}{2}\rho - \frac{3}{2}p + \frac{\Lambda}{8\pi})d^4x - \frac{1}{16\pi}\int (g^{(3)})^{1/2} g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t}d^3x. \tag{74.33}$$

We can take

$$p = (\gamma - 1)\rho \tag{74.34}$$

for the equation of state, where $1 \leq \gamma < 2$. Then (74.33) becomes

$$I_{cl} = \frac{3}{2}\int (-g^{(4)})^{1/2}(2-\gamma)\rho d^4x + \frac{1}{8\pi}\int (-g^{(4)})^{1/2}\Lambda d^4x - \frac{1}{16\pi}\int (g^{(3)})^{1/2} g^{(3)ij}\frac{\partial g_{ij}^{(3)}}{\partial t}d^3x. \tag{74.35}$$

The integral in (74.35) is taken over the past light cone of the event for which we want to calculate an amplitude.

74.7 Spatially homogeneous spacetimes

The integration in (74.20) is an integration over functions g_1 and ϕ_1 defined on S_1. In that sense, it is similar to a path integral. For example, there are six independent functions that define g_1. As in the integration for a path integral, there are approximations that can be made to reduce the number of integrations that must be performed.

Here, we want to consider matter distributions similar to that observed, at least for the large scale in our universe. Thus, we want to restrict the integration in (74.20) to classical spatially homogeneous 4-geometries that have a homogeneous matter distribution in calculating the classical action in the exponential. The integration in (74.20) would then be over the 3-geometries that form the boundary of those 4-geometries on S_1.

As an example, we shall use Einstein's General Relativity for the classical 4-geometries, but the same calculations could be done for other classical gravitational theories, in case it turns out that General Relativity is not the correct theory of gravity. Thus, we want to consider the integration in (74.20) in which the classical 4-geometries used to calculate the action in the exponential are restricted to Bianchi cosmologies.

The appropriate calculation would be to consider the most general Bianchi model, with all of the parameters that describe that model, and carry out the integration over all of those parameters. We notice that the Bianchi parameters (which are time independent) define the initial three geometry, and therefore are valid integration variables in (74.20). On the other hand, if it is suspected that the saddlepoint for the integration will correspond to the Friedmann-Robertson-Walker (FRW) model, then one can restrict consideration to only those Bianchi models that include the FRW model as a special case, and consider integration in (74.20) for only one Bianchi parameter at a time, holding the others fixed at the FRW value. Here, we do that for only one of the Bianchi models for illustration.

In choosing which Bianchi model to use, we would like one that has a parameter that can be varied continuously to give the FRW model. In addition, we would like to choose a parameter that represents rotation of inertial frames relative to the matter distribution.

So far, I have not been able to find a completely satisfactory example. Although the Bianchi IX cosmology is often used to represent anisotropy, it seems inappropriate for the present case because it is only a superposition of gravitational waves on a Friedman-Robertson-Walker background.

The Bianchi VI_h model seems to be a better homogeneous model that has a parameter that represents an angular velocity of inertial frames relative to matter, and setting that parameter to zero seems to give the FRW metric. However, there seem to be some difficulties with the Bianchi VI_h model being able to change continuously into the FRW model, and also a possible problem with the topology. Until I find a better example, however, I shall use this one.

We use the solution for the Bianchi VI_h model from [163] with $h = -1/9$. This cosmological model is relevant here because it has a relative rotation of inertial frames with respect to the matter. Specifically,

$$\Omega(t) = \frac{b}{Y^2(t)Z(t)} \tag{74.36}$$

is the angular velocity in the rest frame of an observer moving with the fluid, of a set of Fermi-propagated axes with respect to a particular inertial triad. The parameter b is an arbitrary constant of the model, and is zero if and only if there is no rotation of inertial frames relative to matter. Thus, we are interested to know the dependence of the classical action on b. $Y(t)$ and $Z(t)$ are functions in the model.

74.8 Surface term

From (74.28) and [163, eq. 6.6], we have

$$K \equiv K_i^i = -\frac{1}{2} g^{(3)ij} \frac{\partial g_{ij}^{(3)}}{\partial t} = -\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z} = -3\frac{\dot{R}}{R}. \tag{74.37}$$

The R in (74.37) is defined by (74.108), and is not the same as the scalar curvature R in (74.29). This gives

$$R^3 K = -3R^2\dot{R}. \tag{74.38}$$

Then,

$$K = -\frac{3}{R^3} \int (R^2\dot{R})^{\cdot} dt. \tag{74.39}$$

Using (74.113) gives

$$K = -\frac{3}{R^3} \int R^3 \left[\Lambda + 4\pi(\rho - p) + \frac{2}{3}\frac{3a_0^2 + q_0^2}{X^2}\right] dt. \tag{74.40}$$

Equation (74.40) allows us to convert the surface integral in (74.26) to a volume integral so that it can be combined with the volume integral in the first term.

So, using $-g^{(4)} = g^{(3)}$ gives

$$I = \int (-g^{(4)})^{1/2} \left[L_{geom} + L_{matter} - \frac{3\Lambda}{8\pi} - \frac{3}{2}(\rho - p) - \frac{3a_0^2 + q_0^2}{4\pi X^2}\right] d^4x, \tag{74.41}$$

Using (74.31) and (74.32) gives

$$I = -\frac{1}{4\pi} \int (-g^{(4)})^{1/2} \left[\Lambda + \frac{3a_0^2 + q_0^2}{X^2}\right] d^4x, \tag{74.42}$$

From (74.28) and [163, eq. 6.6], we have

$$-g^{(4)} = g^{(3)} = X^2Y^2Z^2 e^{-4a_0x^1} = R^6 e^{-4a_0x^1}. \tag{74.43}$$

74.9 Coordinate system

The given coordinate system is not appropriate to integrate over the past light cone. We must change to a new coordinate system to conform to the light cone structure, with two of the coordinates being angles. In the new coordinate system (x'),

$$g'^{(4)} = [\text{determinant}\left(\frac{\partial x}{\partial x'}\right)]^2 g^{(4)}. \tag{74.44}$$

Thus,

$$(-g'^{(4)})^{1/2} = \text{determinant}\left(\frac{\partial x}{\partial x'}\right)(-g^{(4)})^{1/2}. \tag{74.45}$$

Thus,

$$(-g'^{(4)})^{1/2} = \text{determinant}\left(\frac{\partial x}{\partial x'}\right) R^3 e^{-2a_0x^1}. \tag{74.46}$$

Thus, (74.42) becomes

$$I = -\frac{1}{4\pi} \int \text{determinant}\left(\frac{\partial x}{\partial x'}\right) R^3 e^{-2a_0x^1} \left[\Lambda + \frac{3a_0^2 + q_0^2}{X^2}\right] d^4x', \tag{74.47}$$

Or,

$$I = -\frac{1}{4\pi}\int \text{determinant}\left(\frac{\partial x}{\partial x'}\right) e^{-2a_0 x^1} d^3x' \int R^3 \left[\Lambda + \frac{3a_0^2 + q_0^2}{X^2}\right] dt, \tag{74.48}$$

The integrations in the equations above should be over the past light cone. This model is spatially homogeneous. The next step is to change spatial variables to spherical polar coordinates centered at the apex of the light cone, that is centered at the place where the amplitude for something is being calculated (roughly here, now).

To do that, we first calculate null geodesics through the event in question. I have done that for this metric (with one approximation that neglects the off-diagonal terms in the metric) and have found the geodesics passing through the point with angles θ and ϕ. As expected, the velocity of light is independent of direction.

I have not yet calculated the formulas to make the coordinate transformation because the equations are very complicated. I plan to eventually do that, but in the meantime, I will assume that I can represent the metric by a simple geometry of constant spatial curvature on each spacelike hypersurface.

74.10 Light cone

For the purposes of calculating the integral over the past light cone, I assume we can represent the metric as the spatially constant metric of [209, p. 721]. I will check later to see if is is valid to do that, and correct that later if necessary.

$$(-g^{(4)})^{1/2} d^4x = \left(\frac{-6}{R^{(3)}}\right)^{3/2} \Sigma^2 \sin\theta dt d\chi d\theta d\phi, \tag{74.49}$$

where the condition that the integration is limited to being within the light cone gives

$$\chi(t) - \chi_2 = \int_{t_2}^{t} \left(\frac{-R^{(3)}}{6}\right)^{1/2} dt, \tag{74.50}$$

where t_2 is the time at the event for which we want to calculate the amplitude, and is at the apex of the past light cone.

The spatial scalar curvature is (for $k = -1$)

$$R^{(3)} = -2(q^2 + 3a^2) = -2(q_0^2 + 3a_0^2)/X^2 = -24a^2 = -24a_0^2/X^2. \tag{74.51}$$

In that case, (74.49) becomes

$$(-g^{(4)})^{1/2} d^4x = \left(\frac{3X^2}{3a_0^2 + q_0^2}\right)^{3/2} \Sigma^2 \sin\theta dt d\chi d\theta d\phi = \left(\frac{X}{2a_0}\right)^3 \Sigma^2 \sin\theta dt d\chi d\theta d\phi, \tag{74.52}$$

and (74.50) becomes

$$\chi(t) - \chi_2 = \left(\frac{q_0^2 + 3a_0^2}{3}\right)^{1/2} \int_{t_2}^{t} \frac{dt}{X} = 2a_0 \int_{t_2}^{t} \frac{dt}{X} = 2a_0 \int_{t_2}^{t} \frac{(1+\alpha)^{1/2} dt}{R}, \tag{74.53}$$

At this point, it is worthwhile to define some of the variables. R is proportional to the linear size of the universe. It would be zero at the initial singularity. So, R increases with cosmic time. χ, on the other hand is the radial coordinate for the past light cone. χ increases as R decreases.

$$\chi(R) - \chi_2 = 2a_0 \int_{R_2}^{R} \frac{(1+\alpha)^{1/2} dR}{R(dR/dt)}, \tag{74.54}$$

Or, using (74.127),

$$\chi(R) - \chi_2 = 2a_0 \int_{R_2}^{R} \frac{(1+\alpha)^{1/2} dR}{R\sqrt{\frac{\Lambda}{3}R^2 + \frac{8\pi\rho}{3}R^2 + 4a_0^2}}, \tag{74.55}$$

More specifically, for $R > R_1$,

$$\chi(R) - \chi_2 = 2a_0 \int_{R_2}^{R} \frac{(1+\alpha)^{1/2} dR}{R\sqrt{\frac{\Lambda}{3}R_{m2}^2 (R/R_{m2})^2 + 4a_0^2 (R/R_{m2})^{2-3\gamma_2} + 4a_0^2}}, \tag{74.56}$$

and, for $R < R_1$,

$$\begin{aligned}\chi(R) - \chi_2 &= 2a_0 \int_{R_2}^{R_1} \frac{(1+\alpha)^{1/2} dR}{R\sqrt{\frac{\Lambda}{3}R_{m2}^2 (R/R_{m2})^2 + 4a_0^2 (R/R_{m2})^{2-3\gamma_2} + 4a_0^2}} \\ &+ 2a_0 \int_{R_1}^{R} \frac{(1+\alpha)^{1/2} dR}{R\sqrt{\frac{\Lambda}{3}R_{m1}^2 (R/R_{m1})^2 + 4a_0^2 (R/R_{m1})^{2-3\gamma_1} + 4a_0^2}}, \end{aligned}\tag{74.57}$$

Or, for $R > R_1$,

$$\chi(R) - \chi_2 = \int_{R_2/R_{m2}}^{R/R_{m2}} \frac{(1+\alpha)^{1/2} dx}{x\sqrt{\frac{\Lambda}{12a_0^2}R_{m2}^2 x^2 + x^{2-3\gamma_2} + 1}}, \tag{74.58}$$

and, for $R < R_1$,

$$\chi(R) - \chi_2 = \int_{R_2/R_{m2}}^{R_1/R_{m2}} \frac{(1+\alpha)^{1/2} dx}{x\sqrt{\frac{\Lambda}{12a_0^2}R_{m2}^2 x^2 + x^{2-3\gamma_2} + 1}} + \int_{R_1/R_{m1}}^{R/R_{m1}} \frac{(1+\alpha)^{1/2} dx}{x\sqrt{\frac{\Lambda}{12a_0^2}R_{m1}^2 x^2 + x^{2-3\gamma_1} + 1}}, \tag{74.59}$$

Except for very late times, the second term under the radical in (74.58) and (74.59) will dominate. Therefore, we can neglect the other terms to give approximately for $R > R_1$,

$$\chi(R) - \chi_2 \approx \int_{R_2/R_{m2}}^{R/R_{m2}} x^{\frac{3}{2}\gamma_2 - 2}(1+\alpha)^{1/2} dx, \tag{74.60}$$

and, for $R < R_1$,

$$\chi(R) - \chi_2 \approx \int_{R_2/R_{m2}}^{R_1/R_{m2}} x^{\frac{3}{2}\gamma_2 - 2}(1+\alpha)^{1/2} dx + \int_{\frac{R_1}{R_{m1}}}^{\frac{R}{R_{m1}}} x^{\frac{3}{2}\gamma_1 - 2}(1+\alpha)^{1/2} dx, \tag{74.61}$$

for $R > R_1$,

$$\begin{aligned}\chi(R) - \chi_2 \approx\ & (1+\alpha_1)^{1/2} \int_{R_2/R_{m2}}^{R/R_{m2}} x^{\frac{3}{2}\gamma_2 - 2} \left\{ 1 + \frac{1}{3\gamma_2 - 6}\frac{R_1^3}{2a_0 R_{m2}^2}\frac{\dot{\alpha}_1}{1+\alpha_1}(x^{\frac{3}{2}\gamma_2 - 3} - x_1^{\frac{3}{2}\gamma_2 - 3}) \right. \\ & - \frac{6}{3\gamma_2 + 2}\frac{(3\gamma_2 - 6)x^{3\gamma_2 - 2} - (6\gamma_2 - 4)x_1^{\frac{3}{2}\gamma_2 + 1} x^{\frac{3}{2}\gamma_2 - 3} + (3\gamma_2 + 2)x_1^{3\gamma_2 - 2}}{(3\gamma_2 - 2)(3\gamma_2 - 6)} \\ & \left. + \frac{4}{3}\frac{b^2}{(2a_0)^2 R_{m2}^4}\frac{x^{3\gamma_2 - 6} - 2x_1^{\frac{3}{2}\gamma_2 - 3} x^{\frac{3}{2}\gamma_2 - 3} + x_1^{3\gamma_2 - 6}}{(\gamma_2 - 2)(3\gamma_2 - 6)} \right\} dx, \end{aligned}\tag{74.62}$$

and, for $R < R_1$,

$$\begin{aligned}\chi(R)-\chi_2 \quad &\approx \quad (1+\alpha_1)^{1/2}\int_{R_2/R_{m2}}^{R_1/R_{m2}} x^{\frac{3}{2}\gamma_2-2}\left\{1+\frac{1}{3\gamma_2-6}\frac{R_1^3}{2a_0R_{m2}^2}\frac{\dot{\alpha}_1}{1+\alpha_1}(x^{\frac{3}{2}\gamma_2-3}-x_1^{\frac{3}{2}\gamma_2-3})\right.\\
&-\frac{6}{3\gamma_2+2}\frac{(3\gamma_2-6)x^{3\gamma_2-2}-(6\gamma_2-4)x_1^{\frac{3}{2}\gamma_2+1}x^{\frac{3}{2}\gamma_2-3}+(3\gamma_2+2)x_1^{3\gamma_2-2}}{(3\gamma_2-2)(3\gamma_2-6)}\\
&\left.+\frac{4}{3}\frac{b^2}{(2a_0)^2R_{m2}^4}\frac{x^{3\gamma_2-6}-2x_1^{\frac{3}{2}\gamma_2-3}x^{\frac{3}{2}\gamma_2-3}+x_1^{3\gamma_2-6}}{(\gamma_2-2)(3\gamma_2-6)}\right\}dx\\
&+(1+\alpha_0)^{1/2}\int_{\frac{R_1}{R_{m1}}}^{\frac{R}{R_{m1}}} x^{\frac{3}{2}\gamma_1-2}\left\{1+\frac{1}{3\gamma_1-6}\frac{R_0^3}{2a_0R_{m1}^2}\frac{\dot{\alpha}_0}{1+\alpha_0}(x^{\frac{3}{2}\gamma_1-3}-x_0^{\frac{3}{2}\gamma_1-3})\right.\\
&-\frac{6}{3\gamma_1+2}\frac{(3\gamma_1-6)x^{3\gamma_1-2}-(6\gamma_1-4)x_0^{\frac{3}{2}\gamma_1+1}x^{\frac{3}{2}\gamma_1-3}+(3\gamma_1+2)x_0^{3\gamma_1-2}}{(3\gamma_1-2)(3\gamma_1-6)}\\
&\left.+\frac{4}{3}\frac{b^2}{(2a_0)^2R_{m1}^4}\frac{x^{3\gamma_1-6}-2x_0^{\frac{3}{2}\gamma_1-3}x^{\frac{3}{2}\gamma_1-3}+x_0^{3\gamma_1-6}}{(\gamma_1-2)(3\gamma_1-6)}\right\}dx, \qquad (74.63)\end{aligned}$$

Using $\gamma_1 = 4/3$ and $\gamma_2 = 1$ gives for $R > R_1$,

$$\begin{aligned}\chi(R)-\chi_2 \quad &\approx \quad (1+\alpha_1)^{1/2}\int_{R_2/R_{m2}}^{R/R_{m2}} x^{-\frac{1}{2}}\left\{1-\frac{R_1^3}{6a_0R_{m2}^2}\frac{\dot{\alpha}_1}{1+\alpha_1}(x^{-\frac{3}{2}}-x_1^{-\frac{3}{2}})\right.\\
&+\frac{2}{5}(-3x-2x_1^{5/2}x^{-\frac{3}{2}}+5x_1)\\
&\left.+\frac{b^2}{9a_0^2R_{m2}^4}(x^{-3}-2x_1^{-\frac{3}{2}}x^{-\frac{3}{2}}+x_1^{-3})\right\}dx, \qquad (74.64)\end{aligned}$$

and, for $R < R_1$,

$$\begin{aligned}\chi(R)-\chi_2 \quad &\approx \quad (1+\alpha_1)^{1/2}\int_{R_2/R_{m2}}^{R_1/R_{m2}} x^{-\frac{1}{2}}\left\{1-\frac{1}{3}\frac{R_1^3}{2a_0R_{m2}^2}\frac{\dot{\alpha}_1}{1+\alpha_1}(x^{-\frac{3}{2}}-x_1^{-\frac{3}{2}})\right.\\
&+\frac{2}{5}(-3x-2x_1^{5/2}x^{-\frac{3}{2}}+5x_1)\\
&\left.+\frac{b^2}{9a_0^2R_{m2}^4}(x^{-3}-2x_1^{-\frac{3}{2}}x^{-\frac{3}{2}}+x_1^{-3})\right\}dx\\
&+(1+\alpha_0)^{1/2}\int_{\frac{R_1}{R_{m1}}}^{\frac{R}{R_{m1}}}\left\{1-\frac{1}{2}\frac{R_0^3}{2a_0R_{m1}^2}\frac{\dot{\alpha}_0}{1+\alpha_0}(x^{-1}-x_0^{-1})\right.\\
&+\frac{1}{2}(-x^2-2x_0^3x^{-1}+3x_0^2)\\
&\left.+\frac{b^2}{(2a_0)^2R_{m1}^4}(x^{-2}-2x_0^{-1}x^{-1}+x_0^{-2})\right\}dx, \qquad (74.65)\end{aligned}$$

So, (74.42) becomes

$$I=-\int_{t_2}^{t_0}\left(\frac{-6}{R^{(3)}}\right)^{3/2}\left(\Lambda+\frac{3a_0^2+q_0^2}{X^2}\right)\left(\int\Sigma^2d\chi\right)dt, \qquad (74.66)$$

where, for the negative curvature case ($\Sigma = \sinh\chi$),

$$I=-\int_{t_2}^{t_0}\left(\frac{-6}{R^{(3)}}\right)^{3/2}\left(\Lambda+\frac{3a_0^2+q_0^2}{X^2}\right)\left(-\frac{\chi}{2}+\frac{\sinh 2\chi}{4}\right)dt, \qquad (74.67)$$

Using (74.51) gives (since $-R^{(3)} = 24a_0^2/X^2$)

$$I = -\int_{t_2}^{t_0} \frac{X^3}{8a_0^3}\left(\Lambda + \frac{12a_0^2}{X^2}\right)\left(-\frac{\chi}{2} + \frac{\sinh 2\chi}{4}\right) dt, \tag{74.68}$$

If we assume that the light-cone radial coordinate χ is composed of an isotropic part χ_{iso} plus a small anisotropic part χ_{an}, then

$$I = -\int_{t_2}^{t_0} \frac{X^3}{8a_0^3}\left(\Lambda + \frac{12a_0^2}{X^2}\right)\left(-\frac{1}{2}\chi_{iso} + \frac{1}{4}\sinh 2\chi_{iso} - \frac{1}{2}\chi_{an} + \frac{1}{4}\chi_{an}\cosh 2\chi_{iso}\right) dt, \tag{74.69}$$

If we further neglect Λ to get the main contribution, then we have

$$I = -\frac{3}{2a_0}\int_{t_2}^{t_0} X\left(-\frac{1}{2}\chi_{iso} + \frac{1}{4}\sinh 2\chi_{iso} - \frac{1}{2}\chi_{an} + \frac{1}{4}\chi_{an}\cosh 2\chi_{iso}\right) dt, \tag{74.70}$$

This gives

$$I = -\frac{3}{2a_0}\int_{R_2}^{R_0} \frac{R(1+\alpha)^{-\frac{1}{2}}}{\dot{R}}\left(-\frac{1}{2}\chi_{iso} + \frac{1}{4}\sinh 2\chi_{iso} - \frac{1}{2}\chi_{an} + \frac{1}{4}\chi_{an}\cosh 2\chi_{iso}\right) dR, \tag{74.71}$$

74.11 The old wrong way

There are some errors from here on that I have to correct because the following is based on a closed hypersphere instead of the past light cone.

Previously, I incorrectly derived the following formula, which I need to correct.

$$I_{cl} = \frac{3\pi^2}{4a_0}\int_{t_0}^{t} \frac{Y(t)Z(t)}{X(t)} dt + \Lambda part, \tag{74.72}$$

where the spatial part of the 4-volume integration has already been carried out, a_0 is a parameter of the model, t_0 corresponds to the surface S_1 in (74.18) and (74.20) and is enough larger than the Planck time T^* that the semiclassical approximation is valid, the upper limit in (74.72) corresponds to the surface S_2 in (74.18) and (74.20), and $X(t)$, $Y(t)$, and $Z(t)$ are functions of the model that must be determined by differential equations given by [163]. As expected, the surface term in (74.35) has canceled.

If we define

$$r^3(t) = X(t)Y(t)Z(t)\left(\frac{-3k}{3a_0^2 + q_0^2}\right)^{3/2} \tag{74.73}$$

and

$$1 + \alpha(t) = \frac{Y(t)^{2/3}Z(t)^{2/3}}{X(t)^{4/3}} \tag{74.74}$$

then the classical action in (74.72) becomes

$$I_{cl} = \frac{3\pi^2}{4}\left(\frac{3 + q_0^2/a_0^2}{-3k}\right)^{1/2}\int_{r_0}^{r} \frac{(1+\alpha)r}{\dot{r}} dr = \frac{3\pi^2}{4}\frac{2}{(-k)^{1/2}}\int_{r_0}^{r} \frac{(1+\alpha)r}{\dot{r}} dr, \tag{74.75}$$

where $\dot{r} = dr/dt$, r_0 is enough larger than the Planck length L^* that quantum effects can be neglected, and $k = +1$ for a closed universe.

For the $h = -1/9$ case, we have

$$q_0 = -3a_0 \tag{74.76}$$

if and only if $b \neq 0$. Substituting (74.76) into (74.75) gives

$$I_{cl} = \frac{3\pi^2}{2}\left(\frac{-1}{k}\right)^{1/2}\int_{r_0}^{r}\frac{(1+\alpha)r}{\dot{r}}dr, \tag{74.77}$$

The form of the equation of state in (74.34) allows one of the differential equations for the model to be integrated in closed form to give

$$8\pi\rho = 3r_m^{3\gamma-2}r^{-3\gamma} \tag{74.78}$$

where r_m is a constant of integration that depends on the amount of matter in the universe and the speed of expansion relative to the gravitational attraction. Equation (74.78) shows that r_m is a measure of the amount of matter in the universe for a given value of r.

Using (74.78), we have

$$\dot{r}^2 = \frac{1}{3}\Lambda r^2 + \left(\frac{r_m}{r}\right)^{3\gamma-2} - k - k\alpha - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{r^2}{12}\left(\frac{\dot{\alpha}}{1+\alpha}\right)^2. \tag{74.79}$$

For the isotropic case, only the first three terms on the right hand side of (74.79) are nonzero. r_m is the value of r where the second two terms are equal in magnitude. For a closed universe for the isotropic case, r_m is the maximum value of r when Λ is zero.

Equation (74.79) can be written

$$\dot{r} = \sqrt{\frac{1}{3}\Lambda r^2 + \left(\frac{r}{r_m}\right)^{2-3\gamma} - k(1+\alpha) - \frac{k}{(2a_0)^6}\frac{b^2}{3(1+\alpha)^2r^4} + \frac{V^2}{3r^4}}, \tag{74.80}$$

and the remaining differential equations to solve are

$$\dot{V} = 3k(1+\alpha)r - \frac{k}{(2a_0)^6}\frac{2b^2}{(1+\alpha)^2r^3} \tag{74.81}$$

and

$$\frac{\dot{\alpha}}{1+\alpha} = \frac{2V}{r^3}, \tag{74.82}$$

where (74.82) is the definition of $V(t)$, and $r(t)$ represents the size of the universe. The angular rotation (74.36) of an inertial frame relative to local matter is given by

$$\Omega(t) = \left(\frac{-k}{4a_0^2}\right)^{3/2}\frac{b}{[1+\alpha(t)]r^3(t)} \tag{74.83}$$

It is not possible to solve the differential equations exactly in closed form, but we can find approximate solutions. In the limit as b approaches zero, it is valid to neglect all but the second term under the radical in (74.80). The appendix then gives

$$\begin{aligned} 1+\alpha \quad \approx \quad & (1+\alpha_0)exp\left\{\frac{2r_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3} - r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma} + \right. \\ & 12k\frac{(3\gamma-6)r^{3\gamma-2} - (6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3} + (3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}} \\ & \left. -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6} - 2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3} + r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\right\} \end{aligned}$$

$$
\begin{aligned}
&\approx (1+\alpha_0)\left\{1+\frac{2r_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3}-r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma}+\right.\\
&12k\frac{(3\gamma-6)r^{3\gamma-2}-(6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3}+(3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\\
&\left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6}-2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3}+r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\right\}\\
&\approx (1+\alpha_0)\{1+\\
&12k\frac{r^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)r_m^{3\gamma-2}}\\
&\left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\right\}. \qquad (74.84)
\end{aligned}
$$

This solution is valid for r smaller than r_m if b is small enough.

We assume that γ changes from γ_1 to γ_2 at $r=r_1$. To satisfy continuity of ρ at $r=r_1$, we must have r_m change from r_{m1} to r_{m2} at $r=r_1$, where

$$
\frac{r_{m1}}{r_{m2}}=\left(\frac{r_1}{r_{m2}}\right)^{\frac{3\gamma_1-3\gamma_2}{3\gamma_1-2}}. \qquad (74.85)
$$

Equation (74.84) applies for $r\le r_1$ with $\gamma=\gamma_1$ and $r_m=r_{m1}$.

For $r>r_1$, (74.84) gives

$$
\begin{aligned}
1+\alpha &\approx (1+\alpha_1)exp\left\{\frac{2r_1^3}{3\gamma_2-6}\frac{\dot{\alpha}_1}{1+\alpha_1}(r^{\frac{3}{2}\gamma_2-3}-r_1^{\frac{3}{2}\gamma_2-3})r_{m2}^{1-\frac{3}{2}\gamma_2}+\right.\\
&12k\frac{(3\gamma_2-6)r^{3\gamma_2-2}-(6\gamma_2-4)r_1^{\frac{3}{2}\gamma_2+1}r^{\frac{3}{2}\gamma_2-3}+(3\gamma_2+2)r_1^{3\gamma_2-2}}{(3\gamma_2+2)(3\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}}\\
&\left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma_2-6}-2r_1^{\frac{3}{2}\gamma_2-3}r^{\frac{3}{2}\gamma_2-3}+r_1^{3\gamma_2-6}}{(\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}}\right\}\\
&\approx (1+\alpha_1)\left\{1+\frac{2r_1^3}{3\gamma_2-6}\frac{\dot{\alpha}_1}{1+\alpha_1}(r^{\frac{3}{2}\gamma_2-3}-r_1^{\frac{3}{2}\gamma_2-3})r_{m2}^{1-\frac{3}{2}\gamma_2}+\right.\\
&12k\frac{(3\gamma_2-6)r^{3\gamma_2-2}-(6\gamma_2-4)r_1^{\frac{3}{2}\gamma_2+1}r^{\frac{3}{2}\gamma_2-3}+(3\gamma_2+2)r_1^{3\gamma_2-2}}{(3\gamma_2+2)(3\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}}\\
&\left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma_2-6}-2r_1^{\frac{3}{2}\gamma_2-3}r^{\frac{3}{2}\gamma_2-3}+r_1^{3\gamma_2-6}}{(\gamma_2-2)(3\gamma_2-6)r_{m2}^{3\gamma_2-2}}\right\}\\
&\approx (1+\alpha_1), \qquad (74.86)
\end{aligned}
$$

where continuity of α and $\dot{\alpha}$ at $r=r_1$ requires that

$$
\begin{aligned}
1+\alpha_1 &\approx (1+\alpha_0)exp\left\{\frac{2r_0^3}{3\gamma_1-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r_1^{\frac{3}{2}\gamma_1-3}-r_0^{\frac{3}{2}\gamma_1-3})r_{m1}^{1-\frac{3}{2}\gamma_1}+\right.\\
&12k\frac{(3\gamma_1-6)r_1^{3\gamma_1-2}-(6\gamma_1-4)r_0^{\frac{3}{2}\gamma_1+1}r_1^{\frac{3}{2}\gamma_1-3}+(3\gamma_1+2)r_0^{3\gamma_1-2}}{(3\gamma_1+2)(3\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_1^{3\gamma_1-6}-2r_0^{\frac{3}{2}\gamma_1-3}r_1^{\frac{3}{2}\gamma_1-3}+r_0^{3\gamma_1-6}}{(\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}}\Bigg\} \\
\approx\ & (1+\alpha_0)\left\{1+\frac{2r_0^3}{3\gamma_1-6}\frac{\dot\alpha_0}{1+\alpha_0}(r_1^{\frac{3}{2}\gamma_1-3}-r_0^{\frac{3}{2}\gamma_1-3})r_{m1}^{1-\frac{3}{2}\gamma_1}+\right. \\
& 12k\frac{(3\gamma_1-6)r_1^{3\gamma_1-2}-(6\gamma_1-4)r_0^{\frac{3}{2}\gamma_1+1}r_1^{\frac{3}{2}\gamma_1-3}+(3\gamma_1+2)r_0^{3\gamma_1-2}}{(3\gamma_1+2)(3\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}} \\
& \left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_1^{3\gamma_1-6}-2r_0^{\frac{3}{2}\gamma_1-3}r_1^{\frac{3}{2}\gamma_1-3}+r_0^{3\gamma_1-6}}{(\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}}\right\} \\
\approx\ & (1+\alpha_0)\{1+ \\
& 12k\frac{r_1^{3\gamma_1-2}}{(3\gamma_1+2)(3\gamma_1-2)r_{m1}^{3\gamma_1-2}} \\
& -\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}}{(\gamma_1-2)(3\gamma_1-6)r_{m1}^{3\gamma_1-2}}\Bigg\}.
\end{aligned}
\tag{74.87}
$$

We now perform the integration in (74.77). For $r<r_1$, the result is

$$
\begin{aligned}
\frac{I_{cl}}{\hbar} =\ & \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m1}}{L^*}\right)^2(1+\alpha_0)\left\{\frac{2}{3\gamma_1+2}\left[\left(\frac{r}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]\right. \\
& +\frac{24k\left[\left(\frac{r}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}\right]}{(3\gamma_1+2)(3\gamma_1-2)(9\gamma_1-2)} \\
& \left.-\frac{8k}{3}\frac{b^2r_{m1}^{-4}}{(2a_0)^6}\frac{\left[\left(\frac{r}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]}{(\gamma_1-2)^2 3(3\gamma_1+2)}2\left(\frac{r_0}{r_{m1}}\right)^{3\gamma_1-6}\right\}.
\end{aligned}
\tag{74.88}
$$

And for $r>r_1$, the result is

$$
\begin{aligned}
\frac{I_{cl}}{\hbar} =\ & \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\left\{\frac{2}{3\gamma_1+2}\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]\left(\frac{r_{m1}}{r_{m2}}\right)^2\right. \\
& +\frac{24k\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{9}{2}\gamma_1-1}\right]}{(3\gamma_1+2)(3\gamma_1-2)(9\gamma_1-2)}\left(\frac{r_{m1}}{r_{m2}}\right)^2 \\
& -\frac{8k}{3}\frac{b^2r_{m1}^{-4}}{(2a_0)^6}\frac{\left[\left(\frac{r_1}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}-\left(\frac{r_0}{r_{m1}}\right)^{\frac{3}{2}\gamma_1+1}\right]}{(\gamma_1-2)^2 3(3\gamma_1+2)}2\left(\frac{r_0}{r_{m1}}\right)^{3\gamma_1-6}\left(\frac{r_{m1}}{r_{m2}}\right)^2 \\
& +\left[1+12k\frac{r_1^{3\gamma_1-2}r_{m1}^{2-3\gamma_1}}{(3\gamma_1+2)(3\gamma_1-2)}-\frac{8k}{3}\frac{b^2r_{m1}^{-4}}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}r_{m1}^{6-3\gamma_1}}{(\gamma_1-2)(3\gamma_1-6)}\right] \\
& \left.\frac{2}{3\gamma_2+2}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}-\left(\frac{r_1}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1}\right]\right\}.
\end{aligned}
\tag{74.89}
$$

Equations (74.88) and (74.89) neglect all but the second term under the radical in (74.80). In making the calculation, I actually included the other terms under the radical to first order, but then determined after the integration that they could be neglected.

Neglecting some small terms gives for $r > r_1$

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\frac{2}{3\gamma_2+2}\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1} \\ &\left[1-\frac{8k}{3}\frac{b^2 r_{m1}^{-4}}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}r_{m1}^{6-3\gamma_1}}{(\gamma_1-2)(3\gamma_1-6)}\right]. \end{aligned} \tag{74.90}$$

Substituting (74.85) into (74.90) gives for $r > r_1$

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\frac{2}{3\gamma_2+2}\left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}\gamma_2+1} \\ &\left[1-\frac{8k}{3}\left(\frac{r_1}{r_{m2}}\right)^{3\gamma_2-3\gamma_1}\frac{b^2 r_{m2}^{-4}}{(2a_0)^6}\frac{r_0^{3\gamma_1-6}r_{m2}^{6-3\gamma_1}}{(\gamma_1-2)(3\gamma_1-6)}\right]. \end{aligned} \tag{74.91}$$

To get a rough estimate, we take

$$\gamma_1 = 4/3 \tag{74.92}$$

to represent a relativistic early universe and

$$\gamma_2 = 1 \tag{74.93}$$

to represent a matter-dominated late universe. Substituting (74.92) and (74.93) into (74.89) gives for $r > r_1$

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3\pi^2}{2}(-k)^{1/2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\left\{\frac{1}{3}\left[\left(\frac{r_1}{r}\right)^3\left(\frac{r}{r_{m1}}\right)^3-\left(\frac{r_0}{r_{m1}}\right)^3\right]\left(\frac{r_{m1}}{r_{m2}}\right)^2\right. \\ &+\frac{2}{5}\left[\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}-\left(\frac{r_1}{r}\right)^{5/2}\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}\right]\left[1+\left(\frac{r_1}{r}\right)^2 k\left(\frac{r}{r_{m1}}\right)^2\right] \\ &+\frac{k}{5}\left[\left(\frac{r_1}{r}\right)^5\left(\frac{r}{r_{m1}}\right)^5-\left(\frac{r_0}{r_{m1}}\right)^5\right]\left(\frac{r_{m1}}{r_{m2}}\right)^2 \\ &-\frac{2kb^2}{(2a_0)^6 r_{m1}^2 r_0^2} \\ &\left[\frac{1}{3}\left(\frac{r_{m1}}{r_{m2}}\right)^2\left(\left(\frac{r_1}{r}\right)^3\left(\frac{r}{r_{m1}}\right)^3-\left(\frac{r_0}{r_{m1}}\right)^3\right)\right. \\ &\left.\left.+\ \frac{2}{5}\left(\frac{r}{r_{m2}}\right)^{5/2}\left(1-\left(\frac{r_1}{r}\right)^{5/2}\right)\right]\right\}. \end{aligned} \tag{74.94}$$

Neglecting some small terms, letting $k = +1$ for a closed universe, and using (74.85) gives for $r > r_1$

$$\begin{aligned}\frac{I_{cl}}{\hbar} &= \frac{3i\pi^2}{2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0)\left\{\frac{1}{3}\left(\frac{r_1}{r}\right)^3\left(\frac{r}{r_{m2}}\right)^{5/2}\left(\frac{r}{r_1}\right)^{1/2}\right. \\ &+\frac{2}{5}\left[1+\left(\frac{r_1}{r}\right)^2\left(\frac{r}{r_{m2}}\right)^2\frac{r_{m2}}{r_1}\right]\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}+\frac{1}{5}\left(\frac{r_1}{r}\right)^5\left(\frac{r}{r_{m2}}\right)^{7/2}\left(\frac{r}{r_1}\right)^{3/2} \\ &\left.-\ \frac{2}{5}\frac{2b^2}{(2a_0)^6 r_{m2}^2 r_0^2}\left(\frac{r}{r_1}\right)\left(\frac{r}{r_{m2}}\right)^{3/2}\right\} \\ &\approx \frac{3i\pi^2}{2}\left(\frac{r_{m2}}{L^*}\right)^2(1+\alpha_0) \\ &\left\{\frac{2}{5}\left(\frac{r}{r_{m2}}\right)^{\frac{5}{2}}-\frac{2}{5}\frac{2b^2}{(2a_0)^6 r_{m2}^2 r_0^2}\left(\frac{r}{r_1}\right)\left(\frac{r}{r_{m2}}\right)^{3/2}\right\}. \end{aligned} \tag{74.95}$$

Because the parameter b is an initial value for the cosmology, it is one of the variables of integration in (74.20). In making the saddlepoint approximation for that integration, we need to locate the saddlepoint (that is, the value of b that makes the action in (74.90), (74.91), or (74.95) stationary. We see that the action is stationary with respect to variation of b at the isotropic case of $b = 0$, as expected. The range of values of b that contribute significantly to the integral in (74.20) is given by (74.23). That is

$$\left| \frac{I_{cl}(b)}{\hbar} - \frac{I_{cl}(b=0,r)}{\hbar} \right| < 1. \tag{74.96}$$

Thus, substituting (74.91) into (74.96) gives

$$\frac{3\pi^2}{2} \left(\frac{r_{m2}}{L^*}\right)^2 \frac{(1+\alpha_0)b^2}{(2a_0)^6 r_{m2}^4} \left(\frac{r}{r_{m2}}\right)^{\frac{9}{2}\gamma_2 - 3\gamma_1 + 1}$$
$$\frac{16}{(\gamma_1 - 2)^2 9(3\gamma_2 + 2)} \left(\frac{r}{r_1}\right)^{3(\gamma_1 - \gamma_2)} \left(\frac{r_{m2}}{r_0}\right)^{6 - 3\gamma_1} < 1. \tag{74.97}$$

The approximations made so far are valid whenever (74.97) holds.

Substituting (74.92) and (74.93) into (74.97) gives

$$\frac{6\pi^2}{5}(1+\alpha_0) \left(\frac{r}{r_1}\right) \left(\frac{r}{r_{m2}}\right)^{\frac{3}{2}} \left(\frac{L^*}{r_0}\right)^2 \frac{b^2}{(2a_0)^6 L^{*4}} < 1. \tag{74.98}$$

This gives

$$b < \frac{\sqrt{5}(2a_0)^3}{\pi\sqrt{6}} \left(\frac{r_1}{r}\right)^{1/2} \frac{L^{*2}}{(1+\alpha_0)^{1/2}} \left(\frac{r_{m2}}{r}\right)^{\frac{3}{4}} \left(\frac{r_0}{L^*}\right). \tag{74.99}$$

Thus, from (74.83), the rotation rate of inertial frames is

$$|\Omega(t)| < \frac{\sqrt{5}}{\pi\sqrt{6}} \left(\frac{L^*}{r_{m2}}\right)^2 \frac{1}{(1+\alpha_0)^{1/2}[1+\alpha(t)]r_{m2}} \left(\frac{r_{m2}}{r(t)}\right)^{\frac{15}{4}} \left(\frac{r_1}{r}\right)^{1/2} \left(\frac{r_0}{L^*}\right). \tag{74.100}$$

If we now take the Planck length L^* to be $1.6 \times 10{-}33$ cm, use the Hubble distance of 1.7×10^{28} cm for r_{m2}, a tenth of that for r, neglect α and α_0 compared to 1, choose $r_0 = 10L^*$, and take

$$r_1 = \frac{r}{100} \tag{74.101}$$

as an estimate that the universe changed from radiation-dominated to matter-dominated when the universe was about one-hundredth of its present size [13, Section 15.3, p. 481], then we get

$$|\Omega| < 8 \times 10^{-130} \approx 10^{-129} \text{ radians per year,} \tag{74.102}$$

which is much less than the bound set by experiment of 10^{-14} to 7×10^{-17} radians per year if the universe is spatially closed [164].

The rotation rate in (74.100) and (74.102) is so small because the Planck length is so much smaller than the Hubble distance.

That the small value of allowed rotation rate depends mostly on the universe being much larger than a Planck length rather than on details of the model suggests that the result has some generality.

We notice also, that the selection criterion in (74.96) is so sharp that the initial wave function in the integration in (74.20) would have to be very sharply peaked to overcome it.

74.12 Discussion

We see that considerations of quantum cosmology show how a range of classical cosmologies can be selected that contribute significantly to the wave function in the final state. The effect enters through the action. Using semiclassical calculations gives results that should not depend on particular details of the theory of quantum gravity.

For our universe (which is much larger than the Planck length) the selection is very sharp. The initial wave function over 3-geometries would have to be extremely sharp (not a probable occurrence) to dominate over the effect of the action.

The selection process seems to occur very soon in the development of a cosmology. That is, for a broad wave function over 3-geometries in the initial state, the wave function becomes sharply peaked after the universe has become a few orders of magnitude larger than the Planck length.

A different choice than (74.32) [162] is

$$L_{matter} = p. \tag{74.103}$$

The choice in (74.103) gives a third of (74.33) for the total action. A correct theory of quantum gravity will determine which (if either) of these two choices is correct, but for this illustration, a factor of three in the action makes little difference.

It appears likely now, however, that there is not enough matter to keep our universe from expanding forever (e.g. [189, 210, 211, 212, 213]). To accommodate that with a spatially closed universe within General Relativity would require a positive cosmological constant (e.g. [190]). I shall try to include the cosmological constant in a future calculation.

74.13 Appendix: Bianchi VI_h Models

We start with Equations (6.7a) of [163]. These can be written as

$$\frac{\dot\rho}{\rho + p} = -\frac{3\dot R}{R} \tag{74.104}$$

$$\Lambda + 4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot X}{X}\right)^{\cdot} - \frac{2(a_0^2 + q_0^2)}{X^2} + \frac{2b^2}{Y^4Z^2} \tag{74.105}$$

$$\Lambda + 4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot Y}{Y}\right)^{\cdot} - \frac{2(a_0^2 + a_0q_0)}{X^2} - \frac{2b^2}{Y^4Z^2} \tag{74.106}$$

$$\Lambda + 4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot Z}{Z}\right)^{\cdot} - \frac{2(a_0^2 - a_0q_0)}{X^2}, \tag{74.107}$$

where

$$R(t)^3 = X(t)Y(t)Z(t), \tag{74.108}$$

R is not the same R as in (74.29), b, a_0, and q_0 are constants that are parameters of the model, and the variation of ρ, X, Y, and Z with time is determined by (74.104) through (74.107).

A first integral is

$$\begin{aligned}\Lambda + 8\pi\rho + \tfrac{3a_0^2+q_0^2}{X^2} &= \tfrac{1}{2}\left[9\left(\tfrac{\dot R}{R}\right)^2 - \left(\tfrac{\dot X}{X}\right)^2 - \left(\tfrac{\dot Y}{Y}\right)^2 - \left(\tfrac{\dot Z}{Z}\right)^2\right] - \tfrac{b^2}{Y^4Z^2}\\ -2a_0R^{-6}[(3a_0 + q_0)\textstyle\int Y^2Z^2 d\ln(Z/Y) &+ 2a_0\textstyle\int Y^2Z^2 d\ln(XY/Z^2)],\end{aligned} \tag{74.109}$$

which is the generalized Friedmann equation. For the Bbii case, $b \neq 0$ implies that $q_0 + 3a_0 = 0$ and XY/Z^2 is constant, so that the generalized Friedmann equation (74.109) becomes

$$\Lambda + 8\pi\rho + \frac{3a_0^2 + q_0^2}{X^2} = \frac{1}{2}\left[6\left(\frac{\dot{R}}{R}\right)^2 - 2\left(\frac{\dot{R}}{R} - \frac{\dot{X}}{X}\right)^2\right] - \frac{b^2}{Y^4 Z^2}. \tag{74.110}$$

The last term of (74.110) is minus half of that in equation (6.7b) of [163], but direct calculation verifies that (74.110) is consistent with (74.104) through (74.107)[11].

We can add (74.105), (74.106), and (74.107) with coefficients A, B, and C to give

$$(A+B+C)\left[\Lambda + 4\pi(\rho - p) + \frac{2a_0^2}{X^2}\right] = R^{-3}\left(R^3\frac{\dot{U}}{U}\right)^{\cdot} - (B-C)\frac{2a_0 q_0}{X^2} - 2A\frac{q_0^2}{X^2} + (A-B)\frac{2b^2}{Y^4 Z^2}, \tag{74.111}$$

where

$$U \equiv X^A Y^B Z^C = X^{A-B} R^{2B+C}. \tag{74.112}$$

We can take special cases of A, B, and C. Taking $A = B = C = 1/3$ in (74.111) gives

$$\Lambda + 4\pi(\rho - p) = R^{-3}(R^2\dot{R})^{\cdot} - \frac{2}{3}\frac{3a_0^2 + q_0^2}{X^2}, \tag{74.113}$$

where I have used (74.108). For another special case, we take $A = B = 1$ and $C = -2$ in (74.111) to give

$$R^{-3}\left(R^3\frac{\dot{U}}{U}\right)^{\cdot} = -2q_0\frac{3a_0 + q_0}{X^2} = 0, \tag{74.114}$$

where

$$U \equiv \frac{XY}{Z^2} = \text{constant} = 1. \tag{74.115}$$

For a third special case, we take $A = -4/3$ and $B = C = 2/3$ in (74.111) to give

$$R^{-3}\left(R^3\frac{\dot{\alpha}}{1+\alpha}\right)^{\cdot} = \frac{4b^2}{Y^4 Z^2} - \frac{8q_0^2}{3X^2}, \tag{74.116}$$

where

$$1 + \alpha \equiv \frac{Y^{2/3}Z^{2/3}}{X^{4/3}} = \frac{R^2}{X^2}. \tag{74.117}$$

We can use (74.108), (74.115), and (74.117) to determine X, Y, and Z in terms of R, U, and α. This gives

$$X = \frac{R}{(1+\alpha)^{1/2}}, \tag{74.118}$$

$$Y = U^{1/3}(1+\alpha)^{1/2}R = (1+\alpha)^{1/2}R, \tag{74.119}$$

and

$$Z = \frac{R}{U^{1/3}} = R. \tag{74.120}$$

Using (74.118), (74.119), and (74.120) in (74.113), (74.114), (74.116), and (74.110) gives

$$R^{-3}\left(R^2\dot{R}\right)^{\cdot} = \Lambda + 4\pi(\rho - p) + \frac{8a_0^2(1+\alpha)}{R^2}, \tag{74.121}$$

[11] Note added 16 May 2012: MacCallum[191, Note 3, page 60] points out that the last term in (74.110) is the correct term.

$$R^{-3}\left(R^3\frac{\dot{U}}{U}\right)^{\cdot} = -2q_0\frac{(3a_0+q_0)(1+\alpha)}{R^2} = 0, \tag{74.122}$$

$$R^{-3}\left(R^3\frac{\dot{\alpha}}{1+\alpha}\right)^{\cdot} = \frac{4b^2}{(1+\alpha)^2U^{2/3}R^6} - \frac{8q_0^2(1+\alpha)}{3R^2} = \frac{4b^2}{(1+\alpha)^2R^6} - \frac{24a_0^2(1+\alpha)}{R^2}, \tag{74.123}$$

and the generalized Friedmann equation becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4a_0^2}{X^2} + \frac{b^2X^4}{3R^{10}} + \frac{1}{3}\left(\frac{\dot{R}}{R} - \frac{\dot{X}}{X}\right)^2. \tag{74.124}$$

Or,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4a_0^2(1+\alpha)}{R^2} + \frac{b^2}{3(1+\alpha)^2R^6} + \frac{1}{12}\left(\frac{\dot{\alpha}}{1+\alpha}\right)^2. \tag{74.125}$$

Or,

$$\dot{R} = \sqrt{\frac{\Lambda}{3}R^2 + \frac{8\pi\rho}{3}R^2 + 4a_0^2(1+\alpha) + \frac{b^2}{3(1+\alpha)^2R^4} + \frac{1}{12}\left(\frac{\dot{\alpha}}{1+\alpha}\right)^2R^2}. \tag{74.126}$$

For very small anisotropy, we can neglect the anisotropy in this place. We can check this approximation later. So,

$$\dot{R} \approx \sqrt{\frac{\Lambda}{3}R^2 + \frac{8\pi\rho}{3}R^2 + 4a_0^2}. \tag{74.127}$$

Or, using (74.145) gives

$$\dot{R} \approx \sqrt{\frac{\Lambda}{3}R_m^2(R/R_m)^2 + 4a_0^2(R/R_m)^{2-3\gamma} + 4a_0^2}. \tag{74.128}$$

More specifically, for $R < R_1$,

$$\dot{R} \approx \sqrt{\frac{\Lambda}{3}R_{m1}^2(R/R_{m1})^2 + 4a_0^2(R/R_{m1})^{2-3\gamma_1} + 4a_0^2}. \tag{74.129}$$

and, for $R > R_1$,

$$\dot{R} \approx \sqrt{\frac{\Lambda}{3}R_{m2}^2(R/R_{m2})^2 + 4a_0^2(R/R_{m2})^{2-3\gamma_2} + 4a_0^2}. \tag{74.130}$$

If we change variables from R to r, where

$$r = \sqrt{\frac{-3k}{3a_0^2+q_0^2}}R = \frac{(-k)^{1/2}}{2a_0}R = \frac{R}{2a_0}, \tag{74.131}$$

(and where k is +1 for a closed universe and -1 for an open universe), then (74.121) through (74.125) become

$$r^{-3}(r^2\dot{r})^{\cdot} = \Lambda + 4\pi(\rho - p) - \frac{2k(1+\alpha)}{r^2}, \tag{74.132}$$

$$r^{-3}\left(r^3\frac{\dot{U}}{U}\right)^{\cdot} = \frac{6kq_0(3a_0+q_0)(1+\alpha)}{(3a_0^2+q_0^2)r^2} = 0, \tag{74.133}$$

$$r^{-3}\left(r^3\frac{\dot{\alpha}}{1+\alpha}\right)^{\cdot} = \left(\frac{-k}{4a_0^2}\right)^3\frac{4b^2}{(1+\alpha)^2r^6} + \frac{8kq_0^2(1+\alpha)}{(3a_0^2+q_0^2)r^2}, \tag{74.134}$$

and

$$\begin{aligned}\left(\frac{\dot r}{r}\right)^2 &= \frac{1}{3}\left[\left(\frac{1}{2}\frac{\dot\alpha}{1+\alpha}\right)^2\right] \\ &\quad +\frac{\Lambda}{3}+\frac{8\pi\rho}{3}-\frac{k(1+\alpha)}{r^2}+\left(\frac{-k}{4a_0^2}\right)^3\frac{b^2}{3(1+\alpha)^2r^6}. \end{aligned} \tag{74.135}$$

If we define W, V, and K by

$$r^2\dot r = W = \left(\frac{-3k}{3a_0^2+q_0^2}\right)^{3/2} R^2\dot R = \frac{(-k)^{3/2}}{8a_0^3}R^2\dot R = (-k)^{3/2}\frac{R^2\dot R}{8a_0^3}, \tag{74.136}$$

$$\frac{\dot\alpha}{1+\alpha} = \frac{2V}{r^3} = \frac{2V}{R^3}\left(\frac{3a_0^2+q_0^2}{-3k}\right)^{3/2} = \frac{2V}{R^3}\frac{8a_0^3}{(-k)^{3/2}} \tag{74.137}$$

and

$$\frac{\dot U}{U} = \frac{3K}{r^3} = 0, \tag{74.138}$$

then (74.132) through (74.135) become

$$\dot W = \Lambda r^3 + 4\pi(\rho-p)r^3 - 2k(1+\alpha)r, \tag{74.139}$$

$$\dot K = \frac{2kq_0(3a_0+q_0)(1+\alpha)r}{3a_0^2+q_0^2} = 0, \tag{74.140}$$

$$\dot V = \left(\frac{-3k}{3a_0^2+q_0^2}\right)^3\frac{2b^2}{(1+\alpha)^2r^3} + \frac{4kq_0^2(1+\alpha)r}{3a_0^2+q_0^2}, \tag{74.141}$$

$$\begin{aligned} W^2 &= \frac{V^2}{3}+\frac{\Lambda r^6}{3}+\frac{8\pi\rho r^6}{3} \\ &\quad -k(1+\alpha)r^4+\left(\frac{-k}{4a_0^2}\right)^3\frac{b^2}{3(1+\alpha)^2}. \end{aligned} \tag{74.142}$$

If we take

$$p = (\gamma-1)\rho \tag{74.143}$$

for the equation of state, where

$$1 \le \gamma < 2 \tag{74.144}$$

is a constant, then (74.104) can be integrated to give

$$8\pi\rho = 3r_m^{3\gamma-2}r^{-3\gamma} = (3a_0^2+q_0^2)R_m^{3\gamma-2}R^{-3\gamma} = 12a_0^2R_m^{3\gamma-2}R^{-3\gamma} \tag{74.145}$$

where r_m is a constant of integration. Substituting (74.143) and (74.145) into (74.139) and (74.142) gives

$$\dot W = \Lambda r^3 + 3(1-\gamma/2)r_m^{3\gamma-2}r^{3-3\gamma} - 2k(1+\alpha)r, \tag{74.146}$$

$$\begin{aligned} W^2 &= \frac{V^2}{3}+\frac{\Lambda r^6}{3}+r_m^{3\gamma-2}r^{6-3\gamma} \\ &\quad -k(1+\alpha)r^4+\left(\frac{-k}{4a_0^2}\right)^3\frac{b^2}{3(1+\alpha)^2}. \end{aligned} \tag{74.147}$$

Let us now convert to a set of dimensionless variables defined by

$$\tau = t/r_m \tag{74.148}$$

$$x = r/r_m = R/R_m \tag{74.149}$$

$$y = V/r_m^2 \tag{74.150}$$

$$z = W/r_m^2 \tag{74.151}$$

$$C = K/r_m^2 = 0 \tag{74.152}$$

$$D = b/r_m^2 \tag{74.153}$$

Let us also use $'$ for $d/d\tau$. Then (74.136) through (74.141), (74.146), and (74.147) become

$$x^2 x' = z, \tag{74.154}$$

$$\frac{\alpha'}{1+\alpha} = \frac{2y}{x^3} \tag{74.155}$$

$$\frac{U'}{U} = \frac{3C}{x^3} = 0, \tag{74.156}$$

$$C' = \frac{2kq_0(3a_0+q_0)(1+\alpha)x}{3a_0^2+q_0^2} = 0, \tag{74.157}$$

$$y' = \left(\frac{-k}{4a_0^2}\right)^3 \frac{2D^2}{(1+\alpha)^2 x^3} + 3k(1+\alpha)x, \tag{74.158}$$

$$z' = \Lambda r_m^2 x^3 + 3(1-\gamma/2)x^{3-3\gamma} - 2k(1+\alpha)x, \tag{74.159}$$

$$z^2 = \frac{1}{3}\Lambda r_m^2 x^6 + x^{6-3\gamma} - k(1+\alpha)x^4 + \left(\frac{-k}{4a_0^2}\right)^3 \frac{D^2}{3(1+\alpha)^2} + \frac{y^2}{3}. \tag{74.160}$$

For the $h = -1/9$ case, we have

$$q_0 = -3a_0. \tag{74.161}$$

if and only if $b \neq 0$. However, the $b = 0$ case is only a single point. When integrating over b, the behavior for small b is more important than exactly at $b = 0$. Therefore, we shall use (74.161) in any case.

Therefore, from (74.158) we have

$$y' = 3k(1+\alpha)x - \frac{k}{(2a_0)^6}\frac{2D^2}{(1+\alpha)^2 x^3}, \tag{74.162}$$

and from (74.160) we have

$$z^2 = \frac{1}{3}\Lambda r_m^2 x^6 + x^{6-3\gamma} - k(1+\alpha)x^4 - \frac{k}{(2a_0)^6}\frac{D^2}{3(1+\alpha)^2} + \frac{y^2}{3}. \tag{74.163}$$

We notice at this point that if we take the derivative of (74.163) and substitute from (74.154), (74.155), (74.162), and (74.159), that we get an identity, confirming the consistency of the equations at this point.

Combining (74.154) with (74.163) gives

$$x' = \sqrt{\frac{1}{3}\Lambda r_m^2 x^2 + x^{2-3\gamma} - k(1+\alpha) - \frac{k}{(2a_0)^6}\frac{D^2}{3(1+\alpha)^2 x^4} + \frac{y^2}{3x^4}}. \tag{74.164}$$

Combining (74.162) with (74.164) gives

$$\frac{dy}{dx} = \frac{3k(1+\alpha)x - \frac{k}{(2a_0)^6}\frac{2D^2}{(1+\alpha)^2x^3}}{\sqrt{\frac{1}{3}\Lambda r_m^2 x^2 + x^{2-3\gamma} - k(1+\alpha) - \frac{k}{(2a_0)^6}\frac{D^2}{3(1+\alpha)^2x^4} + \frac{y^2}{3x^4}}}. \tag{74.165}$$

Both b and a_0 are constants that are determined by initial conditions on the initial hypersurface. They are either related or independent. We assume first that they are independent. In that case, the fourth term under the radical in (74.164) and (74.165) will get smaller as D gets smaller. We shall assume that we can neglect that term relative to the second term under the radical for all values of x. We can check that assumption later. There is also the possibility of iterating later by assuming that this term is small instead of zero.

We shall also neglect the last term under the radical in (74.164) and (74.165). We can check that approximation later. There is also the possibility of iterating by substituting an approximate solution for y into (74.165). We shall see that it it permissible to neglect the fifth term for all values of x within the integration range if b is small enough.

Neglecting the fourth and fifth terms under the radical in (74.164) and (74.165) gives

$$x' = \sqrt{\frac{1}{3}\Lambda r_m^2 x^2 + x^{2-3\gamma} - k(1+\alpha)}. \tag{74.166}$$

and

$$\frac{dy}{dx} = \frac{3k(1+\alpha)x - \frac{k}{(2a_0)^6}\frac{2D^2}{(1+\alpha)^2x^3}}{\sqrt{\frac{1}{3}\Lambda r_m^2 x^2 + x^{2-3\gamma} - k(1+\alpha)}}. \tag{74.167}$$

We shall assume that

$$\alpha \ll 1. \tag{74.168}$$

We can test that approximation from the solution later and iterate if necessary. In that case, (74.164) and (74.165) become

$$x' = \sqrt{\frac{1}{3}\Lambda r_m^2 x^2 + x^{2-3\gamma} - k}. \tag{74.169}$$

and

$$\frac{dy}{dx} = \frac{3kx - \frac{k}{(2a_0)^6}\frac{2D^2}{x^3}}{\sqrt{\frac{1}{3}\Lambda r_m^2 x^2 + x^{2-3\gamma} - k}}. \tag{74.170}$$

We notice that (74.170) could be integrated numerically to obtain an approximate solution for $y(x)$. Except for later times, we can neglect all but the second term under the radical in (74.169) and (74.170) to give

$$x' = x^{1-\frac{3}{2}\gamma}. \tag{74.171}$$

and

$$\frac{dy}{dx} = 3kx^{\frac{3}{2}\gamma} - 2kE^2x^{\frac{3}{2}\gamma-4}, \tag{74.172}$$

where

$$E \equiv D/(2a_0)^3 = \frac{b/r_{m1}^2}{(2a_0)^3}. \tag{74.173}$$

We can integrate (74.172) to give

$$y = y_0 + \frac{6k}{3\gamma+2}(x^{\frac{3}{2}\gamma+1} - x_0^{\frac{3}{2}\gamma+1}) - \frac{4k}{3}\frac{E^2}{\gamma-2}(x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3}), \tag{74.174}$$

where, without loss of generality, the constant of integration has been chosen such that $y = y_0$ when $x = x_0$.

We notice that on substituting (74.174) into (74.164) and (74.165) the terms it gives that do not involve b have a higher power of x than the dominant term. Therefore, for small b and small x, it was justified to neglect the fourth term under the radical in (74.164) and (74.165), since the terms that involve b are proportional to a positive power of b.

Using (74.174) in (74.155) gives

$$\frac{\alpha'}{1+\alpha} = 2y_0 x^{-3} + \frac{12k}{3\gamma+2}(x^{\frac{3}{2}\gamma-2} - x_0^{\frac{3}{2}\gamma+1}x^{-3}) - \frac{8k}{3}\frac{E^2}{\gamma-2}(x^{\frac{3}{2}\gamma-6} - x_0^{\frac{3}{2}\gamma-3}x^{-3}). \tag{74.175}$$

Combining (74.175) with (74.171) gives

$$\frac{d\ln(1+\alpha)}{dx} = 2y_0 x^{\frac{3}{2}\gamma-4} + \frac{12k}{3\gamma+2}(x^{3\gamma-3} - x_0^{\frac{3}{2}\gamma+1}x^{\frac{3}{2}\gamma-4}) - \frac{8k}{3}\frac{E^2}{\gamma-2}(x^{3\gamma-7} - x_0^{\frac{3}{2}\gamma-3}x^{\frac{3}{2}\gamma-4}). \tag{74.176}$$

We can integrate (74.176) to give

$$\begin{aligned}\ln\frac{1+\alpha}{1+\alpha_0} &= \frac{4y_0}{3\gamma-6}(x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3}) + \\ &\quad \frac{12k}{3\gamma+2}\frac{(3\gamma-6)x^{3\gamma-2} - (6\gamma-4)x_0^{\frac{3}{2}\gamma+1}x^{\frac{3}{2}\gamma-3} + (3\gamma+2)x_0^{3\gamma-2}}{(3\gamma-2)(3\gamma-6)} \\ &\quad -\frac{8k}{3}\frac{E^2}{\gamma-2}\frac{x^{3\gamma-6} - 2x_0^{\frac{3}{2}\gamma-3}x^{\frac{3}{2}\gamma-3} + x_0^{3\gamma-6}}{3\gamma-6},\end{aligned} \tag{74.177}$$

where, without loss of generality, the constant of integration has been chosen such that $\alpha = \alpha_0$ when $x = x_0$. Or, taking $k = -1$,

$$\begin{aligned}1+\alpha &= (1+\alpha_0)exp\left\{\frac{2}{3\gamma-6}\frac{R_0^3}{2a_0R_m^2}\frac{\dot\alpha_0}{1+\alpha_0}(x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3})\right. \\ &\quad -\frac{12}{3\gamma+2}\frac{(3\gamma-6)x^{3\gamma-2} - (6\gamma-4)x_0^{\frac{3}{2}\gamma+1}x^{\frac{3}{2}\gamma-3} + (3\gamma+2)x_0^{3\gamma-2}}{(3\gamma-2)(3\gamma-6)} \\ &\quad \left.+\frac{8}{3}\frac{b^2}{(2a_0)^2R_m^4}\frac{x^{3\gamma-6} - 2x_0^{\frac{3}{2}\gamma-3}x^{\frac{3}{2}\gamma-3} + x_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)}\right\},\end{aligned} \tag{74.178}$$

Also,

$$\begin{aligned}(1+\alpha)^{1/2} &= (1+\alpha_0)^{1/2}exp\left\{\frac{1}{3\gamma-6}\frac{R_0^3}{2a_0R_m^2}\frac{\dot\alpha_0}{1+\alpha_0}(x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3})\right. \\ &\quad -\frac{6}{3\gamma+2}\frac{(3\gamma-6)x^{3\gamma-2} - (6\gamma-4)x_0^{\frac{3}{2}\gamma+1}x^{\frac{3}{2}\gamma-3} + (3\gamma+2)x_0^{3\gamma-2}}{(3\gamma-2)(3\gamma-6)} \\ &\quad \left.+\frac{4}{3}\frac{b^2}{(2a_0)^2R_m^4}\frac{x^{3\gamma-6} - 2x_0^{\frac{3}{2}\gamma-3}x^{\frac{3}{2}\gamma-3} + x_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)}\right\},\end{aligned} \tag{74.179}$$

Or,

$$(1+\alpha)^{1/2} \approx (1+\alpha_0)^{1/2}\left\{1 + \frac{1}{3\gamma-6}\frac{R_0^3}{2a_0R_m^2}\frac{\dot\alpha_0}{1+\alpha_0}(x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3})\right.$$

$$
-\frac{6}{3\gamma+2}\frac{(3\gamma-6)x^{3\gamma-2}-(6\gamma-4)x_0^{\frac{3}{2}\gamma+1}x^{\frac{3}{2}\gamma-3}+(3\gamma+2)x_0^{3\gamma-2}}{(3\gamma-2)(3\gamma-6)}
$$

$$
\left.+\frac{4}{3}\frac{b^2}{(2a_0)^2R_m^4}\frac{x^{3\gamma-6}-2x_0^{\frac{3}{2}\gamma-3}x^{\frac{3}{2}\gamma-3}+x_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)}\right\}, \tag{74.180}
$$

Writing (74.174) and (74.177) in original variables using (74.149), (74.150), (74.173), and (74.137) gives

$$
\begin{aligned}
V \quad \equiv \quad & \frac{r^3}{2}\frac{\dot{\alpha}}{1+\alpha}=\frac{r_0^3}{2}\frac{\dot{\alpha}_0}{1+\alpha_0}+ \\
& [6k\frac{r^{\frac{3}{2}\gamma+1}-r_0^{\frac{3}{2}\gamma+1}}{3\gamma+2}-\frac{4k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{\frac{3}{2}\gamma-3}-r_0^{\frac{3}{2}\gamma-3}}{\gamma-2}]r_m^{1-\frac{3}{2}\gamma},
\end{aligned} \tag{74.181}
$$

$$
\begin{aligned}
1+\alpha \quad \approx \quad & (1+\alpha_0)exp\left\{\frac{2r_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(r^{\frac{3}{2}\gamma-3}-r_0^{\frac{3}{2}\gamma-3})r_m^{1-\frac{3}{2}\gamma}+\right. \\
& 12k\frac{(3\gamma-6)r^{3\gamma-2}-(6\gamma-4)r_0^{\frac{3}{2}\gamma+1}r^{\frac{3}{2}\gamma-3}+(3\gamma+2)r_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)r_m^{3\gamma-2}} \\
& \left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\frac{r^{3\gamma-6}-2r_0^{\frac{3}{2}\gamma-3}r^{\frac{3}{2}\gamma-3}+r_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)r_m^{3\gamma-2}}\right\},
\end{aligned} \tag{74.182}
$$

$$
\begin{aligned}
1+\alpha \quad \approx \quad & (1+\alpha_0)exp\left\{\left(\frac{-3k}{3a_0^2+q_0^2}\right)^{1/2}\frac{2R_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(R^{\frac{3}{2}\gamma-3}-R_0^{\frac{3}{2}\gamma-3})R_m^{1-\frac{3}{2}\gamma}+\right. \\
& +12k\frac{(3\gamma-6)R^{3\gamma-2}-(6\gamma-4)R_0^{\frac{3}{2}\gamma+1}R^{\frac{3}{2}\gamma-3}+(3\gamma+2)R_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)R_m^{3\gamma-2}} \\
& \left.-\frac{8k}{3}\frac{b^2}{(2a_0)^6}\left(\frac{3a_0^2+q_0^2}{-3k}\right)^2\frac{R^{3\gamma-6}-2R_0^{\frac{3}{2}\gamma-3}R^{\frac{3}{2}\gamma-3}+R_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)R_m^{3\gamma-2}}\right\},
\end{aligned} \tag{74.183}
$$

$$
\begin{aligned}
1+\alpha \quad \approx \quad & (1+\alpha_0)exp\left\{\frac{1}{2a_0}\frac{2R_0^3}{3\gamma-6}\frac{\dot{\alpha}_0}{1+\alpha_0}(R^{\frac{3}{2}\gamma-3}-R_0^{\frac{3}{2}\gamma-3})R_m^{1-\frac{3}{2}\gamma}+\right. \\
& -12\frac{(3\gamma-6)R^{3\gamma-2}-(6\gamma-4)R_0^{\frac{3}{2}\gamma+1}R^{\frac{3}{2}\gamma-3}+(3\gamma+2)R_0^{3\gamma-2}}{(3\gamma+2)(3\gamma-2)(3\gamma-6)R_m^{3\gamma-2}} \\
& \left.+\frac{8}{3}\frac{b^2}{(2a_0)^6}(2a_0)^4\frac{R^{3\gamma-6}-2R_0^{\frac{3}{2}\gamma-3}R^{\frac{3}{2}\gamma-3}+R_0^{3\gamma-6}}{(\gamma-2)(3\gamma-6)R_m^{3\gamma-2}}\right\},
\end{aligned} \tag{74.184}
$$

Chapter 75

The criteria for a solution of the field equations to be a classical limit of a quantum cosmology [1]

abstract

If the gravitational field is quantized, then a solution of Einstein's field equations is a valid cosmological model only if it corresponds to a classical limit of a quantum cosmology. To determine which solutions are valid requires looking at quantum cosmology in a particular way. Because we infer the geometry by measurements on matter, we can represent the amplitude for any measurement in terms of the amplitude for the matter fields, allowing us to integrate out the gravitational degrees of freedom. Combining that result with a path-integral representation for quantum cosmology leads to an integration over 4-geometries. Even when a semiclassical approximation for the propagator is valid, the amplitude for any measurement includes an integral over the gravitational degrees of freedom. The conditions for a solution of the field equations to be a classical limit of a quantum cosmology are: (1) The effect of the classical action dominates the integration, (2) the action is stationary with respect to variation of the gravitational degrees of freedom, and (3) only one saddlepoint contributes significantly to each integration.

75.1 Introduction

We normally consider all solutions of Einstein's field equations to be valid cosological models. However, this may not be true if a valid cosmological model is required to be the classical limit of a quantum cosmology.

Section 75.2 points out that we infer the gravitational field from measurements on matter. Therefore, in comparing measurements with theory, it is sufficient to consider the amplitudes for matter fields only, allowing us to integrate over the gravitational degrees of freedom (an integration on a spacelike three-dimensional hypersurface).

Section 75.3 points out that a path-integral representation of the wave function involves an integration over all 3-geometries on an initial spacelike hypersurface. Section 75.4 replaces the integrations over 3-geometries on the two spacelike hypersurfaces by the equivalent integration over the 4-geometries connecting those two hypersurfaces.

Section 75.5 considers the semiclassical approximation for the propagator that takes the wave function for 3-geometries and matter fields from one spacelike hypersurface to another. In that

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approximation, the propagator depends on only one solution of the field equations. Solutions to the field equations fall into two categories:

1. The action for the propagator dominates the behavior of the path integral and a saddlepoint approximation is valid for each integration in the path integral over the geometry.

2. The action for the propagator does not dominate the behavior of the path integral or a saddlepoint approximation is invalid for at least one of the integrations in the path integral over the geometry.

Section 75.6 considers the former case. Section 75.7 considers the latter case. Section 75.8 interprets these examples.[2]

75.2 Measurements in quantum cosmology

We can represent a quantum cosmology by $\langle g, \phi, S|\psi\rangle$, which is the amplitude that on a spacelike hypersurface S, the 3-geometry is g and the matter fields are ϕ.[3] This representation is implicit in the path integral approach to quantum gravity [111].

To relate this amplitude to a measurement of the geometry, we notice that we do not measure the geometry directly. We infer the geometry from measurements using material objects, that is, from measurements on the matter. This allows us to represent any measurement by integrating over the gravitational degrees of freedom to give

$$\langle \phi, S|\psi\rangle = \int \langle g, \phi, S|\psi\rangle \, D(g), \tag{75.1}$$

the amplitude that on a spacelike hypersurface S, the matter fields are ϕ, where $D(g)$ is the measure on g.

75.3 Path-integral representation

The wave function over 3-geometries g_2 and matter fields ϕ_2 on one 3-dimensional spacelike hypersurface S_2 is related to the wave function over 3-geometries g_1 and matter fields ϕ_1 on another 3-dimensional spacelike hypersurface S_1 by an extension of the path-integral [21] formulation of quantum cosmology [111] to give

$$\begin{aligned}&\langle g_2, \phi_2, S_2|\psi\rangle = \\ &\int \langle g_2, \phi_2, S_2|g_1, \phi_1, S_1\rangle \, \langle g_1, \phi_1, S_1|\psi\rangle \, D(g_1) D(\phi_1),\end{aligned} \tag{75.2}$$

where $\langle g_2, \phi_2, S_2|g_1, \phi_1, S_1\rangle$ is the propagator (that is, the amplitude to go from a state with 3-geometry g_1 and matter fields ϕ_1 on hypersurface S_1 to a state with 3-geometry g_2 and matter fields ϕ_2 on hypersurface S_2), $\langle g_1, \phi_1, S_1|\psi\rangle$ is the wave function over 3-geometries g_1 and matter fields ϕ_1 on a spacelike hypersurface S_1, $D(g_1)$ is the measure on g_1, and $D(\phi_1)$ is the measure on ϕ_1. The integration is over all 3-geometries g_1 and matter fields ϕ_1 for which the integral is defined.[4] less

[2] This paper considers the meaning of quantum gravity, especially with regard to interpreting measurements, but does not discuss theories of quantum gravity.

[3] The development here for a pure state represented by a wave function can be generalized to a mixed state, represented by a density matrix.

[4] Because a matter distribution cannot be specified independent of geometry, we take the above integration to mean that we first specify a 3-geometry, then on that 3-geometry, we specify a matter distribution.

Substituting (75.2) into (75.1) gives

$$\langle \phi_2, S_2|\psi\rangle = \int \langle g_2, \phi_2, S_2|g_1, \phi_1, S_1\rangle \langle g_1, \phi_1, S_1|\psi\rangle$$
$$D(g_1)D(\phi_1)D(g_2). \tag{75.3}$$

75.4 Integration over 4-geometries

Because (75.3) involves an integration over all 3-geometries g_1 and g_2 on S_1 and S_2, it is equivalent to an integration over all 4-geometries that connect S_1 and S_2. Thus, (75.3) can be written as

$$\langle \phi_2, S_2|\psi\rangle = \int \left\langle g_2(g^{(4)}), \phi_2, S_2|g_1(g^{(4)}), \phi_1, S_1\right\rangle$$
$$\left\langle g_1(g^{(4)}), \phi_1, S_1|\psi\right\rangle D(g^{(4)})D(\phi_1), \tag{75.4}$$

where $D(g^{(4)})$ is the measure on the 4-geometry $g^{(4)}$. Of course, until we have a full theory of quantum gravity, we do not have formulas to give most of the functions in these integrals. We can, however, make some semiclassical approximations without having a full theory. To justify replacing (75.3) by (75.4), we notice that the integration in (75.3) is an integration over all 4-geometries that connect S_1 and S_2, as is the integration in (75.4).

75.5 semiclassical approximation for the propagator

Making the semiclassical approximation[5] for the propagator gives [214]

$$\left\langle g_2(g^{(4)}), \phi_2, S_2|g_1(g^{(4)}), \phi_1, S_1\right\rangle \approx$$
$$f_a(g_2(g^{(4)}), S_2; g_1(g^{(4)}), \phi_1, S_1)$$
$$e^{\frac{i}{\hbar}I_{classical}[g_2(g^{(4)}),S_2;g_1(g^{(4)}),\phi_1,S_1]}, \tag{75.5}$$

where $I_{classical}[g_2(g^{(4)}), S_2; g_1(g^{(4)}), \phi_1, S_1]$ is the action for the classical spacetime bounded by the two 3-geometries that satisfies the field equations and $f_a(g_2(g^{(4)}), S_2; g_1(g^{(4)}), \phi_1, S_1)$ is a slowly varying function. Explicit dependence on ϕ_2 is not shown, because for classical solutions to the field equations, ϕ_2 is determined from ϕ_1 and $g^{(4)}$. Thus, substituting (75.5) into (75.4) gives

$$\langle \phi_2, S_2|\psi\rangle \approx \int f_b(g^{(4)}, \phi_1)e^{\frac{i}{\hbar}I_{classical}[g^{(4)},\phi_1]}$$
$$\left\langle g_1(g^{(4)}), \phi_1, S_1|\psi\right\rangle D(\phi_1)D(g^{(4)}). \tag{75.6}$$

where $f_b(g^{(4)}, \phi_1)$ is a slowly varying function and the integration is over all classical 4-geometries that connect S_1 and S_2.

The number of functions being integrated over to represent the 4-geometry $g^{(4)}$ is probably an order of infinity greater than that of the real numbers. To test the validity as a cosmology of a given 4-geometry, it is sufficient to restrict consideration to a small subset of cases, such as a family of known exact solutions. This allows us to represent the integration over 4-geometries in (75.6) more explicitly. Solutions to the field equations can be represented by a number of parameters a_i. These are the parameters that specify the 4-geometry that are not constrained by the matter distribution ϕ_1 on the hypersurface S_1. The number of these parameters is usually finite, and in most cases, at

[5] A semiclassical approximation for the propagator is not always valid. Here, we consider only cases where it is valid.

least countable. I shall assume here, that they are finite, and that there are N of these parameters, although I think the development could be extended to even the uncountable case. Thus, we may rewrite (75.6) more explicitly as

$$\begin{aligned} \langle \phi_2, S_2|\psi\rangle \approx \int f_c(a_i, \phi_1) e^{\frac{i}{\hbar} I_{classical}[a_i,\phi_1]} \\ \langle g_1(a_i), \phi_1, S_1|\psi\rangle\, D(\phi_1) d^N a_i. \end{aligned} \tag{75.7}$$

where $f_c(a_i, \phi_1)$ is a slowly varying function that depends explicitly on the parameters a_i that define the 4-geometry, and now we are left with an ordinary Nth order integral to define the integration over the 4-geometries.

75.6 When a saddlepoint approximation is valid

When the behavior of $e^{\frac{i}{\hbar} I_{classical}}$ dominates over that of $\langle g_1(a_i), \phi_1, S_1|\psi\rangle$ and $f_c(a_i, \phi_1)$ in the integration over each a_i in (75.7) and when a saddlepoint approximation for each integration is valid, then we can approximate each of those integrations by a saddlepoint approximation. We analytically continue each function into the complex domain, deform the path of integration in the complex a_i plane for each a_i to go through the saddlepoint, a_{i0}, defined by where $I_{classical}$ is stationary for variation of each of the a_i, that is,

$$\left.\frac{\partial I_{classical}}{\partial a_i}\right|_{a_i=a_{i0}} = 0, \tag{75.8}$$

for each a_i. For each integration, the path must be deformed (without passing over any non-analytic points) onto a steepest descent path or a stationary phase path. Also, to be a valid approximation, there must not be any non-analytic points too close to the saddlepoint. For stationary phase paths, the saddlepoint approximation gives e.g. [215]

$$\begin{aligned} &\langle \phi_2, S_2|\psi\rangle \approx \\ &\int f_c(a_{i0}, \phi_1) e^{\frac{i}{\hbar} I_{classical}[a_{i0},\phi_1]} \langle g_1(a_{i0}), \phi_1, S_1|\psi\rangle \\ &e^{Ni\pi/4} \prod_{i=1}^{N} \left|\frac{2\pi}{\partial^2 I/\partial a_i^2}\right|^{1/2}_{a_i=a_{i0}} D(\phi_1). \end{aligned} \tag{75.9}$$

For steepest descent paths, the formula differs only by a phase.

The usual form for the action I is

$$I = \int (-|g|^{(4)})^{1/2} L d^4x, \tag{75.10}$$

where $|g|$ is the determinant of the metric tensor $g_{\mu\nu}$,

$$L = \underbrace{\frac{R-2\Lambda}{16\pi}}_{geometry} \underbrace{-\frac{1}{2}\rho g_{\mu\nu} U^\mu U^\nu}_{matter} \underbrace{-\frac{\rho_e}{c} A_\mu U^\mu}_{interaction} \underbrace{-\frac{F_{\mu\nu}F^{\mu\nu}}{16\pi}}_{EM} \tag{75.11}$$

is the Lagrangian, R is the Riemann scalar, Λ is the cosmological constant, ρ is the mass density, U^μ is the four-velocity, ρ_e is the electric charge density, A_μ is the electromagnetic 4-vector potential, $F_{\mu\nu}$ is the electromagnetic field tensor, and the usual designation of the four terms is shown.[6]

[6]All of the examples here use Einstein's theory of General Relativity, but the procedure applies to nearly any gravitational theory.

Because the integration in (75.10) must consider the light-cone structure of the propagators, it is more appropriate to derive a formula for the amplitude of observing a particular event instead of deriving a general formula for all possible measurements. The integral for the action in (75.10) must therefore be restricted to the past light cone of the event whose amplitude is being calculated. There is some fuzziness to the light cone,[7] which is taken into account by using the correct propagators [216].

An example of applying such a saddlepoint approximation to a family of solutions to the field equations will be given in a future publication.

75.7 When a saddlepoint approximation is not valid

We consider here several examples where the saddlepoint approximation is either not valid or not applicable. We take Λ, $F_{\mu\nu}$ and A_μ to be zero in these examples. In addition, we take R and ρ to be zero except where there are masses.

75.7.1 Minkowski space

In empty Minkowski space, the Lagrangian is everywhere zero because the scalar curvature R is zero and the matter density is zero, and therefore, the action $I_{classical}$ is zero. Because there is no matter, there is no possibility for measurements, so this case is not applicable.

75.7.2 Schwarzschild metric

The simplest matter distribution added onto Minkowski space-time gives us the Schwarzschild metric. Normally, we use the Schwarzschild metric to represent the local field around a planet or star or black hole, but not for a whole cosmology, and there may be good reason for that.

There are no gravitational degrees of freedom defining the Schwarzschild metric, so there is no integration over 4-geometries. However, formally, we could write (75.7) as

$$\begin{aligned}&\langle \phi_2, S_2|\psi\rangle \approx \\ &\int f_c(\phi_1) e^{\frac{i}{\hbar} I_{classical}[\phi_1]} \langle g_1, \phi_1, S_1|\psi\rangle\, D(\phi_1).\end{aligned} \tag{75.12}$$

75.7.3 Kerr metric

The next simplest model is a symmetric body like a planet that has a rotation rate relative to an inertial frame. We can represent the field outside of the body by the exterior Kerr metric. This metric has three gravitational degrees of freedom to characterize the direction and magnitude of the rotation rate (which I shall refer to as a_1, a_2, and a_3 here). Because the scalar curvature and matter density are everywhere zero outside of the body, the only contribution to the action $I_{classical}$ is from the mass of the body, which does not depend on the rotation rate. Thus, (75.7) becomes

$$\begin{aligned}&\langle \phi_2, S_2|\psi\rangle \approx \int e^{\frac{i}{\hbar} I_{classical}[\phi_1]} f_d(a_1, a_2, a_3, \phi_1) \\ &\langle g_1(a_1, a_2, a_3), \phi_1, S_1|\psi\rangle\, D(\phi_1) da_1 da_2 da_3,\end{aligned} \tag{75.13}$$

where $f_d(a_1, a_2, a_3, \phi_1)$ is a slowly varying function. Because the exponential factor does not dominate the integration, we cannot make a saddlepoint approximation for the integration over a_1, a_2,

[7] There is an interesting similarity between the light cone and the event horizon of a black hole. In both cases, travel across the boundary is classically possible in only one direction (into the light cone or into the black hole), but the prohibition is not absolute in either case, because in the quantum situation a particle can temporarily escape by doing a zigzag path in space-time [216], resulting in Hawking radiation in the case of a black hole.

and a_3. We are left with an integration over various Kerr metrics with various rotation rates. There is no single 4-geometry that dominates the integration.

We normally consider the Kerr metric to represent the local gravitational field around a spinning planet, star, or black hole, rather than for a cosmology. In light of the result here, this seems appropriate.

We want matter in the cosmological model so that we can do measurements. That is, because we cannot directly measure the geometry, we must infer it from measurements on matter. However, the example of a single body represented here by the Kerr metric is not really interesting enough to offer the possibility for measurements of the geometry. If we had a planetary system, we might be able to model possible measurements on the geometry using matter.

75.7.4 Asymptotically flat metrics

Therefore, consider a collection of planets and a star in some star system as the only matter in the universe. We assume we have some solution of the field equations for these. In fact, we will have many solutions, because we have some freedom in applying boundary conditions.

Let us consider a subset of those solutions in which we apply asymptotically flat boundary conditions. Then very far from where all of the matter is concentrated for the star system, the solution will be approximately that of a Kerr metric, in which the solution is characterized by the angular momentum of the matter relative to the flat metric to which the Kerr solution is asymptotic. The angular momentum is characterized by 3 values, say a_1, a_2, and a_3. This leads to the wave function given by (75.13), but we cannot apply a saddlepoint approximation because the action is independent of a_1, a_2, and a_3.

75.8 Interpretation

In summary, the conditions for a solution of the field equations to be a classical limit of a quantum cosmology are: (1) The effect of the classical action dominates the integration, (2) the action is stationary with respect to variation of the gravitational degrees of freedom, and (3) only one saddlepoint contributes significantly to each integration.

As pointed out earlier, we can always represent a measurement of the geometry in terms of the matter; we infer the geometry from measurements on the matter. So, in the above examples, what geometry would we infer from measurements on the matter?

Measurements on the matter in section 75.6 would indicate a geometry that was confined within the limits given by $|I_{saddlepoint} - I_{classical}| < \hbar$.

On the other hand, measurements on the matter in section 75.7.4 would indicate an ambiguous geometry. In fact, the system of bodies would seem very nonclassical. There is an aspect of relativity here. Although it is the background geometry that is quantum, we can infer the geometry and matter only relative to each other. More specifically, we can observe directly, only the matter, so it will appear to an observer that the matter is behaving in a quantum manner.

It should be pointed out that there are no new theories or assumptions here. This is simply an application of standard ideas about quantum theory to cosmology. To falsify the results presented here, it would be sufficient to show that our present cosmology does not satisfy the criteria given here for a valid cosmological model. But unless I have made a logical error, that would also invalidate some of our standard ideas about quantum theory.

75.9 Reviewer's comments

Confidential comments and recommendations to the author(s)

This paper addresses path integrals in quantum gravity, in which an integration over four-geometries has to be performed. The main result is the clarification of criteria which have to be fulfilled in order to achieve a classical limit.

In my opinion, this paper is not suitable for publication in Astronomische Nachrichten, for the following reasons. Firstly, there exists already a huge literature on this subject which addresses these issues and has not been cited by the author. Some examples are the references [1-5] below. Everything that has been indicated by the present author is there discussed and clarified in technical detail; for example, the validity of the saddle point approximation and the choice of contour.

Secondly, some statements are incorrect. The composition law (2) is not valid for quantum gravitational path integrals, since they resemble energy Green functions in quantum mechanics instead of propagators [1,5]. Because there is no Birkhoff theorem for the Kerr metric, it describes only axisymmetric black holes, not the geometry outside a rotating body, in contrast to what is claimed in Sect. 7.3.

[**1**] P. Hajicek, J. Math. Phys. 27 (1986) 1800.

[**2**] J.J. Halliwell, Phys. Rev. D 38 (1988) 2468.

[**3**] J.J. Halliwell and J. Louko, Phys. Rev. D 39 (1989) 2206.

[**4**] J.J. Halliwell and J.B. Hartle, Phys. Rev. D 41 (1990) 1815.

[**5**] C. Kiefer, Ann. Phys. (N.Y.) 207 (1991) 53.

75.10 Afterthoughts - 2008

Some of the part about the Kerr metric is incorrect because there is no Birkhoff theorem for the Kerr metric.

There is also a question about using the correct propagator. I think I may have used the correct propagator in my 1979 manuscript, but was convinced by the reviewer to change. I need to check on that.

Some of what I present here may already be well-known. I need to read very carefully the papers by Hajicek (1986)[217], Halliwell (1988)[218], Halliwell and Louko (1989)[219], Halliwell and Hartle (1990)[220], and Kiefer (1991)[221].

In addition, I need to make the whole calculation, not just show how it goes. I need to specifically calculate the dependence of the action on the relative rotation of matter and inertial frames.

75.11 Afterthoughts - 2009

I think the problem with the propagators is my notation. I need to fix that.

In addition, there is the other problem with talking about varying the initial and/or boundary conditions for the gravitational field while holding the matter distribution fixed. I discuss this below.

75.12 Further Afterthoughts - 2009

Newton noticed that his laws of motion applied in a frame that does not rotate relative to the "fixed stars," and concluded the existence of an "absolute space" with the stars fixed in that absolute space (what we now refer to as an "inertial frame").

Mach denied the existence of absolute space, pointing out that only relative motions are observable. Instead of absolute space, he postulated that inertia is caused by an interaction with the rest of the universe. He suggested that what we now refer to as an inertial frame is caused by the rest of matter in the universe (now referred to as frame dragging).

Einstein based his General Relativity partly on Mach's ideas, and coined the term "Mach's principle" for the idea that inertial frames should be determined by the matter distribution. The term "Mach's principle" now means many things to many people, and there is no general agreement on what it means or on the validity of the various versions.

Although Einstein's General Relativity does include frame dragging, frame dragging is not complete. There are many solutions of Einstein's field equations in which there is inertia in the absence of matter or where there is rotation of inertial frames relative to the average matter distribution. This is because initial and boundary conditions also contribute to inertia.

Some researchers have proposed that boundary and initial conditions be chosen so that inertia and/or the metric be determined by the matter distribution. Some researchers have proposed that Mach's principle be used as a selection principle to discard solutions that have relative rotation of matter and inertial frames or have inertia without matter.

I have argued that if the metric (gravitation) is determined solely by sources (the matter), with no dependence on initial or boundary conditions, then gravitation would be very different from electromagnetic theory, in which the electromagnetic field has its own degrees of freedom, and can have arbitrary initial and boundary conditions. Thus, I have rejected Mach's principle in such a restricted form.

That leaves us still with the problem of explaining why we observe no relative rotation of inertial frames and the visible matter in the universe. I suggested that if we consider quantum gravity, that we could allow all of the solutions to the field equations, and that the non-physical solutions (those with relative rotation of inertial frames and matter) would cancel each other out if we look at the change in the action with variation of the initial and boundary conditions on the gravitational field.

It was at this point that I made a mistake in the present manuscript. I suggested that we vary the initial and boundary conditions on the gravitational field while holding the matter distribution constant. It was pointed out that there is no general procedure for doing that because it is not possible to specify the matter distribution independent of the geometry.

What I should do instead (and what I actually did in my calculations) is consider a family of solutions to the field equations (such as one of the Bianchi cosmologies) in which there is an initial condition that determines the amount of relative rotation of inertial frames and matter, and consider the change in action as that parameter is varied. In this way, we have no problems of the kind I mentioned above. Then, a saddlepoint approximation to the path integral for that parameter will give the solution for zero relative rotation.

Chapter 76

The goal of physics is not for a theory of measurement[1]

The goal of physics: an algorithm for the evolution of the universe, not a theory of measurement.

Before quantum theory, the goal of physics was to derive the equations that controlled the way the universe works, in a way similar to describing the universe as a mechanism, as a clock mechanism. After quantum theory, we lowered our standards because we realized that if quantum theory were right, we could measure the universe only imperfectly, and that a theory was needed to express that imperfection. In fact, we included the possibility that even the universe may not know what it would do in the future. In fact, it may be impossible by any means of measurement to distinguish between those two alternatives. So our equations limit themselves in that way.

However, they are limited further. Now, our equations no longer tell how the world works, but only the results of measurements we may make in experiments. they do not, for example, tell how an electron (if such exists) will choose to do this or that. We do not, for example, know if the wave function represents our knowledge of the system or whether it describes the actual system. We know of no measurement that will distinguish between those two alternatives.

We should now return to the previous goal of deriving a set of equations or algorithms that determine how the world works. It may be argued that there is no unique way to determine such a set of algorithms because our measurements are limited. This is probably true. However, we require only one such algorithm. If we find several algorithms that cannot be distinguished by measurement, that is OK. At present, we have no such algorithm. Finding at least one would be sufficient.

Maybe we already have such a system. Maybe it is only a matter of interpretation. Then we need to explicitly show how that works.

This new theory should satisfy several criteria:

1. It should give an algorithm for how the universe and all its parts evolve. (The algorithm need not be deterministic.)

2. Concepts such as “preparing an initial state” and the measuring process should not be part of the theory. They could be part of a separate theory describing how scientists interact with the universe and make measurements.

3. “Collapse of wave functions” does not seem appropriate for such a theory, although something like “decoherence” might be appropriate.

4. The concept of ensemble needs to be considered carefully.

[1] August 2002, Sarlat, France

5. The measurement theory should be derivable from the main theory.

6. The theory may not be unique.

7. It should be possible to falsify parts of the theory.

8. It may not be possible to falsify all parts of the theory.

9. The theory needs to be consistent with all known valid measurements now.

Until the discovery of the uncertainty principle and the recognition that our measurement of nature is limited, the goal of physics was to discover the rules by which the universe behaves. Since then, the goals have diminished to discover only those parts of the rules that can actually be verified by experiment. This has led to a situation in which the rules we discover are incomplete in that they do not supply a set of rules sufficient to describe the complete evolution of the universe. We have a choice:

1. Discuss only those rules that can be verified, or

2. Find a complete set of rules that can only be partly verified.

Until now, we have done the former. I suggest that instead we do the latter.

I do not have such a theory at present. I only propose that we as a community look for such a theory. It may be that one of the present formulations of quantum theory can be modified to give such a theory. Some people may believe that the many worlds interpretation (possibly renamed "many possibilities" theory) is already such a theory. I am not so sure. It would have to be demonstrated.

21 August 2002.

Dave Peterson suggested that maybe decoherence gives the necessary theory.

Chapter 77

Derivation of the no-boundary approximation[1]

It is shown that the no-boundary conjecture (Hartle and Hawking, Phys. Rev. D28, 2960-2975, 1983)[124] is an approximation that follows from a saddlepoint approximation in quantum gravity. The correct condition is that only those cosmological models whose action is stationary with respect to variation of the boundary conditions contribute significantly to path integrals in quantum cosmology. This condition includes those models having no boundary, but may include others as well. The weaker condition may be sufficient, since it is not important whether there is a boundary, but whether that boundary has any effect.

In addition, cosmological models within the first Fresnel zone around the saddlepoint must also be included, that is, those models for which the action differs by less than $\hbar$ from that at the saddlepoint.

[1] draft: 7 October 2002. Note added, March 2014: I realize this is not a correct derivation.

Chapter 78

Conjectured spacetime structure of the universe[1]

As a result of reading the chapter 17 by Lee Smolin <http://www.edge.org/documents/ThirdCulture/z-Ch.17.html> (some of which I agree with), I have come to some conjectures about the fundamental underlying structure of spacetime. The short answer is that there isn't any. The long answer is more interesting.

I now believe that there is no basic space or time built into the underlying structure of nature. That the apparent 3 spatial dimensions plus one time dimension come about from interactions. I think that there is no fixed dimension, either. Not 11, nor 10, nor 4.

Let us start by considering a bunch of particles. These might be electrons, neutrinos, or quarks, or something more fundamental, not protons. Consider that each of these particles experiences the fields of all of the other particles. I do not know how to calculate these fields. Remember, we have no space or time, yet.

Nowadays, we no longer believe in point particles because of infinities, so we use strings to represent particles because they are extended in at least one dimension instead of zero. These strings vibrate, somewhat like guitar strings, and each vibration mode gives a different particle. Oops, we do not yet have any space or time structure, so there is nothing for the strings to vibrate in or with respect to. (No space, no displacement; no time, no frequencies.)

Also, since there is no space yet, particles are not points. Nor are they strings. Without space, there is no way to distinguish between points and lines or strings. So, no infinities, either.

But also, no laws yet for how to calculate the fields at each particle. That may be the first project.

But, there are other considerations. In quantum theory, particles are also waves. However, without space or time, we have to consider the meaning of waves. We can still define the amplitude for a process, including a phase. And we can still consider phase interference. Also, we can consider the probability for an event. But maybe not. If the wave for wave functions are really waves, maybe there are no wave amplitudes until we get space and time. This is still an open question.

In quantum theory, a particle may not be localized, but be spread out over the whole universe. Without underlying space or time, that is already the case. Each particle is already in contact with the rest of the universe, and there are no distances.

The reason for considering the conjecture that there is no underlying space or time is that I think the string theorists and particle theorists are making a big mistake by not considering the source of the background spacetime. Basically, they are barking up the wrong tree.

Of course, the difficulty to making use of my conjecture is that it seems to start with nothing.

[1]13 March 2003

It is difficult to see how to get something for almost nothing. But, that makes it interesting. If it were not difficult, someone would have probably already done it. On the other hand, it may turn out to be easy and simple. Maybe no one has done it because they did not try this way before.

Chapter 79

Faster-than-light travel[1]

79.1 Introduction

A friend and physicist, Gerd Hartmann, asked me about the following news article that appeared on <http://www.oestereich1.com/science/> about the results of Antoine Suarez. The news article follows, with my comments afterward, taken from my e-mail of 28 June to Dr. Hartmann.

79.2 2.1.2003 :Einzigartiges Experiment zur Widerlegung Einsteins:

Quantenkorrelation zehnmillionenmal schneller als Licht!

Einem Forscherteam in Genf unter Leitung von Antoine Suarez und Valerio Scarani gelang der Nachweis, dass sich die Quanteninformation von korrelierten Lichtteilchen mindestens zehnmillionenmal schneller als Licht ausbreitet.

Sie schickten einen Laserstrahl in einen Kalium-Niobium-Kristall, der den Strahl in zwei korrelierte Strahlen aufspaltete, die über zwei unterschiedliche Glasfaserleitungen in zwei rund zehn Kilometer voneinander entfernte Dörfer (Bernex und Bellevue) geschickt wurden. Durch genaue Messung der korrelierten Größen "Energie" und "Zeit" konnte festgestellt werden, dass die Informationsübertragung mindestens zehnmillionenmal schneller als Licht sein muss.

Einstein sprach von "spukhafter Fernwirkung" und lehnte deswegen die Erkenntnisse der Quantenphysik ab, da in seiner Theorie die Lichtgeschwindigkeit eine obere Grenze darstellt.

Eine raffinierte Variante des Experiments (die Detektoren drehten sich mit 10.000 Umdrehungen pro Minute und erzeugten dadurch relativistische Effekte) sollten zudem Aussagen über die von der speziellen Relativitätstheorie geforderte Zeitdehnung erbringen. Weil sich ein Strahl durch die Rotation wesentlich schneller bewegt als der andere, müsste er für den anderen früher ankommen und damit jünger erscheinen.

Wegen der Einsteinschen Relativität müsste aber auch der andere Strahl dem einen jünger erscheinen, ein Phänomen, das unter dem Namen "Zwillingsparadoxon" bisher viele Deutungen und Lösungen fand (die einander zum Teil widersprechen).

Wie auch immer : Durch eine Zeitdehnung müsste die Quantenkorrelation auf jeden Fall zerstört werden, was bei Zusammenführen der Laserstrahlen eindeutig sichtbar wird.

Das Rotationsexperiment erbrachte das unerwartete Resultat:

Die Strahlen blieben korreliert, eine Zeitdehnung fand nicht statt.

Dieser Versuch ist die bisher eindeutigste Widerlegung der Relativitätstheorie.

In diesem Sinne, euer TopDog ...

[1] 28 June 2004

79.3 My comments

It seems that Suarez has performed a variation of the thought experiment, first proposed in 1935 in a paper by Einstein, Rosen, and Podolsky.

First, however, some background on some of the strange things about quantum theory. As far as we know, quantum theory is good enough to calculate the probability for getting any result in any experiment that can be performed. As is well known, only probabilities can be calculated. We make these calculations using wave functions (sometimes called quantum amplitudes). We do not know if these wave functions represent reality or our knowledge of reality. Some careful thought will convince us that there is no experiment we can perform that will tell us which of these two interpretations is correct.

There are philosophical problems with either interpretatation. If we maintain that a wave function represents only our knowledge of the world, then we do not have a theory that represents how the world really behaves. In fact, this would be a theory that would say nothing about the part of the world we are not watching. Or, to put it another way, "How do we think molecules stay together when we are not observing them?".

On the other hand, if we maintain that a wave function represents reality, then we get into the philosophical difficulties that Suarez mentions.

So, with that introduction, we get into the so-called Einstein-Podolsky-Rosen (EPR) paradox. Let us consider a system that in the beginning has zero angular momentum. We could, for example, consider an electron-positron pair, that happened to have zero total angular momentum. Suppose this electron-positron pair annihilates, producing two photons. In the center-of-mass system, these two photons must have equal energy, and they must have opposite momentum, so they go off in opposite directions to conserve linear momentum. In addition, to conserve angular momtum, they must have opposite spins. The spin of a photon is one, so the component of the spin of one photon must be opposite to that of the other.

However, the spin axis of neither photon is determined. The only thing that is determined is that the two spin axes must be oppositely directed. In fact (and here is a crucial point), it is not that we do not know what the spin axes are. They are really not determined. To the extent that a photon can "know" anything, the two photons do not know what their spins are. The experiments by Alain Aspect demonstrated that the spins in an EPR experiment are not determined until one of the spins is actually measured.

Let's consider more carefully what is happening. There is a wave function that describes the original electron-positron system. There is another wave function that describes the system of two photons. There is no wave function that describes each photon by itself. That is because the two photons are said to be "entangled." Instead, each photon is in a "mixed state" described by a density matrix. Only a "pure state" can be described by wave function.

Now suppose someone measures the x-component of the spin on one of the photons. As soon as that happens, that photon will be in a pure state, described by a wave function, and it will have a spin component of plus or minus one in the x direction. Also, at that same time, the other photon will also be in a pure state, described by a wave function, and it will have a spin component in the x direction the opposite of that of the other photon. The y and z components of both photons will immediately become zero when that measurement is made.

This is the "faster-than-light-speed" process described in the article. That is, each photon changed instantaneously from a mixed state described by a density matrix to a pure state described by a wave function, and the two photons ceased to be entangled.

Notice, however, that nothing really happened. It is not possible to send information this way. There is no faster-than-light communication as far as people are concerned. No energy propagated faster than light. Also notice that this seems like a problem only if we interpret a wave function as representing reality. If we interpret a wave function as representing only our knowledge, then nothing is mysterious. A very good 2-page explanation can be found in a preprint by Asher Peres at <http://arXiv.org/abs/quant-ph/0310010>.

The additional part of the experiment added by Suarez beyond what had been done by Alain Aspect and others does not add significantly to the difficulty in interpreting what is really meant by a wave function.

October 2008 addition: Maybe all that Suarez is really talking about here is the "apparent" faster-than-light transfer of information in the Einstein-Rosen-Podolsky effect, but this is an illusion, as is well known. It is not possible to transfer information faster than light in that way.

On the other hand, it may simply be the failure to recognize that there is a difference between the relative motion of two systems when one of the systems accelerates relative to an inertial frame.

Chapter 80

Does the wave function represent reality or our knowledge?[1]

abstract

There is no experiment, measurement, or observation that can answer the question, "Does the wave function represent reality or our knowledge?", and therefore the question is not a scientific one. We are allowed to use either interpretation to suit our purposes.

80.1 Introduction

The correct interpretation of the wave function has been debated since the beginning of quantum theory. One aspect of that interpretation is whether the wave function represents reality or our knowledge. For example, the many-worlds interpretation assumes that the wave function represents reality, whereas Feynman interpreted the wave function in a more positivist sense, in terms of knowledge. Often it is not explicitly stated whether the wave function is interpreted as representing reality or knowledge. For example, Peres (2004)[222] assumes the wave function represents knowledge to resolve the EPR paradox. If one assumes the wave function represents reality then one has the problem of apparently faster-than-light-speed collapse of the wave function.

We would like to assume that the wave function represents reality, but that leads to the 'collapse of the wave function' difficulty. The simplest example of the collapse problem is illustrated by the measurement of light in an astronomer's telescope from a distant star. Even the closest star, alpha-centauri, four light years away, is far enough away to represent the difficulty. Suppose an atom in an excited state on the surface of that star undergoes a transition and emits a photon, whose wave function expands in all directions as a spherical wave. (Neglect for now that the star may block the wave function in some directions.) That spherical wave then propagates out for four years until it is a sphere eight light years in diameter. Then that photon is detected in the telescope of an astronomer here on Earth. Since that photon is no longer available to be absorbed anywhere else, the eight-light-year-diameter wave function immediately "collapses." Although this simplified picture leaves out some details, it correctly illustrates the problem.

This example illustrates the collapse problem of wave function reality. Another problem is illustrated by the Schroedinger cat paradox, in which a dead and alive cat coexist in a quantum superposition.

One proposal to solve these problems while still keeping the reality of the wave function is the many-worlds interpretation, in which the universe splits at each collapse. Not everyone is convinced

[1] ca August 2004

that this is an improvement.

The alternate solution is the minimalist one, in which we consider the wave function to represent only our knowledge of reality. Feynman was a proponent of this "interpretation." Peres (2004)[222] recently used this interpretation to resolve the EPR paradox.

The question of whose knowledge the wave function represents is illustrated by the 'Wigner's friend' problem, and Rovelli (1996)[223] has suggested a relative-wave-function interpretation to deal with that.

The problem with assuming that the wave function represents only our knowledge is how to explain the operation of the universe when we are not watching it. This then leads to philosophical discussions such as we don't know what the universe is doing when we are not watching it.

This pretty much outlines the problem, which I deal with here as a scientific, rather than a philosophical problem.

80.2 A scientific question

If the question "Does the wave function represent reality or our knowledge?" is a scientific question, then there must be an experiment that will answer the question. Careful consideration shows, however, that there is no experiment that will answer the question. That is because experiment, measurement, and observation do not tell us about reality; these tell us only about our knowledge. Therefore, it is not a scientific question, but possibly a philosophical question. Actually, the same result holds for all of science, not just quantum theory. There are no experiments that can tell us whether our theories represent reality or just our knowledge of reality.

80.3 Reality and the goal of science

We normally think of the goal of science is to discuss the laws of nature. It then becomes disconcerting to realize that whether our theories represent reality or our knowledge of reality is not a scientific question. However, there is a bright side: because there is no experiment, observation, or measurement that can distinguish whether our theories represent reality or knowledge, we are free to assume they represent reality when we want or knowledge when we need to. They are equivalent.

Chapter 81

Perturbation by gravitational waves[1]

Is it possible that the quantum fluctuations that cause identically prepared systems to give different outcomes for identical measurements are fluctuations in the gravitational field (that is, gravitational waves)?

Although gravitational waves may slightly influence electrons, they may not explain why different electrons under apparently the same conditions land at different spots on the screen in the 2-slit diffraction experiment. The problem is that if the interaction with gravitational waves is large enough to influence the electrons, it will change the wave function and change the interference pattern. We need an interaction that will not influence the calculated interference pattern.

I wonder if maybe the gravitational waves are so weak that there is an effect from individual gravitons.

When the electron makes a spot on the screen, that spot is not the electron. It simply is the source of light that tells us that the electron had an interaction there. ψ for the electron doesn't give the probability amplitude for the presence of the electron. It gives the probability amplitude for an interaction.

Added thought in 2008

Maybe interaction with gravitational waves is the reason we have wave functions.

[1] 29 March 2005

Chapter 82

Reconciling gravitation and quantum theory[1]

abstract

It is well known that General Relativity (as a theory of gravitation) and quantum theory are inconsistent with each other. The most important problem facing theoretical physics today is almost certainly that of finding a consistent theory that includes both gravitation and quantum theory.

Although both loop quantum gravity and string theory have given important insights into this problem, there are additional important considerations that are mostly ignored by both of these theories. Although most of these considerations have been looked at before, I do not know of any effort to look at all of them together, especially in this context.

82.1 "Quantum gravity"

The term, "quantum gravity," may be misleading, in that it suggests that part of the solution to the problem is to find a quantum theory of gravitation in the same sense that we have quantum theories of the other three interactions (electromagnetic, weak, and strong). Although that may turn out to be the case, the actual solution might be more radical. Therefore, to use terms such as "quantum gravity" or "quantizing general relativity" may cause us to ignore important paths of research.

82.2 Quantum theory on the background of an inertial frame

The quantum theory we have is on a background of an inertial frame. Usually this background is a flat Minkowski frame, but sometimes extended with some success to a curved background. In any case, it is still a background.

It is now generally recognized that inertia and inertial forces are gravitational forces, not fictitious forces resulting from doing physics in a non-inertial frame. Thus, a proper role of inertia and inertial frames is not as a background, but as an active, dynamical part of the process. I do not know how to accomplish that, but since we observe that our inertial frame does not seem to rotate relative to the visible matter in the universe (and probably does not accelerate as well), the interaction we are looking for is probably a gravitational interaction with the rest of the matter in the universe.

[1] 29 December 2006

I do not know how to write a quantum theory that replaces a background inertial frame with a dynamical gravitational interaction with the rest of the universe. Nor can I even imagine how to write a quantum theory in the absence of a background inertial frame. I suggest, however, that trying to do so will lead to important insights into combining gravitation with quantum theory. Thinking about this aspect shows that gravitation and quantum theory are already closely linked.

82.3 Physics in a sparse universe

If local inertial frames and the bulk of matter in the universe are strongly connected, then what would local inertial frames (and the corresponding quantum theory) look like as the amount of matter in the universe approaches zero? Although we could end up with empty Minkowski spacetime, I suggest something else is more likely. Trying to speculate on what that something else might be may give us insight into the problem.

82.4 Discrete gravitation

Just as the continuous fluid approximation treats the molecules that make up gases and liquids in a statistical way in terms of density, pressure, temperature, and entropy, General Relativity does the same for cosmology. Suppose we were to formulate General Relativity for discrete bodies or particles instead of in terms of pressure and density? This may help us imagine what physics in a sparse universe would be like.

82.5 The anthropic principle

The anthropic principle is sometimes considered to be a cop out, allowing physics to be anything. However, requiring the laws of physics to apply consistently also to those universes where intelligent life did not exist puts some restrictions on what is allowed.

82.6 Wave mechanics as a nonlinearity

Conservation of energy and momentum as $E_1 + E_2 = E_3 + E_4$ and $p_1 + p_2 = p_3 + p_4$ becomes, if we take $E = \hbar\omega$ and $p = \hbar k$, $\omega_1 + \omega_2 = \omega_3 + \omega_4$ and $k_1 + k_2 = k_3 + k_4$, which follows from wave interference. However, we can go one level deeper.

Consider nonlinearities in physics. To first order, these will give sum and difference frequencies, and the same for wave numbers. If we expand a variable, say h, into first-order perturbations plus second-order, etc., then we have $h = h^{(1)} + h^{(2)} + \ldots$. Then by nonlinear wave interaction, we get $\omega_1^{(2)} + \omega_2^{(2)} = \omega_3^{(2)} + \omega_4^{(2)}$ and $k_1^{(2)} + k_2^{(2)} = k_3^{(2)} + k_4^{(2)}$. The point is, that our rules for quantum theory and wave mechanics may simply be a perturbation effect from some kind of nonlinearity.

Chapter 83

Origin of geometry[1]

Einstein's field equations for General Relativity can be derived from using the scalar curvature R as a Lagrangian. This shows that General Relativity is based on curvature and that it is based on a deviation from a flat Minkowski spacetime (where special relativity holds), which has zero curvature. Thus, General Relativity really is the generalization of special relativity in the sense of generalizing to nonzero curvature.

However, if we want a theory of gravitation, rather than a theory of curvature, we must do something more radical. Because gravitation (in the form of inertia) exists already in flat spacetime and special relativity, the origin of gravitation means the origin of geometry.

To develop the origin of geometry, we must be able to make calculations without reference to geometry. We can probably do that with propagators, which can be specified independent of geometry. In the same way that Einstein used curvature (that could vary from zero to any nonzero value continuously) to base his theory, we must do the same for geometry. That is, we must have a parameter that varies from zero to a nonzero value continuously to indicate the absence or presence of geometry, or amount of geometry, analogous to R for curvature.

Possibly, the parameter we want represents the amount by which the system is classical.

[1]draft: 16 January 2007

Chapter 84

Resolving the zero-point energy problem by dropping an assumption in quantum theory[1]

abstract

Dropping the assumption that zero-point energy of $\hbar\omega/2$ for electromagnetic radiation applies to each member of the ensemble in addition to being an ensemble average implies that zero-point energy exists only where such an ensemble actually exists. This implies that zero-point energy exists only in those places and for wavelengths for which there is enough electromagnetic energy in that part of the spectrum. Estimates of zero-point energy from various sources are calculated and shown to be much smaller than estimates of "dark energy."

84.1 Introduction

In quantizing the electromagnetic field, we usually model the radiation at each frequency ω of the field as a quantum oscillator, that has energy levels

$$(n + 1/2)\hbar\omega \ , \tag{84.1}$$

where n is an integer and $\hbar$ is Plank's constant. The ground state energy for each frequency is $\hbar\omega/2$. If such a ground-state energy existed everywhere for all frequencies up to some cutoff, it would lead to an equivalent gravitational mass per unit volume of about[21, p. 246].

$$\rho_{\text{zero-point}} = \frac{E_0/c^2}{\text{unit vol}} = \frac{2}{c^2}\frac{1}{(2\pi)^3}\int_0^{k_{\text{max}}} \frac{\hbar c k}{2} \cdot 4\pi k^2 dk = \frac{\hbar k_{\text{max}}^4}{8\pi^2 c} \ . \tag{84.2}$$

The initial factor of two in (84.2) includes two polarizations. The corresponding formulas [224, eq. (37)] [225, eq. (3.5)] [226, eq. (17)] do not include the factor of two. So, is Feynman correct, or are the other authors correct?

Choosing the value of the cutoff in (84.2) is crucial. The most natural choice is the Plank length[226] (that is, using a reduced Planck mass equal to $(\hbar c/8\pi G)^{1/2} \approx 4.3 \times 10^{-6}$ g $\approx 2.4 \times 10^{18}$ GeV/c^2) would give about 2×10^{71} GeV^4[225] $\approx 5 \times 10^{88}$ g cm^{-3} for the energy density of zero-point energy. That is almost 120 orders of magnitude larger than the observed density of the vacuum of 2×10^{-31} g cm^{-3}[20].

[1] Draft: August 2008, updated 27 September 2016, 3:48pm

"A peculiar and truly quantum mechanical feature of the quantum fields is that they exhibit zero-point fluctuations everywhere in space, even in regions which are otherwise 'empty' (i.e. devoid of matter and radiation)."[227]

The main effect of such a large background mass density (if it actually had a gravitational effect) would be in terms of a cosmological model. Specifically, there is a comment attributed to Wolfgang Pauli (in a cafe discussion)[227, p. 667, footnote 8], that the radius of the world "would not even reach to the moon"[224, p. 571][227, p. 667][228, p. 842]. In his own publications, however, Pauli was more careful, merely noting that the zero-point energy evidently produces no gravitational field[229, p. 250].

The cosmological constant Λ (or dark energy) also contributes to an effective energy density of the background[18, 225, p. 614], which is estimated to be about 72% of the total[230]. Λ also contributes to an effective pressure of the background. In fact, it is the effective pressure that allows the present estimates of the value of Λ through its effect on the rate of change of the Hubbel parameter.

Thus, zero-point energy behaves differently from Λ because it contributes to the average mass or energy density, but not to the pressure. Thus, there is actually no discrepancy between different values for zero-point energy and Λ because they are not the same. The only real problem is the gravitational effect that such a large energy density would have.

It is normally considered that the estimates for zero-point energy of about 10^{118} times the estimates for Λ leads to a difficulty that needs to be explained[225], but as explained above, the only real problem is the large gravitational field that is not observed. This problem, sometimes called the "cosmological constant problem," is one of the main outstanding problems in physical theory[231].

Although some popular authors suggest ideas like using zero-point energy as a possible source of antigravity or similar things[232], serious scientists do not believe that zero-point energy is really as large as the usual estimates indicate. Instead, they consider it to be a problem that needs to be resolved. Weinberg[225] and Peebles and Ratra[224] summarizes the situation with zero-point energy and the cosmological constant, mentioning several possible solutions to the difficulty.

(Equation (84.2) does not include the expansion of the universe. I think that should make a difference, but I am not sure how to include that.)

84.2 Possible Solutions

Some possible solutions to the difficulty are:

1. Taking the cutoff to correspond to a larger wavelength, say the Compton wavelength of the proton ($k_{\text{max}} = m_{proton}c/\hbar$), would give a mass density of zero-point energy of $\approx 2 \times 10^{15}$ g cm^{-3}[21, p. 246], which decreases the zero-point energy, but not enough. Although there is no theoretical justification for choosing the Compton wavelength of the proton as the cutoff, there may be justification for choosing some other cutoff.

2. Allowing the cutoff to vary with location might resolve the difficulty if a theoretical justification can be found.

3. Maybe the electromagnetic field is not quantized as we usually think it is. If so, we would have to explain the formula for black-body radiation and other things which lead us to believe the electromagnetic field is quantized. One point here is to notice that the main effect of "second quantization" is to calculate the back reaction of matter on the electromagnetic field, rather than actual quantization of the field.

In this regard, I suspect that classical electromagnetic radiation in equilibrium with hydrogen at temperature T on the surface of a star would take on a black-body spectrum. I have looked into the possibility of demonstrating that, but have so far been unsuccessful.

We normally think that the wave-particle duality is the same for fields that show some particle properties and for particles that show some wave properties. However, there may be some differences. Maybe a radiation field does not follow the same rules as wave functions for particles. In particular, maybe it is not appropriate to consider a radiation field as a quantum oscillator.

Similarly, if a black-body spectrum of electromagnetic radiation is evidence of quantization, then should gravitational waves have a black-body spectrum? If so, then why? If not, then why not? My guess is no, because gravitational waves are so weakly interacting that they have probably not had time to come to equilibrium with anything.

Except for one thing. They probably feel the existence of an inertial frame. That is, they feel the gravitational interaction with the rest of matter in the universe. This is in the same way that all particles feel the existence of inertial frames, by interaction with the rest of matter in the universe. Of course, that interaction is through gravitons. Massive particles may get their mass through that interaction. If that is so, then the Higgs particle is really the graviton.

Any changes to our model of zero-point energy must be consistent with various observations. This will be dealt with later.

84.2.1 Possible solution to the zero-point-energy problem[2]

Assuming the existence of zero point energy everywhere with a wavelength cutoff at the Planck length is sufficient to guarantee that no measurements involving such radiation can violate the uncertainty principle, but it is not necessary.

A much weaker assumption regarding zero point energy is also sufficient to guarantee that no measurements involving radiation can violate the uncertainty principle. Namely: Assuming the existence of zero point energy everywhere that there exists apparatus that could be used to indicate a violation of the uncertainty principle is sufficient to guarantee that no measurements involving such radiation can violate the uncertainty principle. This assumption shows that zero point energy could probably be much less than the usual assumption, but it would be difficult to calculate.

A stronger assumption, but still weaker than the usual assumption regarding zero point energy is also sufficient to guarantee that no measurements involving radiation can violate the uncertainty principle. Namely: Assuming the existence of zero point energy everywhere that there exists matter, such as atoms, that could absorb or radiate electromagnetic radiation at that frequency is sufficient to guarantee that no measurements involving such radiation can violate the uncertainty principle.

This assumption shows that zero point energy could be much less than the usual assumption, and it is easier to calculate. Notice first, that it is not necessary to have zero point energy in free space, that is, anywhere that is devoid of matter. Second, notice that there is a natural wavelength cutoff at a wavelength much larger than the Planck length. Namely at the shortest wavelength that can be absorbed or emitted by matter. With this assumption, zero point energy does not have to be larger than the energy of ordinary matter, and could probably be smaller, and still avoid being able to violate the uncertainty principle.

Although zero point energy can have a similar effect to the cosmological constant, it is not necessary that they are the same. That is, it is not necessary that the cause of the present acceleration of the expansion of the universe is caused by zero point energy. When the calculation of zero point energy was 120 orders of magnitude greater than the value of the cosmological constant

[2] Added 26 January 2015

that would be consistent with the observed acceleration, that was a contradiction. But there is no contradiction if the zero point energy is less than the cosmological constant. The cosmological constant could have a different source.

84.3 A wave function represents an ensemble

A crucial issue is whether (84.1) applies to only an ensemble or whether it applies also to each member of the ensemble. Although wave functions, transition probabilities, and energy levels can be verified only for ensembles, it is often assumed that they apply to each member of the ensemble. Making that assumption for (84.1) leads to the usual difficulty in zero-point energy. Let us consider whether dropping that assumption can help us resolve the difficulty.

If we instead assume that (84.1) applies only to an ensemble, then it will apply only when the appropriate ensemble actually exists. That is, for each frequency ω, at some location, there must be an ensemble of systems that actually have the possibility of having the energy levels given by (84.1). To be more specific, there must be significant electromagnetic energy at that frequency at that location or there must be some mechanism at that location to create electromagnetic energy at that frequency.

This is clearly a controversial hypothesis, but let us consider whether such a hypothesis resolves the zero-point energy problem. To do this, we shall estimate the frequency spectrum of electromagnetic radiation and mechanisms for generating such radiation in various regions of the universe. We start by multiplying (84.2) by one to give

$$\rho_{\text{zero-point}} = \frac{\hbar k_{\text{max}}^4}{8\pi^2 c} \frac{\overline{\rho}_{\text{Baryons}}}{m_{\text{Baryons}}} \int d^3x \ , \tag{84.3}$$

with obvious notation for the average density and mass of ordinary matter (Baryons) in the universe. Suppose k_{max} varies with location. Then (84.3) becomes

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = \frac{\hbar}{8\pi^2 c} \frac{1}{m_{\text{Baryons}}} \int k_{\text{max}}^4 d^3x \ , \tag{84.4}$$

where, (84.4) gives zero-point mass density relative to Baryon mass density, which can be compared with estimates of $\rho_\Lambda/\overline{\rho}_{\text{Baryons}} = 16 \pm 1.26$ derived from a cosmic energy inventory[230], where ρ_Λ is the density of dark energy. Before proceeding further, we notice that if we are going to consider the sources of zero-point energy to be from ordinary matter, then $\rho_{\text{zero-point}}/\overline{\rho}_{\text{Baryons}} < 1$. However, all of the zero-point electromagnetic energy should already be included in the estimate of all electromagnetic radiation of $\rho_\gamma/\overline{\rho}_{\text{Baryons}} \approx 10^{-3}$ from the cosmic energy inventory[230].

Although it is not really necessary to actually estimate zero-point energy directly, to demonstrate how it works, I will estimate some sources. To correctly estimate ground-state energy, we need to consider contributions from various sources. In the vacuum, we have contributions from the microwave background radiation and radiation from the stars. In the interior of stars, planets, and interstellar gas, there are mechanisms to locally generate electromagnetic radiation. We shall consider each of these in turn. Notice at this point that (84.4) is only a generalization of (84.2). Using a constant cutoff everywhere would still recover (84.2).

Here, I shall try to distinguish between ground-state energy for a specific system, and its ensemble average for an ensemble of similar systems, which I shall refer to as zero-point energy.

The hypothesis here, that energy-level calculations apply only to ensembles, but not to members of the ensembles, does not change anything measurable in quantum theory. That is, this hypothesis has no consequences for measurements made here on Earth. In fact, (84.2) will apply here, but with k_{max} determined by physical processes here on Earth that generate electromagnetic radiation or by radiation that arrives here from outside.

84.4 Radiation

Under our hypothesis, in the vacuum of free space, where there is no significant matter, we would have no zero-point energy except from the cosmic microwave background radiation and radiation from the stars.

84.4.1 Cosmic microwave background radiation

Consider the cosmic microwave background radiation first. There are two frequency regimes. For $k < k_{\mathrm{max}}$, where the radiation has more than half a photon at each frequency, we can use (84.2) to estimate the energy density. Above that, the energy density would be less than that of a half of a photon per mode, and we can calculate the total energy density in that tail. Adding the two contributions together gives a total of

$$\rho_{\text{zero-point}} = \frac{\hbar k_{\mathrm{max}}^4}{8\pi^2 c} + \frac{\hbar}{c\pi}\int_{k_{\mathrm{max}}}^{\infty} \frac{k^3 dk}{e^{\frac{\hbar c k}{k_B T}} - 1}\ , \tag{84.5}$$

where k_B is Boltzman's constant, $T \approx 2.7$ K is the absolute temperature, and $k_{\mathrm{max}} = \ln(1 + 2\pi)k_B T/(c\hbar) \approx 2k_B T/(c\hbar)$. This estimate for k_{max} is made by comparing the integral in (84.5) with that in (84.2). We notice that the zero-point energy $\hbar\omega/2 = \hbar ck/2$ in (84.2) is replaced by $(\hbar ck/2)2\pi/(e^{\frac{\hbar ck}{k_B T}} - 1)$ in (84.5). The value of k_{max} is the value of k where the factor of $\hbar ck/2$ falls below one.

It does not mean that there is no electromagnetic energy above k_{max}, nor that it is impossible to absorb photons above that frequency. Only that the effective number of photons per photon state is less than a half.

We can neglect the one in the denominator of the integral to give about 1.3×10^{-33} g cm^{-3}, or about two orders of magnitude below the average matter density of the universe. Thus, the contribution of the cosmic microwave background radiation to zero-point energy is small.

84.4.2 Solar Radiation

If we assume that the Sun is a black body with a surface temperature T, then the Solar radiation in the wavenumber range from k to $k+dk$ (in, say, watts per square cm) a distance r from the Sun is

$$\frac{R_\odot^2}{r^2}\frac{c^2\hbar}{4\pi^2}\frac{k^3 dk}{e^{\frac{\hbar ck}{k_B T}} - 1}\ . \tag{84.6}$$

(To get the total radiated power from the Sun in that wavenumber range, we would multiply by $4\pi r^2$.) So, the energy density (in, say, erg per cubic cm) in that wavenumber range is

$$\frac{R_\odot^2}{r^2}\frac{\hbar c}{4\pi^2}\frac{k^3 dk}{e^{\frac{\hbar ck}{k_B T}} - 1}\ . \tag{84.7}$$

This radiation would be enough to give the usual formula for zero-point energy below some maximum frequency, depending on the distance from the Sun. Thus, we get the usual half a photon for zero-point energy up to some k_{max}, and then less than that for larger wavenumbers. Except where there are planets or other bodies (which we shall consider separately), this gives

$$\rho_{\text{zero-point}} = \frac{\hbar k_{\mathrm{max}}^4}{32\pi^3 c} + \frac{R_\odot^2}{r^2}\frac{\hbar}{4c\pi^2}\int_{k_{\mathrm{max}}}^{\infty} \frac{k^3 dk}{e^{\frac{\hbar ck}{k_B T}} - 1}\ , \tag{84.8}$$

where $k_{\text{max}} = \ln(1 + 2\pi R_\odot^2/r^2)k_BT/(c\hbar)$. At the Earth's distance from the Sun, this would be $k_{\text{max}} \approx 3.5\ \text{cm}^{-1}$.

Notice the factor of 4π in (84.5) that is not in (84.8). That is because the radiation in (84.5) is in all directions, whereas the radiation in (84.8) is directed only away from the Sun. (I am wondering if simply a factor of 4π is enough to make that distinction. I wonder how I justify that.)

For the rough estimates we are making here, we can split the integral into two regimes, depending on the exponent in the denominator. For the exponent smaller than one, we expand and keep only two terms. For the exponent larger than one, we neglect the one in the denominator. This gives

$$\rho_{\text{zero-point}} = \frac{\hbar k_{\text{max}}^4}{32\pi^3 c} + \frac{R_\odot^2}{r^2}\frac{\hbar}{4c\pi^2}\int_{k_{\text{max}}}^{k_BT/(\hbar c)} \frac{k^3 dk}{e^{\frac{\hbar ck}{k_BT}} - 1} + \frac{R_\odot^2}{r^2}\frac{\hbar}{4c\pi^2}\int_{k_BT/(\hbar c)}^{\infty} \frac{k^3 dk}{e^{\frac{\hbar ck}{k_BT}} - 1}\ , \tag{84.9}$$

That is,

$$\rho_{\text{zero-point}} = \frac{\hbar k_{\text{max}}^4}{32\pi^3 c} + \frac{R_\odot^2}{r^2}\frac{\hbar}{4c\pi^2}\int_{k_{\text{max}}}^{k_BT/(\hbar c)} \frac{k^3 dk}{\frac{\hbar ck}{k_BT}} + \frac{R_\odot^2}{r^2}\frac{\hbar}{4c\pi^2}\int_{k_BT/(\hbar c)}^{\infty} \frac{k^3 dk}{e^{\frac{\hbar ck}{k_BT}}}\ . \tag{84.10}$$

Or,

$$\rho_{\text{zero-point}} = \frac{\hbar k_{\text{max}}^4}{32\pi^3 c} + \frac{R_\odot^2}{r^2}\frac{\hbar}{4c\pi^2}\frac{k_BT}{\hbar c}\int_{k_{\text{max}}}^{k_BT/(\hbar c)} k^2 dk + \frac{R_\odot^2}{r^2}\frac{\hbar}{4c\pi^2}\int_{k_BT/(\hbar c)}^{\infty} e^{-\frac{\hbar ck}{k_BT}} k^3 dk\ . \tag{84.11}$$

For several Solar radii away from the Sun, this is approximately

$$\rho_{\text{zero-point}} = \frac{\hbar}{\pi^2 c}\left(\frac{k_BT}{\hbar c}\right)^4 \frac{R_\odot^2}{r^2}\left(\frac{1}{16} + \frac{1}{12} + \frac{4}{e}\right)\ . \tag{84.12}$$

At the Earth's distance from the Sun, this would be about $\times 10^{-26}$ g cm^{-3}. To get the total contribution of the Sun to zero-point radiation, we can average (84.12) to give

$$\rho_{\text{zero-point}} = \frac{3\hbar}{\pi^2 c}\left(\frac{k_BT}{\hbar c}\right)^4 \frac{R_\odot^2}{R_{\text{max}}^2}\left(\frac{1}{16} + \frac{1}{12} + \frac{4}{e}\right)\ . \tag{84.13}$$

where R_{max} is the radius of the sphere over which the averaging is done.

84.4.3 Star light

To get the contribution of all stellar radiation would require summing terms like (84.13) over all of the stars. In doing this, we have to be careful of Olber's paradox, but I think the contribution is small.

84.5 Interior of stars and atmospheres of planets

Within planets and stars, the usual formula (84.2) holds, but with a different cutoff, which may vary with location within the star or planet. Once we have estimated the ground-state energy within the star or planet, we would need to average it over the whole universe to get its contribution to zero-point energy using (84.4).

84.5.1 On the Earth

For the most part, here on the Earth, we would notice no difference based on my hypothesis. That is, we would still have the usual formula for zero-point energy, but the cutoff would vary with location (and possibly with time) and would not be the Planck length, but rather would depend on the physics of the situation with regard to emission and absorption of radiation.

84.5.2 Interior of the Sun

Let us consider the contribution of just the interior of the sun to (84.4). We assume the sun to be spherically symmetric. This gives

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = \frac{\hbar}{8\pi^2 c}\frac{1}{\overline{\rho}_\odot}\frac{\overline{\rho}_\odot}{m_{\text{Baryons}}}\int_0^{R_\odot} k_{\text{max}}^4 4\pi r^2 dr \ . \tag{84.14}$$

If we now change to a relative radial coordinate for integration, (84.14) becomes

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = \frac{\hbar}{8\pi^2 c}\frac{m_\odot}{m_{\text{Baryons}}}\frac{1}{\overline{\rho}_\odot}\int_0^1 3k_{\text{max}}^4 (r/R_\odot)^2 d(r/R_\odot) \ . \tag{84.15}$$

Fusion in the Sun is through the proton-proton cycle, which generates photons with energies of about 4.5 Mev, which corresponds to a k_{max} of about 1.6×10^{11} ev, or a temperature of about 6×10^{11} K, but [233, p. 21] the temperature at the center of the sun is only about 11.5×10^6 K. When the photons are produced, they are not in equilibrium with their surroundings, but they come to equilibrium in about a thousand collisions, which takes only a fraction of a second to cool to about 11.5 million Kelvin, or about 10^3 ev. This corresponds to

$$k_{\text{max}} = \left(\frac{k_B T}{\hbar c}\right) \ , \tag{84.16}$$

or $k_{\text{max}} \approx 5 \times 10^7$ cm^{-1}, so that (84.15) becomes

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = \frac{\hbar}{8\pi^2 c}\frac{m_\odot}{m_{\text{Baryons}}}\frac{1}{\overline{\rho}_\odot}\int_0^1 3\left(\frac{k_B T}{\hbar c}\right)^4 (r/R_\odot)^2 d(r/R_\odot) \ . \tag{84.17}$$

Relative to the temperature T_{c} at the center of the sun, (84.17) becomes

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = \frac{\hbar}{8\pi^2 c}\frac{m_\odot}{m_{\text{Baryons}}}\frac{1}{\overline{\rho}_\odot}\left(\frac{k_B T_{\text{c}}}{\hbar c}\right)^4\int_0^1 3\left(\frac{T}{T_{\text{c}}}\right)^4 (r/R_\odot)^2 d(r/R_\odot) \ . \tag{84.18}$$

The energy of the photons decreases from the center of the Sun to its surface, following the temperature of the Sun. We can take the temperature to be roughly a Gaussian function of the radius ([234, p. 419]), so that (84.18) becomes

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = \frac{\hbar}{8\pi^2 c}\frac{m_\odot}{m_{\text{Baryons}}}\left(\frac{k_B T_{\text{c}}}{\hbar c}\right)^4\frac{1}{\overline{\rho}_\odot}\int_0^1 3\alpha^2 e^{-36\alpha^2} d\alpha \ . \tag{84.19}$$

We can estimate the integral by extending the upper limit to infinity, which adds a negligible amount, to give

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = \frac{\hbar}{8\pi^2 c}\frac{m_\odot}{m_{\text{Baryons}}}\left(\frac{k_B T_{\text{c}}}{\hbar c}\right)^4\frac{1}{\overline{\rho}_\odot}\frac{\sqrt{\pi}}{288} \ . \tag{84.20}$$

We can put in numbers to give

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_\odot}{m_{\text{Baryons}}}\frac{T_{\text{c}}^4}{\overline{\rho}_\odot} \ . \tag{84.21}$$

Taking $m_\odot = 1.989 \times 10^{33}$ g cm^{-3} and $R_\odot = 6.96 \times 10^{10}$ cm[233, p. 432] gives $\overline{\rho}_\odot = 1.4$ g cm^{-3}. Combining that with a central temperature for the sun $T_\odot = 11.5 \times 10^6$ K[233, p. 21] gives about $\rho_{\text{zero-point}}/\overline{\rho}_{\text{Baryons}} \approx 0.125 m_\odot/m_{\text{Baryons}}$. If all stars were like the Sun, and this were the only source of zero-point energy, then this would be only about a factor of 100 below the value of $\rho_\Lambda/\overline{\rho}_{\text{Baryons}} = 16 \pm 1.26$ derived from a cosmic energy inventory[230].

I still have to estimate the number of photons that have not yet thermalized, and calculate their contribution to the zero-point energy.

In the next section, we consider the contributions of other stars in the Main Sequence.

84.5.3 Main Sequence Stars

Stars hotter than the Sun can use the carbon cycle for fusion from hydrogen to helium. This produces three photons with energies of about 1.9 Mev, 7.5 Mev, and 7.3 Mev. 7.5 Mev gives a k_{max} of about 2.7×10^{11} ev. These photons are confined to the center of the star. The temperature of 7.5 Mev photons is about 10^{12} K, but the temperature at the center of main sequence stars is considerably smaller than that.

If we assume that all stars in the main sequence have the temperature falling off from the center as a Gaussian (as in the Sun), then I can use (84.21) for other stars as well. This give us

$$\frac{\rho_{\text{zero-point}}}{\bar{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_\star}{m_{\text{Baryons}}}\frac{T_{\text{c}}^4}{\bar{\rho}_\star}\ . \tag{84.22}$$

We can apply this to two other stars for comparison. From [235] we have for Altair, 13 million K, density of 1/8, gives $\rho_{\text{zero-point}}/\bar{\rho}_{\text{Baryons}} \approx 2.28 m_\star/m_{\text{Baryons}}$. From [235] we have for Cygni B, 5 million K, density of 2, gives $\rho_{\text{zero-point}}/\bar{\rho}_{\text{Baryons}} \approx 0.003 m_\star/m_{\text{Baryons}}$.

Equation (84.22) gives the contribution of just one star. To estimate the contribution from all of the stars in the main sequence, we need to have an estimate for how many stars there are as a function of their mass, and also how the central temperature and average density (or alternatatively, the radius) of the star vary as a function of mass.

For the former, we can use an initial mass function given by Fukugita and Peebles[230].

$$dN/dm \propto m^{-(x+1)}\ , \tag{84.23}$$

where the exponent x takes on different constant values in each of three mass regimes.

For the latter, we use similar models that give central star temperature (with an exponent of 0.22) and star radius (with an exponent of 0.75)[233, p. 28]. We can then rwrite (84.22) in terms of star radius

$$\frac{\rho_{\text{zero-point}}}{\bar{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{m_\odot}{\bar{\rho}_\odot R_\odot^3}\frac{\int T_{\text{c}}^4 R_\star^3 m_\star^{-1-x} dm_\star}{\int m_\star^{-x} dm_\star}\ , \tag{84.24}$$

where m_{MS} is the total mass of the main sequence stars (to cancel the integral in the denominator), we have normalized to some solar values, and the two integrals will be performed in three segments. Now we put in the models for temperature and radius to get

$$\frac{\rho_{\text{zero-point}}}{\bar{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4 m_\odot^{2.25-.88+1}}{\bar{\rho}_\odot}\frac{\int m_\star^{0.88-2.25-1-x} dm_\star}{\int m_\star^{-x} dm_\star}\ , \tag{84.25}$$

where the two integrals must still be performed in three segments for the three values of the exponent x. Separating the integrals into three segments gives

$$\frac{\rho_{\text{zero-point}}}{\bar{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4 m_\odot^{2.37}}{\bar{\rho}_\odot}$$
$$\frac{\int_{0.01m_\odot}^{0.1m_\odot} m_\star^{-2.37-x} dm_\star + c_1\int_{0.1m_\odot}^{1m_\odot} m_\star^{-2.37-x} dm_\star + c_2\int_{1m_\odot}^{100m_\odot} m_\star^{-2.37-x} dm_\star}{\int_{0.01m_\odot}^{0.1m_\odot} m_\star^{-x} dm_\star + c_1\int_{0.1m_\odot}^{1m_\odot} m_\star^{-x} dm_\star + c_2\int_{1m_\odot}^{100m_\odot} m_\star^{-x} dm_\star}\ , \tag{84.26}$$

where c_1 and c_2 are constants to make the initial mass function continuous at the boundaries. There are two models. Here is the first

$$\frac{\rho_{\text{zero-point}}}{\bar{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4 m_\odot^{2.37}}{\bar{\rho}_\odot}$$

$$\frac{\int_{0.01m_\odot}^{0.1m_\odot} m_\star^{-2.37+.5}dm_\star + c_1\int_{0.1m_\odot}^{1m_\odot} m_\star^{-2.37-.25}dm_\star + c_2\int_{1m_\odot}^{100m_\odot} m_\star^{-2.37-1.35}dm_\star}{\int_{0.01m_\odot}^{0.1m_\odot} m_\star^{.5}dm_\star + c_1\int_{0.1m_\odot}^{1m_\odot} m_\star^{-.25}dm_\star + c_2\int_{1m_\odot}^{100m_\odot} m_\star^{-1.35}dm_\star}\ , \tag{84.27}$$

and here is the second

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4 m_\odot^{2.37}}{\overline{\rho}_\odot}$$

$$\frac{\int_{0.01m_\odot}^{0.08m_\odot} m_\star^{-2.37+.7}dm_\star + c_3\int_{0.08m_\odot}^{0.5m_\odot} m_\star^{-2.37-.3}dm_\star + c_4\int_{0.5m_\odot}^{100m_\odot} m_\star^{-2.37-1.3}dm_\star}{\int_{0.01m_\odot}^{0.08m_\odot} m_\star^{.7}dm_\star + c_3\int_{0.08m_\odot}^{0.5m_\odot} m_\star^{-.3}dm_\star + c_4\int_{0.5m_\odot}^{100m_\odot} m_\star^{-1.3}dm_\star}\ , \tag{84.28}$$

$$c_1 = (0.1m_\odot)^{0.5+0.25} = (0.1m_\odot)^{0.75}\ , \tag{84.29}$$

$$c_2 = (0.1m_\odot)^{0.5+0.25}(1m_\odot)^{-0.25+1.35} = (0.1)^{0.75}(m_\odot)^{1.85}\ , \tag{84.30}$$

$$c_3 = (0.08m_\odot)^{0.7+0.3} = 0.08m_\odot\ , \tag{84.31}$$

$$c_4 = (0.08m_\odot)^{0.7+0.3}(0.5m_\odot)^{-0.3+1.3} = 0.04(m_\odot)^2\ . \tag{84.32}$$

So the first model gives

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4 m_\odot^{2.37}}{\overline{\rho}_\odot}$$

$$\frac{\int_{0.01m_\odot}^{0.1m_\odot} m_\star^{-1.87}dm_\star + c_1\int_{0.1m_\odot}^{1m_\odot} m_\star^{-2.62}dm_\star + c_2\int_{1m_\odot}^{100m_\odot} m_\star^{-3.72}dm_\star}{\int_{0.01m_\odot}^{0.1m_\odot} m_\star^{.5}dm_\star + c_1\int_{0.1m_\odot}^{1m_\odot} m_\star^{-.25}dm_\star + c_2\int_{1m_\odot}^{100m_\odot} m_\star^{-1.35}dm_\star}\ , \tag{84.33}$$

and the second model gives

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4 m_\odot^{2.37}}{\overline{\rho}_\odot}$$

$$\frac{\int_{0.01m_\odot}^{0.08m_\odot} m_\star^{-1.67}dm_\star + c_3\int_{0.08m_\odot}^{0.5m_\odot} m_\star^{-2.67}dm_\star + c_4\int_{0.5m_\odot}^{100m_\odot} m_\star^{-3.67}dm_\star}{\int_{0.01m_\odot}^{0.08m_\odot} m_\star^{.7}dm_\star + c_3\int_{0.08m_\odot}^{0.5m_\odot} m_\star^{-.3}dm_\star + c_4\int_{0.5m_\odot}^{100m_\odot} m_\star^{-1.3}dm_\star}\ . \tag{84.34}$$

So, the first model gives

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4}{\overline{\rho}_\odot}231.2\ , \tag{84.35}$$

and the second model gives

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 10^{-29}\ \text{g cm}^{-3}\ \text{K}^{-4}\ \frac{m_{\text{MS}}}{m_{\text{Baryons}}}\frac{T_\odot^4}{\overline{\rho}_\odot}146.7\ . \tag{84.36}$$

Taking $m_\odot = 1.989 \times 10^{33}$ g cm^{-3} and $R_\odot = 6.96 \times 10^{10}$ cm[233, p. 432] gives $\overline{\rho}_\odot = 1.4$ g cm^{-3}. Combining that with a central temperature for the sun $T_\odot = 11.5 \times 10^6$ K[233, p. 21] gives

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 28.9\frac{m_{\text{MS}}}{m_{\text{Baryons}}} \tag{84.37}$$

for the first model and

$$\frac{\rho_{\text{zero-point}}}{\overline{\rho}_{\text{Baryons}}} = 18.3\frac{m_{\text{MS}}}{m_{\text{Baryons}}} \tag{84.38}$$

for the second model. Taking $m_{\text{MS}}/m_{\text{Baryons}} = 0.06$[230] gives $\rho_{\text{zero-point}}/\overline{\rho}_{\text{Baryons}} = 1.73$ for the first model and 1.1 for the second model. These values seem a little large.

84.5.4 Warm intergalactic plasma

Warm intergalactic plasma has about 89% of the Baryon mass[230], so it is important to include that in the calculations. However the low temperature probably significantly decreases the importance of intergalactic plasma as a source of zero-point energy.

84.5.5 Giant stars

Fukugita and Peebles[230] apparently include giant stars in the main sequence, so I have to find another source for their abundance. The interior of giant stars can be calculated the same as for the main sequence, except that they have a higher interior temperature (T_c as high as 10^8 K) and are much larger (10 times the surface area, or so). However, the high interior temperature seems to be confined to a central core.

84.5.6 Dwarf stars

White dwarfs make up about 0.8% of the Baryon mass[230]. White dwarfs have about 0.6 ± 0.1 $M_\odot$, and a radius about $10^{-2} R_\odot$. That gives a density of about 10^6 g cm^{-3}.

84.5.7 Neutron stars

Neutron stars make up about 0.11% of the Baryon mass[230]. In neutron stars, we probably have some possibility of generating some energetic gamma rays.

84.5.8 Black holes

Black holes make up about 0.16% of the Baryon mass[230]. I am not sure how to estimate the effect of black holes.

84.6 Reasons for Zero-point Energy

In spite of the inconvenient problem of the large zero-point energy, there are usually three reasons for keeping zero-point energy. First, it is often considered the source of spontaneous emission [236]. Second, it is deemed necessary for the uncertainty principle[237]. Third, the only direct effect that has been attributed to the zero-point energy is the Casimir effect. Each of these three arguments are considered below.

84.6.1 Spontaneous emission

The ground-state energy of half a photon is often considered the source of spontaneous emission, with an effectiveness equal to that of a whole photon in stimulated emission, but no effectiveness for stimulated absorption when making a semiclassical calculation[236]. However, a full quantum calculation using quantum electrodynamics[238] shows that the term for spontaneous emission comes from the commutator relations for the creation and annihilation operators.

84.6.2 Uncertainty relations

The electric E and magnetic B fields cannot be measured simultaneously precisely. A simple explanation is that a measurement of E will disturb B and vise-versa. The corresponding commutator relations[239, p. 33][240, p. 76] are equivalent to[239, p. 33] $\Delta N \Delta \phi \gtrsim 1$, where N is the photon number and ϕ is the phase, which is consistent with my hypothesis (discussed below) of a varying ground-state energy within the ensemble.

For those modes where I claim there is very little energy, however, the above uncertainty relation still holds with $\Delta\phi$ very large.

84.6.3 Casimir force

The Casimir force is a force of attraction between two parallel plates that has nothing to do with gravitational or electrostatic force. Between the two plates, there are fewer zero-point modes because the wavelengths larger than the plate separation are missing. Outside of the plates, we have all of the zero-point modes pushing against the plates. Thus, there is a net force pushing the plates together, and measurements agree with the usual calculation of this effect.

The Casimir force is usually considered to be the only direct evidence for zero-point energy, but Jaffe[241] argues that the Casimir effect is not really an indication of zero-point energy.

Even if Jaffe were wrong, however, the Casimir force would be an indication of zero-point energy only for wavelengths longer than the separation between the plates, where there is no cancellation, but is insensitive to the shorter wavelengths, because their effect cancels. Thus, using the distance between the plates as the cutoff in (84.2) instead of the Planck length would reduce the discrepancy of 120 orders of magnitude or so.

84.7 What determines ground-state energy?

What would determine the ground-state energy for a given frequency in such a hypothesis? It could be simply a random initial condition. That is, the energy at a given frequency at some initial time could take on an arbitrary value. The ground state for members of an ensemble would then be uniformly distributed between zero and $\hbar\omega$, as discussed below.

In addition, because the radiation absorbed or emitted during an atomic (or other) transition takes a finite time, the spectrum of the associated radiation has a finite width. Further, because absorption or emission lines can be close together, emission or absorption at one line can change the amount of radiation available to affect absorption or emission at an adjacent line. Thus, the ground-state energy for a given line can change with time or location.

The ensemble average of the ground-state energy for each frequency ω would still be $\hbar\omega/2$, just as it is in standard quantum theory today. Since wave functions describe only ensembles, and only energy differences are directly measurable, this hypothesis could not be easily distinguished from standard quantum theory.

84.8 A nearly classical electromagnetic field

Suppose that the ground-state energy at each frequency ω could have any value between zero and $\hbar\omega$ with nearly uniform probability if there really is an ensemble. In fact, that is what we would expect classically if the initial energy at any frequency could take on nearly any arbitrary value. After removing energy at that frequency by atomic (or other) transitions, the ground state that remains would have any value between zero and $\hbar\omega$ with usually nearly uniform probability.

The ensemble average of the ground-state energy would still be $\hbar\omega/2$, just as it is in standard quantum theory today. Since wave functions describe only ensembles, and only energy differences are directly measurable, this hypothesis could not be easily distinguished from standard quantum theory.

So far, this hypothesis leads to nothing new. Not only is it not distinguishable from standard quantum theory, but it does not solve the vacuum zero-point energy problem.

Let us consider the nature of electromagnetic quanta. Are they really distinguishable particles that do not blend into one another? Although they can certainly be produced localized in time and space, there is nothing to keep quanta of electromagnetic radiation from combining with other quanta when they share the same region of spacetime because they are bosons. In other words, some aspects of electromagnetic radiation are classical.

Compton scattering, for example, can be explained by representing both the photon and the particle as plane waves, the latter with the appropriate DeBroglie wavelength. However, one way in which we cannot consider the electromagnetic field to be classical is uncertainty relations, which was considered in a previous section.

84.9 quantum transitions

If we do time-dependent perturbation theory [182, Chapter VIII], then we get a formula like [182, (29.8) on page 197], which shows that the transition amplitude is proportional to the Fourier frequency component corresponding to the transition energy of the transition matrix element. The formulas for the transition matrix element for interaction with the electromagnetic field are given by [182, (35.14) on page 249], which show that the interaction of the atom with the electromagnetic field depends only on the local value of the field at the atom. (But it depends very specifically on the initial and final atomic states.)

In a very crude way, we can think of this for emission as a radio antenna transmitting a radio wave with a very specific antenna pattern. For absorption, we can again think of a radio receiving antenna with a specific antenna pattern interacting with the incoming radio wave. That is, the interaction is not with a plane wave in either case, although it is a nearly monochromatic wave in both cases.

However, this is a gross simplification. First, the calculation is of the transition amplitude for the atomic states only. It is not meant to give the characteristics of absorbed or emitted wave. To infer the characteristics of the absorbed or emitted wave from the part of the field that is responsible for the transition is not necessarily valid.

In addition, the calculation ignores the back reaction of the atom on the field. It is this back reaction that must be taken into account for a correct quantum calculation, especially in the situation that we are most interested, where the field is very weak. That is, where there is only a fraction of a photon available to interact.

What I would like to say is that we should look at that matrix element that involves an integral of the field over the size of the atom to see what fraction of a photon is available.

Suppose there is a region of spacetime where there is not much electromagnetic radiation at some frequency. Then there may not be enough radiation to make a ground state of even a tenth or a hundredth of a quantum. Maybe not even 10^{-15} of a quantum. In that case, the ground-state energy for that frequency in that region of spacetime would be correspondingly small and would no longer be uniformly distributed between zero and one quantum.

I need to be really careful here.

Although it might be possible to devise a situation where the radiation in the appropriate frequency band is weak enough to give a nonuniform distribution of electromagnetic ground-state energy within the ensemble, but still strong enough to make measurements, I cannot actually think of how to explicitly do that.

Although the hypothesis here gives a good physical reason for why the ensemble average of the ground-state energy should be $\hbar\omega/2$, it gives no explanation for why standard quantum theory (in terms of the quantum oscillator) should give the same result. The prediction of standard quantum theory that the ground state energy for the quantum oscillator is $\hbar\omega/2$ seems to apply when there really is an ensemble, but not otherwise. Looking into this may give insight into the true meaning of wave functions in quantum theory. Although the hypothesis proposed here is a hidden-variables theory, it is not a local hidden-variables theory because the hidden variable (ground-state energy) is associated with the system, not with a specific particle.

84.10 Conclusion

The conclusion, then, is that the zero-point radiation is a half quantum only for frequencies for which there are ways of producing and absorbing such radiation. And even then, only in regions of spacetime where there exist atoms or particles that can take part in such transitions.

Thus, Any place or time where we can make measurements, the results are the same as with the present theory. However, any place else (and that includes most of spacetime), the ground-state energy (and therefore, its ensemble average, the zero-point energy) is no more than what we would expect classically. Thus, there is no need for an enormous vacuum zero-point energy.

Chapter 85

The criteria for a solution of the field equations to be a classical limit of a quantum cosmology[1]

abstract[2]

If the gravitational field is quantized, then a solution of Einstein's field equations is a valid cosmological model only if it corresponds to a classical limit of a quantum cosmology. To determine which solutions are valid requires looking at quantum cosmology in a particular way. Because we infer the geometry by measurements on matter, we can represent the amplitude for any measurement in terms of the amplitude for the matter fields, allowing us to integrate out the gravitational degrees of freedom. Combining that result with a path-integral representation for quantum cosmology leads to an integration over 4-geometries. Even when a semiclassical approximation for the propagator is valid, the amplitude for any measurement includes an integral over the gravitational degrees of freedom. The conditions for a solution of the field equations to be a classical limit of a quantum cosmology are: (1) The effect of the classical action dominates the integration, (2) the action is stationary with respect to variation of the gravitational degrees of freedom, and (3) only one saddlepoint contributes significantly to each integration.

85.1 Mach's Principle (Further Afterthoughts from chapter 75) [3]

Newton noticed that his laws of motion applied in a frame that does not rotate relative to the "fixed stars," and concluded the existence of an "absolute space" with the stars fixed in that absolute space (what we now refer to as an "inertial frame").

Mach denied the existence of absolute space, pointing out that only relative motions are observable. Instead of absolute space, he postulated that inertia is caused by an interaction with the rest of the universe. He suggested that what we now refer to as an inertial frame is caused by the rest of matter in the universe (now referred to as frame dragging).

Einstein based his General Relativity partly on Mach's ideas, and coined the term "Mach's principle" for the idea that inertial frames should be determined by the matter distribution. The term "Mach's principle" now means many things to many people, and there is no general agreement on what it means or on the validity of the various versions.

[1] This is the version I was revising to re-submit somewhere. I need to start over. April 2012
[2] file jun06abs.tex
[3] file qcos09.tex, Tuesday 14 April 2009

Although Einstein's General Relativity does include frame dragging, frame dragging is not complete. There are many solutions of Einstein's field equations in which there is inertia in the absence of matter or where there is rotation of inertial frames relative to the average matter distribution. This is because initial and boundary conditions also contribute to inertia.

Some researchers have proposed that boundary and initial conditions be chosen so that inertia and/or the metric be determined by the matter distribution. Some researchers have proposed that Mach's principle be used as a selection principle to discard solutions that have relative rotation of matter and inertial frames or have inertia without matter.

I have argued that if the metric (gravitation) is determined solely by sources (the matter), with no dependence on initial or boundary conditions, then gravitation would be very different from electromagnetic theory, in which the electromagnetic field has its own degrees of freedom, and can have arbitrary initial and boundary conditions. Thus, I have rejected Mach's principle in such a restricted form.

That leaves us still with the problem of explaining why we observe no relative rotation of inertial frames and the visible matter in the universe. I suggested that if we consider quantum gravity, that we could allow all of the solutions to the field equations, and that the non-physical solutions (those with relative rotation of inertial frames and matter) would cancel each other out if we look at the change in the action with variation of the initial and boundary conditions on the gravitational field.

It was at this point that I made a mistake in the present manuscript. I suggested that we vary the initial and boundary conditions on the gravitational field while holding the matter distribution constant. It was pointed out that there is no general procedure for doing that because it is not possible to specify the matter distribution independent of the geometry.

What I should do instead (and what I actually did in my calculations) is consider a family of solutions to the field equations (such as one of the Bianchi cosmologies) in which there is an initial condition that determines the amount of relative rotation of inertial frames and matter, and consider the change in action as that parameter is varied. In this way, we have no problems of the kind I mentioned above. Then, a saddlepoint approximation to the path integral for that parameter will give the solution for zero relative rotation.

The next section should be titled:

Gravitational Induction[4]

85.2 Introduction (Sunday 20 August 2006)

It is necessary to explain why we observe no relative rotation between our local inertial frame and the bulk of visible matter in the universe.

Sciama (1953)[12] showed that inertia could be explained in analogy to electromagnetic induction. Although his argument had a germ of truth, in that frame dragging may be analogous to induction, his actual calculation was not strictly rigorous even in electromagnetic theory, because he carried out his calculation in a non-inertial frame.

Sciama, Waylen, and Gilman (1969)[16] remedied that problem by deriving an integral formulation of Einstein's field equations valid in an arbitrary frame. However, that did not explain why there is no relative rotation between inertial frames and matter, because the surface term allowed arbitrary initial and boundary conditions.

Pfister (1994)[242] showed that rotating cylindrical shells have 100% frame dragging when $v = c$, where v is the velocity of the shell.

[4] file intro3.tex

Barbour (1994)[153] shows that General Relativity is Machian in the sense that it can be formulated in terms of relative configurations, but does not explain why there is no observed relative rotation between matter and inertial frames.

Although frame dragging is probably an induction effect, we still need a quantitative mechanism to explain why no relative rotation is observed between matter an inertial frames.

In addition, we need more than just rules to choose which solutions of General Relativity are valid; we need a specific mechanism.

Haisch, Rueda, and Puthoff (1998)[243] suggest that zero-point energy (ZPE) is somehow related to inertia. However, does the zero-point energy determine the inertial frame, or the reverse? In addition, does electromagnetic induction then have a similar mechanism?

The next section should be titled:

Local Inertial Frame as a Background for Quantum Theory[5]

85.3 Introduction (Saturday 13 May 2006)

That the present form of quantum theory and General Relativity (or any similar metric theory) are incompatible is well known. Removing that incompatibility is one of the most important problems of theoretical physics today. To Remove that incompatibility, we have to decide how much to alter both quantum theory and gravitation.

Expressing the problem as "quantizing gravitation" suggests very little change in quantum theory. Some reflection suggests why that emphasis may be misplaced. First, that approach has been unsuccessful for a long time. Second, and more important, is looking into the origins of quantum theory. It is mainly based on assuming a flat Minkowski background.

Although some progress has been made to generalize to a curved background, that misses the main point. The correct quantum theory may not be based on any background. But to throw away the idea of a background for quantum theory does not leave much to build on.

If we take such an approach, we need to ask about the nature of the background upon which we have been basing our quantum theory. It is a "local inertial frame." And on the level of most laboratory experiments, including accelerator experiments, a flat Minkowski frame is an adequate approximation.

But if that local inertial frame is not intrinsic, but instead arises from something more basic, then it would be appropriate to find out what that more basic thing is. In General Relativity, local inertial frames are just a local part of a solution to Einstein's field equations. But there is something else. We observe that our local inertial frame (excluding Solar System effects) does not rotate relative to visible matter to very great accuracy.

As there are solutions of the field equations that allow relative rotation of inertial frames and the bulk of matter, something else is needed. Boundary conditions and initial conditions, but what criteria?

The idea that large-scale behavior of the universe, may influence local behavior, specifically inertia, goes back at least to Ernst Mach, but the term "Mach's principle" does not seem appropriate, except in a very general or vague sense, because there is no principle, except the principle that there may be some connection between local physics and larger-scale structure of the universe.

Now let us return to quantum theory in light of the above discussion. It seems that we need to first discover the nature of local inertial frames and how they relate to the large-scale structure before we can derive a proper quantum theory. Part of that is to discover a mechanism to explain

[5] file intro1.tex

why local inertial frames do not appear to rotate relative to visible matter. To find and demonstrate that mechanism is the purpose of this paper.

The next section should also be titled:

Local Inertial Frame as a Background for Quantum Theory[6]

85.4 Introduction (26 December 2005)

We know that the relative rotation of local inertial frames (after taking out local effects of the solar system) and the bulk of distant matter is less than 10^{-14} radians per year. However, we cannot explain that quantitatively, other than assume it is somehow due to frame dragging. Although frame dragging is known to be part of GR, and that there cases when it is 100%, in general, because of boundary and initial conditions, there are solutions of the field equations where there is relative rotation. We assume that a better understanding will explain why the relative rotation is so small.

Extrapolation to a nearly empty spacetime is probably not justified, although we assume that GR applies to our universe, where we have enough mass to somehow generate an inertial frame, we do not know if or how it would apply to a universe with less matter. Trying to answer that question should give us insight. We assume that the Schwarzschild and Kerr solutions for nearly empty space really apply to black holes or stars embedded in a full universe, not to a nearly empty one.

QM as we know it requires a local inertial frame, usually a flat Minkowski space. Although extended to curved space, not to possibly sparse space. Considering the question of "How would QM look in a sparse universe?" should give us insight on the relationship of GR and QM.

Note [7]

85.5 Introduction

We normally consider all solutions of Einstein's field equations to be valid cosmological models. However, this may not be true if a valid cosmological model is required to be the classical limit of a quantum cosmology.

Section 85.6 points out that we infer the gravitational field from measurements on matter. Therefore, in comparing measurements with theory, it is sufficient to consider the amplitudes for matter fields only, allowing us to integrate over the gravitational degrees of freedom (an integration on a spacelike three-dimensional hypersurface).

Section 85.7 points out that a path-integral representation of the wave function involves an integration over all 3-geometries on an initial spacelike hypersurface. Section 85.8 replaces the integrations over 3-geometries on the two spacelike hypersurfaces by the equivalent integration over the 4-geometries connecting those two hypersurfaces.

Section 85.9 considers the semiclassical approximation for the propagator that takes the wave function for 3-geometries and matter fields from one spacelike hypersurface to another. In that approximation, the propagator depends on only one solution of the field equations. Solutions to the field equations fall into two categories:

[6] file intro2.tex, The part about "Kerr solutions" is wrong.

[7] Sections 85.5, 85.6, 85.7, 85.8, 85.9, 85.10, 85.11, and 85.12 are in the file jun06bod.tex

1. The action for the propagator dominates the behavior of the path integral and a saddlepoint approximation is valid for each integration in the path integral over the geometry.
2. The action for the propagator does not dominate the behavior of the path integral or a saddlepoint approximation is invalid for at least one of the integrations in the path integral over the geometry.

Section 85.10 considers the former case. Section 85.11 considers the latter case. Section 85.12 interprets these examples.[8]

85.6 Measurements in quantum cosmology

We can represent a quantum cosmology by $\langle g, \phi, S|\psi\rangle$, which is the amplitude that on a spacelike hypersurface S, the 3-geometry is g and the matter fields are ϕ.[9] This representation is implicit in the path integral approach to quantum gravity [111].

To relate this amplitude to a measurement of the geometry, we notice that we do not measure the geometry directly. We infer the geometry from measurements using material objects, that is, from measurements on the matter. This allows us to represent any measurement by integrating over the gravitational degrees of freedom to give

$$\langle \phi, S|\psi\rangle = \int \langle g, \phi, S|\psi\rangle \, D(g), \tag{85.1}$$

the amplitude that on a spacelike hypersurface S, the matter fields are ϕ, where $D(g)$ is the measure on g.

85.7 Path-integral representation

The wave function over 3-geometries g_2 and matter fields ϕ_2 on one 3-dimensional spacelike hypersurface S_2 is related to the wave function over 3-geometries g_1 and matter fields ϕ_1 on another 3-dimensional spacelike hypersurface S_1 by an extension of the path-integral [21] formulation of quantum cosmology [111] to give

$$\begin{aligned}&\langle g_2, \phi_2, S_2|\psi\rangle = \\ &\int \langle g_2, \phi_2, S_2|g_1, \phi_1, S_1\rangle \, \langle g_1, \phi_1, S_1|\psi\rangle \, D(g_1) D(\phi_1),\end{aligned} \tag{85.2}$$

where $\langle g_2, \phi_2, S_2|g_1, \phi_1, S_1\rangle$ is the propagator (that is, the amplitude to go from a state with 3-geometry g_1 and matter fields ϕ_1 on hypersurface S_1 to a state with 3-geometry g_2 and matter fields ϕ_2 on hypersurface S_2), $\langle g_1, \phi_1, S_1|\psi\rangle$ is the wave function over 3-geometries g_1 and matter fields ϕ_1 on a spacelike hypersurface S_1, $D(g_1)$ is the measure on g_1, and $D(\phi_1)$ is the measure on ϕ_1. The integration is over all 3-geometries g_1 and matter fields ϕ_1 for which the integral is defined.[10] less

Substituting (85.2) into (85.1) gives

$$\begin{aligned}&\langle \phi_2, S_2|\psi\rangle = \int \langle g_2, \phi_2, S_2|g_1, \phi_1, S_1\rangle \, \langle g_1, \phi_1, S_1|\psi\rangle \\ &D(g_1) D(\phi_1) D(g_2).\end{aligned} \tag{85.3}$$

[8] This paper considers the meaning of quantum gravity, especially with regard to interpreting measurements, but does not discuss theories of quantum gravity.

[9] The development here for a pure state represented by a wave function can be generalized to a mixed state, represented by a density matrix.

[10] Because a matter distribution cannot be specified independent of geometry, we take the above integration to mean that we first specify a 3-geometry, then on that 3-geometry, we specify a matter distribution.

85.8 Integration over 4-geometries

Because (85.3) involves an integration over all 3-geometries g_1 and g_2 on S_1 and S_2, it is equivalent to an integration over all 4-geometries that connect S_1 and S_2. Thus, (85.3) can be written as

$$\begin{aligned}\langle \phi_2, S_2|\psi\rangle = \int &\left\langle g_2(g^{(4)}), \phi_2, S_2|g_1(g^{(4)}), \phi_1, S_1\right\rangle \\ &\left\langle g_1(g^{(4)}), \phi_1, S_1|\psi\right\rangle D(g^{(4)})D(\phi_1), \end{aligned} \tag{85.4}$$

where $D(g^{(4)})$ is the measure on the 4-geometry $g^{(4)}$. Of course, until we have a full theory of quantum gravity, we do not have formulas to give most of the functions in these integrals. We can, however, make some semiclassical approximations without having a full theory. To justify replacing (85.3) by (85.4), we notice that the integration in (85.3) is an integration over all 4-geometries that connect S_1 and S_2, as is the integration in (85.4).

85.9 semiclassical approximation for the propagator

Making the semiclassical approximation[11] for the propagator gives [214]

$$\begin{aligned}&\left\langle g_2(g^{(4)}), \phi_2, S_2|g_1(g^{(4)}), \phi_1, S_1\right\rangle \approx \\ &f_a(g_2(g^{(4)}), S_2; g_1(g^{(4)}), \phi_1, S_1) \\ &e^{\frac{i}{\hbar} I_{classical}[g_2(g^{(4)}), S_2; g_1(g^{(4)}), \phi_1, S_1]}, \end{aligned} \tag{85.5}$$

where $I_{classical}[g_2(g^{(4)}), S_2; g_1(g^{(4)}), \phi_1, S_1]$ is the action for the classical spacetime bounded by the two 3-geometries that satisfies the field equations and $f_a(g_2(g^{(4)}), S_2; g_1(g^{(4)}), \phi_1, S_1)$ is a slowly varying function. Explicit dependence on ϕ_2 is not shown, because for classical solutions to the field equations, ϕ_2 is determined from ϕ_1 and $g^{(4)}$. Thus, substituting (85.5) into (85.4) gives

$$\begin{aligned}\langle \phi_2, S_2|\psi\rangle \approx \int &f_b(g^{(4)}, \phi_1) e^{\frac{i}{\hbar} I_{classical}[g^{(4)}, \phi_1]} \\ &\left\langle g_1(g^{(4)}), \phi_1, S_1|\psi\right\rangle D(\phi_1)D(g^{(4)}). \end{aligned} \tag{85.6}$$

where $f_b(g^{(4)}, \phi_1)$ is a slowly varying function and the integration is over all classical 4-geometries that connect S_1 and S_2.

The number of functions being integrated over to represent the 4-geometry $g^{(4)}$ is probably an order of infinity greater than that of the real numbers. To test the validity as a cosmology of a given 4-geometry, it is sufficient to restrict consideration to a small subset of cases, such as a family of known exact solutions. This allows us to represent the integration over 4-geometries in (85.6) more explicitly. Solutions to the field equations can be represented by a number of parameters a_i. These are the parameters that specify the 4-geometry that are not constrained by the matter distribution ϕ_1 on the hypersurface S_1. The number of these parameters is usually finite, and in most cases, at least countable. I shall assume here, that they are finite, and that there are N of these parameters, although I think the development could be extended to even the uncountable case. Thus, we may rewrite (85.6) more explicitly as

$$\begin{aligned}\langle \phi_2, S_2|\psi\rangle \approx \int &f_c(a_i, \phi_1) e^{\frac{i}{\hbar} I_{classical}[a_i, \phi_1]} \\ &\langle g_1(a_i), \phi_1, S_1|\psi\rangle D(\phi_1)d^N a_i. \end{aligned} \tag{85.7}$$

[11] A semiclassical approximation for the propagator is not always valid. Here, we consider only cases where it is valid.

where $f_c(a_i, \phi_1)$ is a slowly varying function that depends explicitly on the parameters a_i that define the 4-geometry, and now we are left with an ordinary Nth order integral to define the integration over the 4-geometries.

85.10 When a saddlepoint approximation is valid

When the behavior of $e^{\frac{i}{\hbar}I_{classical}}$ dominates over that of $\langle g_1(a_i), \phi_1, S_1|\psi\rangle$ and $f_c(a_i, \phi_1)$ in the integration over each a_i in (85.7) and when a saddlepoint approximation for each integration is valid, then we can approximate each of those integrations by a saddlepoint approximation. We analytically continue each function into the complex domain, deform the path of integration in the complex a_i plane for each a_i to go through the saddlepoint, a_{i0}, defined by where $I_{classical}$ is stationary for variation of each of the a_i, that is,

$$\left.\frac{\partial I_{classical}}{\partial a_i}\right|_{a_i=a_{i0}} = 0, \tag{85.8}$$

for each a_i. For each integration, the path must be deformed (without passing over any non-analytic points) onto a steepest descent path or a stationary phase path. Also, to be a valid approximation, there must not be any non-analytic points too close to the saddlepoint. For stationary phase paths, the saddlepoint approximation gives e.g. [215]

$$\begin{aligned}&\langle \phi_2, S_2|\psi\rangle \approx \\ &\int f_c(a_{i0}, \phi_1) e^{\frac{i}{\hbar}I_{classical}[a_{i0},\phi_1]} \langle g_1(a_{i0}), \phi_1, S_1|\psi\rangle \\ &e^{Ni\pi/4}\prod_{i=1}^{N}\left|\frac{2\pi}{\partial^2 I/\partial a_i^2}\right|^{1/2}_{a_i=a_{i0}} D(\phi_1).\end{aligned} \tag{85.9}$$

For steepest descent paths, the formula differs only by a phase.

The usual form for the action I is

$$I = \int (-|g|^{(4)})^{1/2} L d^4x, \tag{85.10}$$

where $|g|$ is the determinant of the metric tensor $g_{\mu\nu}$,

$$L = \underbrace{\frac{R-2\Lambda}{16\pi}}_{geometry} \underbrace{-\frac{1}{2}\rho g_{\mu\nu}U^\mu U^\nu}_{matter} \underbrace{-\frac{\rho_e}{c}A_\mu U^\mu}_{interaction} \underbrace{-\frac{F_{\mu\nu}F^{\mu\nu}}{16\pi}}_{EM} \tag{85.11}$$

is the Lagrangian, R is the Riemann scalar, Λ is the cosmological constant, ρ is the mass density, U^μ is the four-velocity, ρ_e is the electric charge density, A_μ is the electromagnetic 4-vector potential, $F_{\mu\nu}$ is the electromagnetic field tensor, and the usual designation of the four terms is shown.[12]

Because the integration in (85.10) must consider the light-cone structure of the propagators, it is more appropriate to derive a formula for the amplitude of observing a particular event instead of deriving a general formula for all possible measurements. The integral for the action in (85.10) must therefore be restricted to the past light cone of the event whose amplitude is being calculated. There is some fuzziness to the light cone,[13] which is taken into account by using the correct propagators [216].

[12] All of the examples here use Einstein's theory of General Relativity, but the procedure applies to nearly any gravitational theory.

[13] There is an interesting similarity between the light cone and the event horizon of a black hole. In both cases, travel across the boundary is classically possible in only one direction (into the light cone or into the black hole), but the prohibition is not absolute in either case, because in the quantum situation a particle can temporarily escape by doing a zigzag path in space-time [216], resulting in Hawking radiation in the case of a black hole.

An example of applying such a saddlepoint approximation to a family of solutions to the field equations will be given in a future publication.

85.11 When a saddlepoint approximation is not valid

We consider here several examples where the saddlepoint approximation is either not valid or not applicable. We take Λ, $F_{\mu\nu}$ and A_μ to be zero in these examples. In addition, we take R and ρ to be zero except where there are masses.

85.11.1 Minkowski space

In empty Minkowski space, the Lagrangian is everywhere zero because the scalar curvature R is zero and the matter density is zero, and therefore, the action $I_{classical}$ is zero. Because there is no matter, there is no possibility for measurements, so this case is not applicable.

85.11.2 Schwarzschild metric

The simplest matter distribution added onto Minkowski space-time gives us the Schwarzschild metric. Normally, we use the Schwarzschild metric to represent the local field around a planet or star or black hole, but not for a whole cosmology, and there may be good reason for that.

There are no gravitational degrees of freedom defining the Schwarzschild metric, so there is no integration over 4-geometries. However, formally, we could write (85.7) as

$$\begin{aligned} &\langle \phi_2, S_2 | \psi \rangle \approx \\ &\int f_c(\phi_1) e^{\frac{i}{\hbar} I_{classical}[\phi_1]} \langle g_1, \phi_1, S_1 | \psi \rangle \, D(\phi_1). \end{aligned} \tag{85.12}$$

85.11.3 Kerr metric

The next simplest model is a symmetric body like a planet that has a rotation rate relative to an inertial frame. We can represent the field outside of the body by the exterior Kerr metric. This metric has three gravitational degrees of freedom to characterize the direction and magnitude of the rotation rate (which I shall refer to as a_1, a_2, and a_3 here). Because the scalar curvature and matter density are everywhere zero outside of the body, the only contribution to the action $I_{classical}$ is from the mass of the body, which does not depend on the rotation rate. Thus, (85.7) becomes

$$\begin{aligned} \langle \phi_2, S_2 | \psi \rangle \approx & \int e^{\frac{i}{\hbar} I_{classical}[\phi_1]} f_d(a_1, a_2, a_3, \phi_1) \\ & \langle g_1(a_1, a_2, a_3), \phi_1, S_1 | \psi \rangle \, D(\phi_1) da_1 da_2 da_3, \end{aligned} \tag{85.13}$$

where $f_d(a_1, a_2, a_3, \phi_1)$ is a slowly varying function. Because the exponential factor does not dominate the integration, we cannot make a saddlepoint approximation for the integration over a_1, a_2, and a_3. We are left with an integration over various Kerr metrics with various rotation rates. There is no single 4-geometry that dominates the integration.

We normally consider the Kerr metric to represent the local gravitational field around a spinning planet, star, or black hole, rather than for a cosmology. In light of the result here, this seems appropriate.

We want matter in the cosmological model so that we can do measurements. That is, because we cannot directly measure the geometry, we must infer it from measurements on matter. However, the example of a single body represented here by the Kerr metric is not really interesting enough to offer the possibility for measurements of the geometry. If we had a planetary system, we might be able to model possible measurements on the geometry using matter.

85.11.4 Asymptotically flat metrics

Therefore, consider a collection of planets and a star in some star system as the only matter in the universe. We assume we have some solution of the field equations for these. In fact, we will have many solutions, because we have some freedom in applying boundary conditions.

Let us consider a subset of those solutions in which we apply asymptotically flat boundary conditions. Then very far from where all of the matter is concentrated for the star system, the solution will be approximately that of a Kerr metric, in which the solution is characterized by the angular momentum of the matter relative to the flat metric to which the Kerr solution is asymptotic. The angular momentum is characterized by 3 values, say a_1, a_2, and a_3. This leads to the wave function given by (85.13), but we cannot apply a saddlepoint approximation because the action is independent of a_1, a_2, and a_3.

85.12 Interpretation

In summary, the conditions for a solution of the field equations to be a classical limit of a quantum cosmology are: (1) The effect of the classical action dominates the integration, (2) the action is stationary with respect to variation of the gravitational degrees of freedom, and (3) only one saddlepoint contributes significantly to each integration.

As pointed out earlier, we can always represent a measurement of the geometry in terms of the matter; we infer the geometry from measurements on the matter. So, in the above examples, what geometry would we infer from measurements on the matter?

Measurements on the matter in section 85.10 would indicate a geometry that was confined within the limits given by $|I_{saddlepoint} - I_{classical}| < \hbar$.

On the other hand, measurements on the matter in section 85.11.4 would indicate an ambiguous geometry. In fact, the system of bodies would seem very nonclassical. There is an aspect of relativity here. Although it is the background geometry that is quantum, we can infer the geometry and matter only relative to each other. More specifically, we can observe directly, only the matter, so it will appear to an observer that the matter is behaving in a quantum manner.

It should be pointed out that there are no new theories or assumptions here. This is simply an application of standard ideas about quantum theory to cosmology. To falsify the results presented here, it would be sufficient to show that our present cosmology does not satisfy the criteria given here for a valid cosmological model. But unless I have made a logical error, that would also invalidate some of our standard ideas about quantum theory.

85.13 Afterthoughts - 2009

I think the problem with the propagators is my notation. I need to fix that.

In addition, there is the other problem with talking about varying the initial and/or boundary conditions for the gravitational field while holding the matter distribution fixed. I discuss this in the 14 April 2009 version of the Introduction.

Finally, I need to include an actual calculation, not just talk about it.

Chapter 86

An integral form of Einstein's gravity[1]

abstract

I attempt to write an integral form of Einstein's field equations, in analogy with the integral form of Maxwell's equations.

86.1 Introduction

If we could write Einstein's field equations in an integral form, as is possible for Maxwell's equations, it might be possible to convert that formulation to a formulation that did not refer to the geometry. There are obvious advantages to being able to make that separation.

86.2 The integal form of Maxwell's equations

The usual integral form of Maxwell's equations is[244, e.g., equation (10.10), p.191]

$$\oint E \cdot \mathrm{d}s + \int \frac{\partial B}{\partial t} \cdot \mathrm{d}S = 0 \tag{86.1}$$

$$\oint H \cdot \mathrm{d}s - \int \left(J + \frac{\partial D}{\partial t} \right) \cdot \mathrm{d}S = 0 \tag{86.2}$$

$$\int B \cdot \mathrm{d}S = 0 \tag{86.3}$$

$$\int D \cdot \mathrm{d}S = \int \rho \, \mathrm{d}v \ , \tag{86.4}$$

where, $B = \mu_0(H + M)$ and $D = \epsilon_0 E + P$, and I use rationalized MKS units here to make it easier to distinguish the roles of E versus D and B versus H.

The distinction between E and D is the matter of polarization charges. The idea is that E has as its source all charges, including polarization charges, whereas D has as its source only "true" charges (that is, not including polarization charges). Similarly, B has as its source all currents, including the currents in magnets, whereas H has only "true" currents as its source.

Notice that (86.1) and (86.3) are in terms of E and B, whereas (86.2) and (86.4), which have the sources (ρ and J), are in terms of D and H. The significance of these comments is that D and H are related to the sources (ρ and J), whereas E and B are defined in terms of the accelerations that will happen to test charges placed in those fields. If we want to understand

[1] 9 September 2009, revised: January - December 2011

how the gravitational interaction with the rest of matter in the universe through Einstein's field equations causes the inertial frames in which electromagnetic waves propagate, it is necessary to make these distinctions.

The above set of equations is not in a pure integral form because of the differential terms, $\frac{\partial B}{\partial t}$ and $\frac{\partial D}{\partial t}$. They actually form a mixed integral-differential form. To get a pure integral form, we begin with the differential form of Maxwell's equations in 4-dimensional covariant form.

The differential form for (86.1) and (86.3) can be written [20, e.g., equation (4.10)]

$$dF = 0 \ , \tag{86.5}$$

where d is an exterior derivative and F is the electromagnetic field tensor, defined as

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} . \tag{86.6}$$

In a vacuum if we were using a system of units in which ϵ_0 and μ_0 were unity, then the differential form for (86.2) and (86.4) would be [20, e.g., equations (4.11)]

$$d^*F = 4\pi^*J \ , \tag{86.7}$$

where

$$^*F = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix} . \tag{86.8}$$

is the dual of F, J is the 4-current, and $^*J$ is the dual of J. Here, however, the differential form of (86.2) and (86.4) can be written

$$dF' = 4\pi^*J \ , \tag{86.9}$$

where the components of F' are given by

$$F' = \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & D_z & -D_y \\ -H_y & -D_z & 0 & D_x \\ -H_z & D_y & -D_x & 0 \end{pmatrix} . \tag{86.10}$$

To put Maxwell's equations in integral form, we use [20, p. 96]

$$\int_V d\sigma = \int_{\partial V} \sigma \ , \tag{86.11}$$

where σ is a p-form, V is a $p+1$-dimensional surface, and ∂V is the closed p-dimensional boundary of V.

Applying (86.11) to (86.5) and (86.9) gives Maxwell's equations in purely integral form:

$$\int_{\partial V} F = 0 \tag{86.12}$$

and

$$\int_{\partial V} F' = 4\pi \int_V {}^*J \ , \tag{86.13}$$

where V is a 3-dimensional hypersurface and ∂V is the closed 2-dimensional boundary of that surface. F and F' are 2-forms and ${}^*J$ is a 3-form.

Although I said that putting Maxwell's equations in integral form might help to separate gravitation from geometry, that has not yet worked. Somehow the integral form of Maxwell's equations still have gravitation in terms of inertial frames and geometry, but the gravitation is not explicit.

Maxwell's equations already implicitly include inertia because electromagnetic waves move in straight lines in inertial frames, but not otherwise. However, trying to see how the inertia enters is difficult.

To get some insight, we write (86.12) explicitly using (86.6). This gives

$$-\int_{\partial V} E_x\,\mathrm{d}t\wedge\mathrm{d}x-\int_{\partial V} E_y\,\mathrm{d}t\wedge\mathrm{d}y-\int_{\partial V} E_z\,\mathrm{d}t\wedge\mathrm{d}z+\int_{\partial V} B_x\,\mathrm{d}y\wedge\mathrm{d}z+\int_{\partial V} B_y\,\mathrm{d}z\wedge\mathrm{d}x+\int_{\partial V} B_z\,\mathrm{d}x\wedge\mathrm{d}y=0\ . \tag{86.14}$$

Similarly, we can write (86.13) explicitly using (86.10). This gives

$$\begin{aligned}&\textstyle\int_{\partial V} H_x\,\mathrm{d}t\wedge\mathrm{d}x+\int_{\partial V} H_y\,\mathrm{d}t\wedge\mathrm{d}y+\int_{\partial V} H_z\,\mathrm{d}t\wedge\mathrm{d}z+\int_{\partial V} D_x\,\mathrm{d}y\wedge\mathrm{d}z+\int_{\partial V} D_y\,\mathrm{d}z\wedge\mathrm{d}x+\int_{\partial V} D_z\,\mathrm{d}x\wedge\mathrm{d}y\\ &\quad\textstyle=4\pi\int_V \rho\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z-4\pi\int_V J_x\,\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z-4\pi\int_V J_y\,\mathrm{d}t\wedge\mathrm{d}z\wedge\mathrm{d}x-4\pi\int_V J_z\,\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y\end{aligned} \tag{86.15}$$

Equation (86.14) is an integral over the 2-dimensional surface of a 3-dimensional volume. If we take that volume to be purely spatial ($\mathrm{d}t=0$), then we get exactly (86.3). If instead, we take that volume to include the time coordinate, but successively eliminate $\mathrm{d}x$, then $\mathrm{d}y$, then $\mathrm{d}z$, we get three equations whose sum is exactly the time integral of (86.1). Similarly, the left side of (86.15) is an integral over the 2-dimensional surface of a 3-dimensional volume, and the right side is a volume integral over that volume. If we take that volume to be purely spatial ($\mathrm{d}t=0$), then we get exactly (86.4). If instead, we take that volume to include the time coordinate, but successively eliminate $\mathrm{d}x$, then $\mathrm{d}y$, then $\mathrm{d}z$, we get three equations whose sum is exactly the time integral of (86.2). Summarizing, gives for the integral form of Maxwell's equations

$$\begin{aligned}\int\oint E\cdot\mathrm{d}s\,\mathrm{d}t+\int B\cdot\mathrm{d}S&=0 &&\text{(a)}\\ \int\oint H\cdot\mathrm{d}s\,\mathrm{d}t-\int\int J\cdot\mathrm{d}S\,\mathrm{d}t-\int D\cdot\mathrm{d}S&=0 &&\text{(b)}\\ \int B\cdot\mathrm{d}S&=0 &&\text{(c)}\\ \int D\cdot\mathrm{d}S&=\int\rho\,\mathrm{d}v &&\text{(d)}\ .\end{aligned} \tag{86.16}$$

The above form of the equations (86.16a) and (86.16b) is not accurate because of the way the time integrals are expressed. It is not clear in that form that the time and space integrals should be combined to form an integral over a closed two-dimensional surface.

To express things more clearly, we write [20, e.g., equation (4.1), p. 99]

$$F=\frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \tag{86.17}$$

and

$${}^*J=\frac{1}{6}{}^*J_{\alpha\beta\gamma}\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma\ . \tag{86.18}$$

Equation (86.12) then becomes

$$\frac{1}{2}\int_{\partial V} F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=0\ , \tag{86.19}$$

which can also be written

$$\frac{1}{2}\int_{\partial V} g_{\mu\alpha}F^{\alpha}{}_{\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=0\ . \tag{86.20}$$

We also have [20, e.g., box 4.1, p. 97]

$$^*F_{\mu\nu}=\frac{1}{2}F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}\ , \tag{86.21}$$

and

$$F_{\mu\nu}=\frac{1}{2}{}^*F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}\ , \tag{86.22}$$

and

$$^*J_{\alpha\beta\gamma}=J^{\mu}\epsilon_{\mu\alpha\beta\gamma}\ . \tag{86.23}$$

So, applying (86.11) to (86.7) gives

$$\int_{\partial V}{}^*F=4\pi\int_V{}^*J\ , \tag{86.24}$$

which can be rewritten

$$\frac{1}{2}\int_{\partial V}{}^*F_{\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=\frac{4\pi}{6}\int_V{}^*J_{\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma}\ , \tag{86.25}$$

which can be rewritten

$$\frac{1}{2}\int_{\partial V}F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=\frac{4\pi}{6}\int_V J^{\mu}\epsilon_{\mu\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma}\ , \tag{86.26}$$

which can be rwritten

$$\frac{1}{2}\int_{\partial V}g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}\epsilon_{\alpha\beta\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=\frac{4\pi}{6}\int_V J^{\mu}\epsilon_{\mu\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma}\ . \tag{86.27}$$

Summarizing, we have from (86.19) and (86.27),

$$\begin{aligned}&\frac{1}{2}\int_{\partial V}F_{\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=0 && \text{(a)}\\ &\frac{1}{2}\int_{\partial V}g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}\epsilon_{\alpha\beta\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=\frac{4\pi}{6}\int_V J^{\mu}\epsilon_{\mu\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma} && \text{(b)}\end{aligned} \tag{86.28}$$

and

$$\begin{aligned}&\frac{1}{4}\int_{\partial V}\left(g^{\alpha\gamma}g^{\beta\delta}\right){}^*F_{\gamma\delta}\epsilon_{\alpha\beta\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=0 && \text{(a)}\\ &\frac{1}{2}\int_{\partial V}{}^*F_{\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=\frac{4\pi}{6}\int_V{}^*J_{\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma} && \text{(b)}\end{aligned} \tag{86.29}$$

and

$$\begin{aligned}&\frac{1}{2}\int_{\partial V}g_{\mu\alpha}F^{\alpha}{}_{\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=0 && \text{(a)}\\ &\frac{1}{2}\int_{\partial V}g^{\beta\delta}F^{\alpha}{}_{\delta}\epsilon_{\alpha\beta\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=\frac{4\pi}{6}\int_V J^{\mu}\epsilon_{\mu\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma} && \text{(b)}\end{aligned} \tag{86.30}$$

and

$$\frac{1}{4}\int_{\partial V}\left(g^{\beta\delta}\right){}^*F^{\alpha}{}_{\delta}\epsilon_{\alpha\beta\mu\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=0 \qquad \text{(a)}$$

$$\frac{1}{2}\int_{\partial V}(g_{\mu\alpha}){}^*F^{\alpha}{}_{\nu}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}=\frac{4\pi}{6}\int_{V}{}^*J_{\alpha\beta\gamma}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\wedge\mathrm{d}x^{\gamma} \qquad \text{(b)} \tag{86.31}$$

To show inertia explicitly in this form of Maxwell's equations, we use (92.24) or (92.33).

$$g^{\alpha'\beta'}(x')=2\kappa\int_{\Omega}G^{-\alpha'\beta'\nu}{}_{\mu}\left(T^{\mu}{}_{\nu}-\frac{1}{2}T^{\lambda}{}_{\lambda}\delta^{\mu}{}_{\nu}-\frac{\Lambda}{\kappa}\delta^{\mu}{}_{\nu}\right)[-g(x)]^{1/2}d^4x+\int_{\partial\Omega}G^{-\alpha'\beta'\nu}{}_{\nu}{}^{;\sigma}[-g(x)]^{1/2}dS_{\sigma}\ , \tag{86.32}$$

where $G^{-}(x',x)$ is the retarded Green's functional that gives the contribution to the metric at x' from the stress-energy tensor at x, and $\kappa\equiv 8\pi G/c^2$. Equation (92.24) shows how $g^{\alpha\beta}$ here depends on the matter distribution in the universe through the volume integral. The surface integral accounts for sources outside of the volume plus initial values and boundary values. That the relative rotation of local inertial frames and distant matter is so far unmeasurable (in the absence of local Lense-Thirring effects) suggests that for our universe, the volume integral dominates over the surface integral in (92.24). We can then substitute $g^{\alpha'\beta'}(x')$ (dropping the primes) directly into equations (86.28) through (86.31).

So, we have four different versions of the possibilities, (86.28), (86.29), (86.30), or (86.31). The versions having superscripts on g have the advantage of being exactly what is given in (86.32), but the disadvantage of having a product of those terms. The other versions reverse these advantages and disadvantages. However, we can probably handle having the products better than having subscripts on g.

As I said earlier, somehow Maxwell's equations already include inertia, but it is not clear how. After finding out how, I want to re-express the equations to put in inertia explicitly.

It seems likely that inertia arises not from geometry, but from the geometry of spacetime. That is, from mixing space and time into a single entity. Thus, it is implicit in special relativity, and also in Maxwell's equations. So, the place to look in Maxwell's equations is where derivatives (or integrals) of space and time are mixed. Also, between current and charge, since a moving charge mixes space and time. Actually any motion mixes space and time.

So, to put inertia and gravity explicitly into Maxwell's equations, we consider a sparse universe. With very little matter in the universe, the speed of light will probably be very large. If we write Maxwell's equations with units, then derivatives with respect to time will be multiplied by $1/c$. Also, J will be multiplied by $1/c$. Thus, to put inertia explicitly into Maxwell's equations, we need to multiply J and derivatives with respect to time by some gravitational variable that might be proportional to the amount of matter in the visible universe.

Will it be good enough to use g^{00}? Does this go to zero for a sparse universe? It might be necessary to multiply other terms by some other gravitational variable, maybe g_{00}? That may not be quite right, since I think the dimension is wrong. Maybe g^{11}, g^{22}, or g^{33}?

There are also some questions about how general some of these equations are. Some are clearly valid only in cartesian coordinates. Some only in inertial frames. I need to get that sorted out.

There are also some questions about the magnitude and normalization of the metric tensor. Usually, we take it to be diagonal, and often have those values to be plus or minus unity. That may not be a good idea in general.

86.3 Einstein's field equations in integral form

If we can put Einstein's field equations in a form like (86.5) and (86.9) involving exterior derivatives, then we can find an integral form for General Relativity. Notice that this would be a different kind of integral formulation for Einstein's field equations from that given by Sciama, Waylen, and Gilman [16], which was a Green's function formulation.

Suppose we could write Einstein's field equations in terms of an exterial derivative d, such as:

$$dF = 0 \tag{86.33}$$

and

$$d^*F = 8\pi^*T \ , \tag{86.34}$$

where F is a 3-form, $^*F$ (a 1-form) is the dual of F, T is the stress-energy tensor, and $^*T$ is the dual of T. Then we could put Einstein's field equations in integral form. See Misner, Thorne, and Wheeler [20, pp. 362-363] for a discussion of $^*T$.

Actually, we already have half of this in terms of the Bianchi identities [20, p. 362]:

$$\mathbf{d}\mathcal{R} = 0 \ , \tag{86.35}$$

where

$$\mathcal{R} = \frac{1}{2}\mathbf{e}_\mu \wedge \mathbf{e}_\nu \mathcal{R}^{\mu\nu} \ , \tag{86.36}$$

and

$$\mathcal{R}^{\mu\nu} = \mathbf{d}\omega^{\mu\nu} - \omega^\mu{}_\alpha \wedge \omega^{\nu\alpha} \ . \tag{86.37}$$

or

$$\mathcal{R}^{\mu\nu} = R^{\mu\nu}{}_{|\alpha\beta|}\omega^\alpha \wedge \omega^\beta \ , \tag{86.38}$$

where the sum over α and β is restricted to $\alpha < \beta$. Then, the integral form of the Bianchi identities is

$$\int_{\partial V} \mathcal{R} = 0 \tag{86.39}$$

where ∂V is a closed 2-dimensional boundary. See Misner, Thorne, and Wheeler [20, pp. 379-380] for integral forms for $^*T$ and $^*G$. In particular, [20, equation (15.24), p. 380] gives another (equivalent) integral form for the Bianchi identities:

$$\int_{\partial V} {}^*G = 0 \tag{86.40}$$

Equations (86.39) and (86.40) are probably equivalent, since they both are equivalent to the Bianchi identities. Either would work well for the homogeneous part of the integral form of Einstein's gravity.

For the inhomogeneous part, that is, the part equivalent to Einstein's field equations, there is still work to be done. One possibility is the tensor F with three indices discussed in chapter 89. However, there are two problems: First, is that formulation valid? Second, can we put it in the form (86.34)?

I think one thing I need to do is write Einstein's field equations as first-order equations, rather than as second-order equations as they now are. This means writing the definition of Γ as one equation, and writing the definition of the Riemann tensor in terms of Γ as another equation. The second part, I would like to write as an exterior derivative. However, there are at least four problems. First, Γ is not a tensor, so including it in an equation would not be covariant. Second, Γ is not only not a tensor, but it is not a form. Third, I can't quite see how to make the second equation look like an exterior derivative. The first two terms in the definition of the Riemann equation have the right form, but we come back to Γ not being a form. Fourth, even if I could do that part, I don't know how to express the Riemann tensor in terms of the stress-energy tensor.

Chapter 87

An electron is waves of what?

abstract[1]

If an electron wave function represents a real field rather than simply information that allows a forecast to be made about the outcome of a future measurement, then we need to inquire about the nature of that field. Considerations of Compton scattering and beta decay suggest that electrons are electroweak Yang-Mills fields, which contain both an electromagnetic and a weak-field component. Similar considerations suggest that neutrinos are also electroweak Yang-Mills fields in which the electric and magnetic components are zero.

87.1 Introduction

A photon is an electromagnetic field, but what kind of wave is an electron? Mermin (2009)[245] suggests that such questions are bad habits, that should be gotten rid of. He includes thinking of photons as electromagnetic waves to be in the category of bad habits.

However, Looking at the electromagnetic interaction in terms of creation and annihilation operators, for example, is just a mathematical artifice, probably bearing little relation to nature. Even if he is correct, the electromagnetic wave description for photons is useful for many purposes. A similar description for electron waves might also be useful if we could discover that correct description.

We knew about electric and magnetic fields before we knew about electromagnetic waves and photons. Because the photon is massless, we can have photons or electromagnetic waves of zero frequency. These are simply static fields, which can (and were) easily studied.

Electrons, however, have mass, and therefore a rest frequency (of about 1.2×10^{20} Hz). If we bring an electron to rest, it will still be oscillating at it's rest frequency (the lowest frequency it can have), and still too large a frequency to be able to directly observe what kind of a field is oscillating. I submit that it is this difficulty that has kept us from discovering what kind of waves make up an electron. We, therefore, must consider indirect investigations.

87.1.1 Additions, 2-9 October 2017

Considering conservation of energy and momentum in Compton scattering, electron diffraction by crystals, beta decay, and the relations $E = \hbar\omega$ and $p = \hbar k$ requires that whatever kinds of waves make up particles, all of the various kinds of waves for a given particle in a given case have the same frequency and wavelength. Thus, if an electron is made up of both electromagnetic waves and weak waves, they all have the same frequency and wavelength.

[1] Revised 9 October 2017

Considering electron diffraction by crystals (Bragg scattering) suggests that electron waves must include electromagnetic waves. Compton scattering leads to the same conclusion. That neutrinos seem to have no electromagnetic interaction at all suggests that neutrino waves are pure weak waves. In beta decay, a neutron decays into a proton, an electron, and an anti-neutrino. Or, in more modern terminology, a down quark decays into an up quark plus a W^-, the latter then decaying into an electron and an anti-neutrino. Because conservation of energy and momentum are wave-interference effects, the electron and the W^- must have waves that include weak waves, since the neutrino contains only a weak wave. For the same reason, both the up- and down-quarks must contain weak waves.

Let us try to determine some general rules for determining what kinds of waves make up various particles.

1. If a particle has a charge associated with one of the fundamental four interactions, then it must include a wave associated with that type of interaction.

2. If the particle is one of the carriers of one of the fundamental interactions, then it must include a wave associated with that type of interaction.

Let's see if these rules work and are sufficient. An electron has both an electric charge and a weak charge, so it should have both an electromagnetic wave and a weak wave. A neutrino has a weak charge, but no electric charge, so it should be a pure weak wave. Up and down quarks have electric charges and weak charges, so they should have both an electromagnetic wave and a weak wave. The W^- has an electric charge and is a carrier for the weak interaction, so it should include both an electromagnetic wave and a weak wave. Actually, there are two kinds of weak charges: weak isospin and weak hypercharge. The W^- and the W^+ both have weak isospin charge, but not weak hypercharge. Since there are two kinds of weak charges, there may be two kinds of weak waves. A photon is the carrier of the electromagnetic interaction, so it should be an electromagnetic wave. It seems OK so far.

Moving on further, quarks and gluons have color charge, so they should also contain a strong wave. The Z^0 has no charge of any kind, but since it is one of the carriers of the weak interaction, it is probably a weak wave. Also, all of the quarks have an electric charge, so they must also include an electromagnetic waves.

One question that needs answering, is whether an electron, say, always has the same ratio of electromagnetic wave to weak wave. Until we figure out an experiment to test that hypothesis, let us assume that to be true. So, if we can calculate or measure the electromagnetic wave for an electron, then we assume we can calculate the weak wave for the electron.

Let us assume we have solved the Dirac equation for an electron in a given situation, so that we now have a solution for the spinor wave function for the electron in that situation. Only the electromagnetic field is involved in the calculation to solve the Dirac equation (plus the gravitational field in terms of inertia), so we should somehow be able to convert the spinor wave function to an electromagnetic wave representation for the wave function. I do not know how to do that. That is what I need to figure out how to do.

The Dirac equation by itself seems insufficient to do that. I need to find a replacement for the Dirac equation that will give a solution for electromagnetic wave representation for the electron. After that, I need to figure out how to calculate the weak wave components.

The following is an attempt to try to calculate the electromagnetic wave representation of an electron wave function by applying Maxwell's equations to the electron. However, that is probably incorrect because that will give the electromagnetic wave for a massless photon. However, the Dirac wave function is for a massive electron. Also, we already have the wave function as a solution to the Dirac equation. The electromagnetic wave representation for the electron wave function will not satisfy Maxwell's equation. We need to simply find out what factors we need to multiply the Dirac wave function by to get the electric and magnetic fields. However, there is also the static field of the point charge that needs to be included, and also the magnetic moment of the electron.

In addition, the effect of gravity (in terms of inertia) has to be somehow included. Thus, the electron wave must somehow include not only an electromagnetic and a weak component, but also a gravitational component. The Dirac equation gives the usual wave function for the electron, but we need a related equation that will give the electromagnetic, weak, and gravitational components.

A related problem with regard to atomic wave functions, is that the radial component is the relative distance between the electron and the nucleus, not the absolute position. So, maybe the wave function is not the wave function of the electron, but of the whole atom.

To get a rough estimate of what the electric and magnetic fields might be that are associated with an electron wave function, we notice that the energy density associated with an electron wave function is $mc^2\psi^*\psi$ if we are considering that the total energy associated with an electron is equal to its rest mass energy. Also, the energy density of an electromagnetic field (in cgs units) is $(E^2 + B^2)/8\pi$. In addition (in cgs units), we have $|E| = |B|$ for an electromagnetic wave, so that the **E** and **B** fields contribute equally to the energy density. Thus, as a first estimate, we can take $|E| = \sqrt{4\pi mc^2}\psi$ and $|B| = \sqrt{4\pi mc^2}\psi$ to give the magnitude of **E** and **B**. We would need something more sophisticated to get the direction of **E** and **B**. We also have to consider spin and the magnetic moment of the electron. Maybe for an electron at rest, we could assume the electric field points radially, and we could take the magnetic field to be a dipole field. For a moving electron, we might be able to simply make a Lorentz transformation of the **E** and **B** fields.

There is a problem here. The wave function is time harmonic, but the static electric field of a point charge is always in the same direction. Similarly for the dipole magnetic field of an electron. It seems that we need to have two contributions to the electric and magnetic fields. A static contribution plus a wave-function contribution.

However, these are just rough estimates. I expect that the actual relations will involve differential equations that relate components of the electric and magnetic fields with components of the electron spinor field.

87.2 Compton scattering

In Compton scattering, we consider the scattering of a photon by an electron. That is, we have a photon and an electron of given momenta in the initial state, and after scattering, a photon and an electron in the final state with resulting momenta. We apply conservation of energy and momentum to solve the scattering problem. There is a continuum of solutions.

However, we know that photons and electrons are waves with wavenumbers k and frequencies ω corresponding to the momenta $p = \hbar k$ and energies $E = \hbar\omega$. When looked at in terms of waves, conservation of energy and momentum correspond exactly to wave scattering, diffraction, and refraction. For example, Snell's law, giving the bending of light when propagating from one medium to another, corresponds to conservation of momentum, but the wave representation through Snell's law is more fundamental.

It is the same with Compton scattering. The wave representation for conservation of energy and momentum is more fundamental. Thus, Compton scattering of a photon by an electron corresponds to a wave interference phenomenon. For such interference to occur, the electron wave and the electromagnetic wave must be the same kind of wave. (Or at least what is taking part in the interference process must be the same kind of wave.)

Thus, the wave function for an electron must consist (at least in part) of an electric or a magnetic wave. However, Maxwell's equations then require (in either case) that the electron be (at least in part) an electromagnetic wave.

87.3 Beta decay

In beta decay, a neutron decays into a proton, an electron, and an antineutrino. As with Compton scattering, conservation of energy and momentum yields a continuum of solutions. As with Compton scattering, a more fundamental description of that process is in terms of wave interference. As with Compton scattering, we have the result that electrons and neutrinos must be (at least in part) the same kind of wave.

Neutrinos do not interact with photons, as far as we know. In fact, as far as we know, neutrinos interact only through the weak interaction (and also gravitation because of their energy and momentum). Therefore, the wave function for a neutrino does not have an electromagnetic component.

Whatever kind of wave the wave function for a neutrino corresponds to, an electron wave function must also contain that component in order for there to be interference between electron waves and neutrino waves.

87.4 Electroweak theory

We know, since the work of Weinberg and Salam, that the weak interaction and electromagnetism is unified into an electroweak theory as a Yang-Mills field. From that, we know that the wave function for an electron or a neutrino is an electroweak Yang-Mills field.

From the above considerations, the wave function for a neutrino must be an electroweak Yang-Mills field in which the electromagnetic components are zero. Also, that the wave function for an electron must be an electroweak Yang-Mills field in which both the electromagnetic and weak fields are non zero.

Just as we know an exact description of a photon as an electromagnetic wave, we need to discover an exact description of the wave functions for an electron and a neutrino in terms of electric, magnetic and weak components. That is, we want to know, for the electron, for example, the values of the components of the electric and magnetic fields and the components of the weak fields.

87.5 A free electron

We first consider the case of a free electron. It is sufficient to consider the rest frame of the electron, since the fields in any other inertial frame can be found by Lorentz transformation. The free electron has many solutions in wave mechanics, but we consider first, a plane wave.

In the rest frame, the wavenumber is zero, so the wavelength is infinite. This corresponds to a field which has no spatial variation, but simply oscillates at the rest frequency. This will be true of each component of the electric field, the magnetic field, and the weak field.

87.5.1 Maxwell's equations

Applying Maxwell's equations to the case of no spatial variation of electric or magnetic field implies that B is time independent in addition to being independent of position. Thus, only the electric field oscillates at the rest frequency, and is independent of position.

However, Our original assumption about the wave function for the free electron was that all components were time harmonic. If any component of the magnetic field is part of the wave function, then at least that component of the magnetic field must be zero.

There is still a question about which components are chosen to be part of the wave function. We could have components of the electric or magnetic field or the 4-vector potential. If we choose the 4-vector potential, then having all four components time harmonic and spatially constant would give a spatially constant electric field and zero magnetic field. That seems like a good choice.

So far, I have neglected possible charge distribution in applying Maxwell's equations. The Dirac equation uses the electron charge only with regard to its interaction with external fields. I could include charge distribution as a perturbation to slightly alter the electromagnetic fields associated with an electron.

87.5.2 Yang-Mills equations

At first glance, the Yang-Mills equations for the SU(2) weak part look similar to Maxwell's equations except for some nonlinear terms. The major effect of the nonlinear terms might be that for a plane wave in the rest frame with $\exp(-i\omega t)$ time dependence for all wave components, the nonlinear terms would have an $\exp(-2i\omega t)$ time dependence. The nonliear terms, however, are proportional to the coupling constant, which is small. As a first approximation, we can therefore, neglect the nonlinear terms. In that case, the result is the same as for Maxwell's equations. That is, there will be several electric-field-like terms, but no magnetic-field-like terms in the electron's rest frame. Specifically, for an SU(2) weak field, there are three electric-type fields, each with three components.

We could then include the nonlinear terms as a perturbation, which would give terms with higher harmonics of the rest frequency. The series would converge fairly quickly.

It appears that a complete wave function for an electron would include both the electromagnetic components and the weak-field components. I suspect, however, that solutions of the Dirac equation include only the electromagnetic components. It is then tempting, to guess that the four components of the 4-vector potential could be used as the components for a solution of the Dirac equation. However, so far, we have not included spin, whereas, the Dirac equation includes spin. The next section will consider spin.

87.6 Spin

So far, we have not included spin. Spin of a particle is defined as the total angular momentum in the rest frame of the particle. Just as linear momentum corresponds to a wavenumber in a linear direction, spin corresponds to an angular wavenumber. That is, the wave function for a spin half particle must be multiplied by a factor $\exp(\pm i\phi/2)$, where ϕ is the angle about the spin axis. (To begin, we ignore subtle complications.) Thus, including spin introduces a spatial variation into the situation, making the analysis of the previous two subsections invalid.

If we represent the electromagnetic part of the free electron wave function in its rest frame by a 4-vector potential as constants A_t, A_x, A_y, and A_z, each multiplying $\exp(-i\omega t)\exp(\pm i\phi/2)$ to represent the time harmonic factor and spin, then Maxwell's equations give the following formulas for the electric and magnetic fields for the free electron in its rest frame.

$$E_x = \left(i\omega A_x \pm i\frac{A_t}{2}\frac{y}{x^2+y^2} \right) e^{-i\omega t} e^{\pm i\phi/2} \ , \tag{87.1}$$

$$E_y = \left(i\omega A_y \mp i\frac{A_t}{2}\frac{x}{x^2+y^2} \right) e^{-i\omega t} e^{\pm i\phi/2} \ , \tag{87.2}$$

$$E_z = i\omega A_z e^{-i\omega t} e^{\pm i\phi/2} \ , \tag{87.3}$$

$$B_x = \pm i\frac{A_z}{2}\frac{x}{x^2+y^2} e^{-i\omega t} e^{\pm i\phi/2} \ , \tag{87.4}$$

$$B_y = \pm i\frac{A_z}{2}\frac{y}{x^2+y^2} e^{-i\omega t} e^{\pm i\phi/2} \ , \tag{87.5}$$

$$B_z = \pm\frac{i}{2}\frac{A_x x + A_y y}{x^2+y^2}e^{-i\omega t}e^{\pm i\phi/2} \; . \tag{87.6}$$

The upper and lower signs represent the two spin states. To get negative energy solutions, we would change the sign on ω everywhere. This is not meant to be an exact representation of the electromagnetic field of a free electron in its rest frame, but it is probably approximately correct. I could have chosen A_t, A_x, A_y, and A_z to have a slight spatial variation instead of being constants.

In addition, when those parameters are constant, what should be the relation among them? Should they have a fixed relation, so that all electrons are the same except for two spin states and two (positive and negative) energy states, or might there be some variation from one electron to the next? (Does that sound like a hidden variable theory?) Then there is the question of normalization. In this case, the normalization should be such as to represent the true strengths of the fields.

If we need to include charge and current distributions for the electron, that can be added as a perturbation.

For the weak-field components, we can make the same calculations, using the Yang-Mills equations instead of Maxwell's equations, with similar results, but three times as many components. The effect of the nonlinear terms can be included as a perturbation.

It seems that we should try to make the fields symmetric about the spin axis (z axis). We can do this by taking $A_y = \mp i A_x$. This gives

$$E_x = \left(i\omega A_x \pm i\frac{A_t}{2}\frac{y}{x^2+y^2}\right)e^{-i\omega t}e^{\pm i\phi/2} \; , \tag{87.7}$$

$$E_y = \left(\pm\omega A_x \mp i\frac{A_t}{2}\frac{x}{x^2+y^2}\right)e^{-i\omega t}e^{\pm i\phi/2} \; , \tag{87.8}$$

$$E_z = i\omega A_z e^{-i\omega t}e^{\pm i\phi/2} \; , \tag{87.9}$$

$$B_x = \pm i\frac{A_z}{2}\frac{x}{x^2+y^2}e^{-i\omega t}e^{\pm i\phi/2} \; , \tag{87.10}$$

$$B_y = \pm i\frac{A_z}{2}\frac{y}{x^2+y^2}e^{-i\omega t}e^{\pm i\phi/2} \; , \tag{87.11}$$

$$B_z = \pm i\frac{A_x}{2}(x^2+y^2)^{-1/2}e^{-i\omega t}e^{\mp i\phi/2} \; . \tag{87.12}$$

Notice the switch on the sign of ϕ for B_z in (87.12). The electric field transverse to the spin axis has two components. One component has constant magnitude, but rotates in direction with the rest frequency. The other component points in a circle around the spin axis. Both have a factor that rotates about the spin axis. The electric field along the spin axis contains the factor that rotates about the spin axis. The component of magnetic field normal to the spin axis is a radial field that rotates about the spin axis, but also contains the factor that rotates about the spin axis at twice that rate. The component of magnetic field along the spin axis has a magnitude that depends only on the distance from the spin axis and has a pattern that rotates about the spin axis.

As mentioned previously, these are only preliminary results, but may be a good first approximation to the electric and magnetic fields associated with a plane-wave electron wave function for a free electron. It is still not clear what other criteria should be imposed to choose the remaining constants.

No, this is wrong. The electromagnetic field is a spin one field, but an electron is a spin half particle. So the spin must be in the weak part of the field. I also need to include the charge. Maybe in the rest frame, the electric field is a radial field that has two parts. First, a constant field, appropriate for the charge. Second a radial mean-zero field that oscillates at the rest frequency.

The magnitude of that second field might be the same as the magnitude of the first field, so that the field goes to zero once per cycle.

No that cannot be correct. It has no magnetic field and there is no magnetic moment. This needs more careful thought.

Notice that polarization of an EM wave is not the same as spin, although they share some properties.

87.7 Review of the properties of the electron and neutrino

Here are some of their properties:

1. Both electrons and neutrinos have a rest frequency.

2. Whatever is oscillating can interfere with each other, and for the electron, with photons.

   Actually, this is incorrect. A more careful investigation of the process of Compton scattering shows that wave interference between an electron and a photon is not a significant part of the process. Similarly for beta decay.

   Further consideration by considering the process in momentum space shows that wave interference does occur.

3. Both have spin 1/2.

4. The electron has an electric charge.

5. The electron has a magnetic moment.

6. For the photon, we know why it oscillates. We can derive it from Maxwell's equations. Maybe for the electron and neutrino, we can derive the oscillation from the Yang-Mills equations for the electroweak field. The equations are like Maxwell's equations, with an additional nonliear term. Maybe that term allows for oscillations in the rest frame. That might tell us what is oscillating for the electron and neutrino in the rest frame.

Maybe the correct model for the electron is an electric charge that is independent of time plus a charge that oscillates at the rest frequency, the latter also having an $\exp{(i\phi/2)}$ factor for spin. No, that may still not be quite right. I think that will not give a time-average magnetic moment.

87.8 Connection with Dirac wave functions

The Dirac equation can be considered to be a set of four coupled linear differential equations that couple the four components (two energy states, two spin states) (or linear combinations of these four, depending on the representation) of the wave function. The Dirac equation does not deal with possible internal structure of the electron.

For the free electron, the four components are not coupled by the Dirac equation. A solution to the Dirac equation for an electron in its rest frame consists of simply assigning relative amplitudes to the four components with the correct harmonic time dependence, and then applying normalization.

What I am suggesting here, is that once we have found the correct representation of the electron in terms of electroweak fields, and we have found a solution to the Dirac equation, then combining them to get the electroweak field distribution corresponding to that solution.

87.9 Normalization

Because we are looking at plane waves here, the normalization must be in terms flux.

87.10 Comments by Dave Peterson

1. It isn't clear that electroweak is a theory by itself. Kaku (p 380) says that the theory of leptons given by the Weinberg-Salam model is actually flawed by the presence of anomalies. The true model requires quarks to cancel the anomalies. Anomalies can destroy renormalization. The photon apparently is a mix of W and B fields after higgs breaking. In that sense, the photon is electroweak. However, the "weak field" is usually thought of as just the massive vector bosons W by themselves.

2. Electromagnetic waves might be considered just as A waves without mentioning E and B fields.

3. Masses have their origin in confined energy from a variety of sources (base mass from Higgs, electromagnetic fields, weak interactions, strong gluon fields, relativistic confined motion). Despite this, the basic mass vibration or rest frequency is simply $\nu = E$ (or mc^2) $/h$. It wouldn't make sense to say that this vibration is electromagnetic or electroweak. Gravity might even be involved. I have no clue what is vibrating.

4. It is possible to imagine converters where one type of wave comes in and another goes out (e.g., plasmons to optical).

5. Penrose says that a spinor is an object which turns into its negative when it undergoes a complete rotation through 2π radians. No standard text seems to repeat this, but that is the meaning of the half angles for SU(2).

6. A quantum field is an entity existing at every point in space which regulates the creation and annihilation of particles. ψ is not a probability amplitude but operators which create and destroy particles in various normal modes. Particle numbers are elementary excitations of the quantum matter field. They can interact with photons which are the excitations of the quantized Maxwell field. QFT identifies a wave with the superposition of an indefinite number of particles.

7. The concept that mass acts like a rest frequency goes back at least to 1925 (deBroglie). How far out can this basic vibration act? Messiah (II, p948) says, "Thus in the non-relativistic limit, the Dirac electron appears not as a point charge, but as a distribution of charge and current extending over a domain of linear dimensions $\hbar/mc$." And, in general, effects go out to at least a wavelength. "For an atom at rest, the quantum mechanical amplitude to find the atom at a place is the same everywhere; it does not depend on position." But the phase can still vary from point to point like $\exp(-imc^2t/\hbar)$ – Feynman [246] [247] [248, e.g., p. 7-2].

8. The Compton effect results in an increase in wavelength from the incident X-rays on electrons. Conservation of momentum then leads to $p = h/\lambda$. Relativity says that the four-vectors (E, p) and (ω, k) are proportional. Quantum mechanics quantifies the proportionality.

9. Energies are additive, so frequencies are also additive (perhaps only for entangled systems of constituent particles).

10. How is it that relative velocity generates a wavelength, lambda? A derivation of this using rest-frequency depends on having a phase velocity of $c^2/v \gg c$ - this hardly sounds physical. Deep and intangible waters.

11. It is more likely that the QM vibration is not identifiable with standard fields having energy content but rather are "pre-energy" fields. In the case of the massless real photon, the "pre-energy" vibration happens to correspond to the EM vibration in wavelength and frequency but not in space-time extent. The vacuum behaves more like a set of mathematical computer rules that randomly show themselves as material energy. This mathematical existence lies below the energy of existence.

My responses:

1.

2. The problem with that is that we can change A by a gauge transformation.

3. I am suggesting that for the electron, it is electric and magnetic fields vibrating plus also some weak fields vibrating. However, this brings up an interesting point. Why is it that there are vibrations associated with the sources of gravitation (mass, energy, momentum), but not with the sources of electric and magnetic fields (charge, current)?

4.

5.

6. Yes, that is the orthodox view in quantum field theory. However, I suspect that all of the usual mathematical baggage of anihilation and creation operators is simply a way for us to make calculations. I doubt that the underlying physics behaves that way.

7.

8.

9.

10.

11.

87.11 Generalization to other particles

Clearly, these ideas apply to other particles as well. Any charged particle would be considered to be electromagnetic waves plus waves of something else.

There has always been a search for the fundamental building blocks of matter. Representing particles as being made up of quarks is an example. An intriguing example is the representation of quarks as strongly bound leptons. [249]

Chapter 88

A wave function applies to an ensemble[1]

abstract

Assuming that a wave function applies unconditionally to an ensemble but only in a restricted sense to a member of the ensemble provides an alternative to wave function collapse.

88.1 Introduction

The title is, of course, obvious. The question is whether a wave function applies only to an ensemble or whether it applies also to members of the ensemble. In the orthodox interpretation, it is usually assumed that it applies to each member of the ensemble.

88.2 The problem

For example, in the orthodox interpretation, let us assume there is an atom in an excited state. It undergoes a transition, and emits a photon. The wave function for that photon is assumed to be a spherical wave that propagates in all directions. Some years later, and an equal number of light-years away, the photon is absorbed in a telescope or some other instrument operated on Earth by an astronomer. When that happens, the spherical wave function of that photon collapses, so that that photon cannot be detected or absorbed on the other side of the original atom, and it is then realized that the original atom must have had a recoil when the original photon was emitted to conserve momentum.

This is all standard. It is also weird, as is generally acknowledged. Collapse of wave functions, although generally accepted, is also generally considered a problem in the interpretation of quantum theory. Interpretation of entanglement and the EPR (Einstein-Podolsky-Rosen) paradox is in the same category, although slightly more complicated.

[1] David Peterson pointed out that what I am proposing in this chapter has long been known as the "ensemble interpretation" of quantum mechanics. A description of it can be found in Wikipedia, including both the pros and cons of the interpretation. As far as I can tell, all of the cons listed there can be easily refuted. Albert Einstein was one of the greatest supporters of the ensemble interpretation. I am starting to suspect, however, that what I am suggesting here differs somewhat from the ensemble interpretation.

88.3 The solution

To try to resolve this weirdness, I will relax one of the postulates of quantum mechanics, namely the assumption that a wave function applies to each member of the ensemble in addition to applying to the ensemble as a whole.

To revisit the above example with a revised interpretation, we have the following. An atom in an excited state decays, emitting one quantum of electromagnetic radiation as a spherical wave. There is a wave function associated with this radiation, which we shall call $\psi(\mathbf{x}, t)$, which depends on position and time.

There is also a quantity, $\psi^*\psi(\mathbf{x}, t)$, which is variously referred to as a probability density or "intensity of presence," (the latter according to George Gamow), and which also depends on position and time. This quantity $\psi^*\psi(\mathbf{x}, t)$ (when multiplied by a small volume) is usually interpreted as being the probability of finding the photon within that small volume at the position $\mathbf{x}$ at time t. A more exact definition of probabilities is in terms of fluxes.

However, this quantity, $\psi^*\psi(\mathbf{x}, t)$, is not directly observable without specifying some sort of apparatus to make the observation. Therefore, we consider another quantity $\psi_1(\mathbf{x}, t)$, which we refer to as the amplitude for observing the photon (or more accurately for observing a photon) in a specified kind of apparatus given the wave function $\psi(\mathbf{x}, t)$. The probability of observing a photon in that apparatus is then $\psi_1^*\psi_1(\mathbf{x}, t)$. This is no longer a probability density, in that the apparatus is now included in the calculation.

Now, suppose that an astronomer on Earth observes a photon in his apparatus. What happens next? Does the wave function collapse? That is, does either $\psi^*\psi(\mathbf{x}, t)$ or $\psi_1^*\psi_1(\mathbf{x}, t)$ change? No, neither of these quantities change as a result of the detection, because both of these quantities refer to a probability estimate having to do with the probability of detection of a photon for an ensemble of atoms in the same situation. That estimate is independent of anything that may happen to any member of the ensemble.

What about some other astronomer on the opposite side of the star from Earth? Assuming he has a measuring instrument identical to that of the astronomer on Earth, his probability for detecting a photon from that atom is also $\psi_1^*\psi_1(\mathbf{x}, t)$. Remembering that this probability applies to an ensemble of such atoms, it is still the correct estimate for that astronomer to detect a quantum of radiation from such an atom. The fact that he cannot detect radiation from that atom because an astronomer on Earth has detected that radiation is irrelevant. The probability estimate for the ensemble still stands, but it no longer applies to this specific case.[2]

Third, what about the original atom? Does it recoil as a result of the detection of a photon on Earth? We have no way of knowing without some sort of measuring device to measure such recoil. If we were to install such a devise, we would certainly measure a recoil, which would tell us toward which direction the radiation from that atom was launched, but that would be a different experiment from the one we are describing.

88.4 More examples

The above example is useful in that it is simple enough to demonstrate fundamental principles, but it does not represent anything that would actually be observed in reality. A more practical example involves radiation from a star that is then observed by an astronomer on Earth.

Radiation from stars starts with nuclear fusion in the interior of a star to convert 4 hydrogen nuclei to a helium nucleus by either the carbon cycle or the proton-proton cycle. In the carbon

[2]Any interpretation must deal with the non-local character of quantum mechanics. As with wave-function collapse, or with Cramer's transactional interpretation, my alternative provides no way for people to transmit information faster than light, and does not violate causality.

cycle, three quanta of radiation are produced, while for the proton-proton cycle, four quanta of radiation are produced, the quanta being of various energies.

In either case, the radiation combines with that being produced by other identical nuclear reactions, so that the quanta (being bosons) lose their identity, giving rise to a single electromagnetic field that is a function of position and time. (Of course, there is a difference between coherent and incoherent radiation. The radiation from stars is incoherent.) The radiation requires thousands of years to reach the surface of the star because of collisions with the atoms and nuclei within the star.

Once the radiation reaches the surface of the star, it travels at the speed of light, radiating in all directions. Given that radiation, we can estimate the amplitude for detection of radiation from this star by an astronomer on Earth. Call that amplitude $\psi_2(\mathbf{x}, t)$, which depends on the amplitude and characteristics of the radiation. It also depends on the characteristics of the apparatus. The probability for detection of one quantum of radiation in that apparatus is then $\psi_2^*\psi_2(\mathbf{x}, t)$.

In this case, the radiation probably contains so many quanta that it already makes up an ensemble. But, even if it does not, the interpretation is clear as long as we realize that probability estimates apply only to an ensemble. In that way, there are no collapsing wave packets.

88.5 EPR paradox

In the EPR paradox, we have an initial system in a spin-zero state. It emits two photons or an electron and a positron (or something similar) in opposite directions. To conserve angular momentum, the pairs of particles must have opposite spins. However, there are experiments that show that the individual spins are not determined, only that they must be opposite. The states of the two particles are said to be entangled. Although the combined state of the two particles is considered to be a pure state that can be described by a wave function, the state of each particle is a mixed state that must be described by a density matrix, not a wave function.

As soon as one of the spins has been detected, we immediately know the state of the other particle. The density matrix for that particle then immediately collapses to a wave function, representing a pure state.

88.6 Solution

Again, the solution is similar to the first example. A wave function (or a density matrix in the case of a mixed state) represents an ensemble. Although it can be used to estimate probabilities for detecting certain things for a single member of the ensemble, such estimates always apply to the ensemble.

Thus, the wave functions and density matrices in the EPR experiment apply to an ensemble of such identical systems. When the spin of one of the particles is measured, the original wave function does not collapse. It is still in existence because it applies to the ensemble of identical systems.

What about conservation of angular momentum? As with the first example, in any experiment in which conservation of angular momentum is checked, it will be found to be obeyed. However, if it is not checked, than we do not know the situation. Thus, measuring only one spin is a different experiment from measuring both spins. Measuring one spin does not change the state of the other particle.

However, once we have measured one spin, we have set new initial conditions for what may happen next. From those new initial conditions, we can make estimates of the future. That does not constitute a collapse of a wave function. We are simply looking at a different situation at that point.

It is also to be pointed out that wave functions have nothing to do with knowledge. They give the amplitude for something occurring in certain situations. People are not required.

Notice that letting the wave function apply to the ensemble, but not to members of the ensemble, also solved the zero-point energy problem in Chapter 84.

88.7 Clarification

Let me be perfectly clear. The wave function $\psi(x,t)$ is an electromagnetic wave, but it combines with whatever other electromagnetic fields are around. $\psi_1(x,t)$ or $\psi_2(x,t)$ (which also includes the detection apparatus) also includes whatever other fields that are around. If a detection occurs, it depends only on that local field, and nothing happens anyplace else.

As to conservation of energy and momentum, whenever they are checked, they will be found to hold. If not, then who knows?

88.8 Further thoughts

I think I need to review Bell's theorem and the experimental evidence for entanglement. Also, Joy Christian claims to have found a flaw in Bell's theorem.

I am thinking now that particles are real waves. For example, electrons are probably electromagnetic waves plus weak waves. See Chapter 87. The amplitudes of such waves would be proportional to the usual wave function.

To calculate detection of an electron, we would estimate the probability amplitude for a particular process. That would depend on the apparatus and the local value of the wave function, which would be proportional to the local amplitude of the associated wave fields. We would get the usual result as in ordinary quantum theory.

Now consider an EPR experiment. A state of zero angular momentum and zero linear momentum decays into two (in most respects) equal particles, such as an electron and a positron or two photons. Each of these particles expands as a spherical wave in such a way that the total linear momentum and total angular momentum remains constant.

If Alice and Bob are equally distant away in opposite directions, then the probability of either of them detecting anything is very small because of the spherical waves. However, we can run the experiment repeatedly until one of them detects something. If Alice detects one of the particles, then there is an increased probability for Bob to also detect a particle because of zero linear momentum, and vise-versa. We can consider the wave function (actually a density matrix because of entanglement between the particles) to be a spherical wave. In momentum space, the wave function would be independent of momentum.

We would like to treat angular momentum in the same way, that is, as a realistic wave. In angular momentum space, we would like the wave function of each particle to be independent of angular momentum. We can do this as being proportional to something like $e^{iS\theta}$, where S is the spin, and we leave unspecified about what direction θ is measured. This is analogous to the linear-momentum case where we take the wave function to be proportional to e^{ikr}/r, where k is just a magnitude. We still require the two particles to be entangled, so that their spins are opposite. To indicate this is simply a matter of notation. We can still consider these to be real waves of something. This seems like a better representation than the usual one where we consider simply a superposition of spin up and spin down, or something similar.

Then for probabilities of detection, we use the usual formula, but now it is proportional to the absolute square of local fields, which we take to be real (reality, not the opposite of complex). So fields and wave functions are real, but probabilities of occurance of any measurement still give the same values as in ordinary quantum mechanics.

There still seems to be a problem of wave-function collapse. After the particle has interacted and been detected, soes the spherical wave continue to expand, or does it disappear?

Emilios Santos [250] presents a realistic interpretation of quantum mechanics. Travis Norsen [251] gives Bell's concept of local causality.

88.9 Additional thoughts, 19 February 2015

As we shall be discussing "The age of entanglement" [252] by Louisa Gilder next Monday, I thought it would be useful to mention my views on entanglement. In the EPR experiment (or thought experiment until Aspect performed it), we have an initial state with zero angular momentum that decays into two identical spin-half particles that (because of conservation of momentum) travel in opposite directions, and (because of conservation of angular momentum) have opposite spin.

Whenever spin components in the same direction for the two particles are measured, they are always found to be opposite. One explanation for that fact might be that the spins of the two particles were determined at the time of decay of the original state. Quantum theory, on the other hand, predicts that the individual spins are not determined, but only correlated. This correlation is expressed in quantum theory as being in a superposition state of two ways of having the spins of the two particles opposite.

Bell's inequality calculates the probabilities of various measurement configurations, when spin measurements on the two particles are not necessarily along the same direction, assuming that the spins of the two particles were determined at the time of decay. These probabilities disagree with the probabilities predicted by quantum theory. Aspect's 1981 or 1982 experiment gave agreement with the predictions of quantum theory. The conclusion is that the individual spins of the two particles cannot be determined at the time of decay, but only correlations are determined.

The standard way to represent this situation in quantum theory is to say that the two particles are represented by a single wave function in which the spins of the two particles are entangled. It is not possible to represent either of the two particles by a wave function even though the particles may be very far apart. We say that the two particles taken together are in a pure state and can be represented by a wave function, but each particle by itself is in a mixed state and can be represented by a density matrix, not a wave function.

If a measurement is made on the spin of one of the particles, then instantaneously each of the two particles switches from a mixed state to a pure state and each particle can be represented by a wave function. (However, there is no way to transmit information faster than light, no way to signal.) If the wave function represents a physical field, then we have a nonlocal change in a physical field. Here is the way that we get into the philosophical question about reality versus causality versus locality.

A realistic interpretation of the wave function would mean that the wave function represents a real physical field, like an electric or magnetic field. Schrödinger believed that the wave function represented a real physical field. Therefore, if we have a realistic interpretation of quantum theory, then it appears that quantum behavior is either a causal (faster that light travel) or nonlocal (instantaneous action at a distance).

The only reasonable way I have found out of this trap is to avoid assuming that a wave function represents a real physical field. There are no experiments that give evidence that a wave function represents a real physical field. All of the experiments regarding wave functions either give energy-level differences for stationary states (in an atom, for example), or give probabilities for the outcomes of various measurements (which can only be tested by performing the same experiment repeatedly under the same conditions).

If we accept that a wave function is only a method for making forecasts of the outcomes of various experiments, similar to weather forecasts, then we no longer have problems like "collapse of the wave function" of philosophical problems concerning reality, locality, or causality. The only problem we are left with is the one we would like to answer, which is "What is really happening?"

My guess for the answer to that question is that the particle (an electron, say) is on the average similar to the wave function we calculate for it, but superimposed on top of that are fluctuations. Further, because of the uncertainty principle, it is impossible to determine any significant details of those fluctuations. In addition, it is very unlikely that we shall ever have a theory that would tell us the details of those fluctuations, and even if we did come up with such a theory, we would not be able to test it by any experiment because of the uncertainty principle.

Chapter 89

Gravitational Field representation of General Relativity[1]

abstract

Einstein's theory of General Relativity is a geometric representation of gravity. Three attempts to form a "gravitational field representation" of General Relativity are described. These attempts all represent the gravitational field by a 3-index tensor $F_{\alpha\beta\gamma}$ analogous to the electromagnetic field tensor. In this representation, the gravitational force on a body of non-zero mass in arbitrary motion would be zero if and only if all components of the gravitational field tensor are zero.

In the first formulation (ca 2012), I defined a 3-index tensor $F_{\alpha\beta\gamma}$ that was the negative of the affine connection in the frame of a particular body, but, because I defined it to be a tensor, I required it to transform as a tensor. Unfortunately, because I also required it to have derivatives that matched the (negative of) the derivatives of the affine connection in that frame, that led to contradictions.

The second formulation (December 2014) was based on a preprint [253, Stephen M. Barnett, December 2014] that generates a gravitational field tensor by subtracting two versions of an affine connection. Although this formulation led to some possibly reasonable results, the dependence of this formulation on the existence of the second affine connection (which seems to have no physical bearing on the situation) brought in something that should have been irrelevant to the problem.

The third formulation (January 2015) returned to the first formulation, but without requiring equality of derivatives of the 3-index tensor $F_{\alpha\beta\gamma}$ and the (negative of) the derivatives of the affine connection in that frame. This seems to lead to consistent results, but it still has to be checked. Direct calculation, however, shows that $F_{\alpha\beta\gamma}$ does not transform as a tensor.

This leaves the character of gravitation in doubt. All three other fundamental forces can be expressed as a multi-component tensor field in a formula that is covariant. All of these on a background that is provided by the gravitational metric tensor field. The closest we can come to a reasonable tensor formula for gravitational force is in the covariant divergence of the energy-momentum tensor. But, even then the formula is not analogous to the form of the interaction of any of the other fundamental forces. And, we still have gravitation providing the background for the other interactions.

[1]December 2009, revised March 2013 through January 21, 2016

89.1 Introduction

That classical physics could be derived from a variational principle was considered to be a mathematical curiosity until the discovery of quantum theory showed that classical physics was based on wave interference in the same way that geometrical optics was based on wave interference through Fermat's principle.

However, efforts to find an underlying quantum theory for gravitation have so far failed. There are several possibilities to explain why that may be so, and some of those possibilities have been investigated by various people in a reference list that is much too long to put here. However, see [254, Kiefer, 2013] for a review.

One of the possibilities that has been investigated is that the formulation of General Relativity, although very successful, may be in error, and needs to be replaced by a different theory. Again, the list of alternative theories that has been investigated is too long to put here. However, one alternative representation for gravitation that has not to my knowledge been tried is a gravitational field representation to replace Einstein's geometrical representation.

Newton's theory of gravity can be expressed as a field theory. That is, the gravitational field a distance r from a mass M is a vector given by

$$\mathbf{g}(\mathbf{r}) = -\frac{GM\mathbf{r}}{|r^3|} , \tag{89.1}$$

where G is Newton's constant of gravitation. That is, the gravitational field points toward the position of the mass M. It is a field because it has a value at every position. If there are more masses, the gravitational fields simply add as vectors to give the total gravitational field vector. The gravitational force on a body of mass m is then given by

$$\mathbf{F} = m\mathbf{g}(\mathbf{r}) . \tag{89.2}$$

The gravitational force on a body is zero if and only if all three components of the gravitational field are zero.

Maxwell's theory of electromagnetism is also a field theory. The electric field E and the magnetic field B are given by Maxwell's equations, and the electromagnetic force on a body of charge q moving with velocity $\mathbf{v}$ is given by the Lorentz force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) . \tag{89.3}$$

The electromagnetic force on a charged body in arbitrary motion is zero if and only if all three components of the electric field and all three components of the magnetic field are zero. In more modern tensor notation, the electric and magnetic fields are combined into an antisymmetric electromagnetic field tensor $F_{\mu\nu}$, in which the (0,i) components (that is, the mixed time and space components) give the three components of the electric field and the (i,j) components give the three components of the magnetic field. In tensor notation, charge and current (that is, moving charge) are combined into a 4-current J^μ. The Lorentz force is then given by

$$F^\mu = F^\mu{}_\nu J^\nu . \tag{89.4}$$

Again, the electromagnetic force on a charged body in arbitrary motion is zero if and only if all components of $F_{\mu\nu}$ are zero.

Einstein's representation of General Relativity, however, is not a field theory. There is nothing in General Relativity that could be called a gravitational field. There is nothing for which the gravitational force will be zero if and only if all components of that something are zero.

The problem arises because Einstein correctly recognized that inertial forces are also gravitational forces. However, although inertial forces are included in General Relativity, they are treated in a slightly different way than other gravitational forces because General Relativity is not a field theory. It is a geometric theory.

To be a field theory, there must be a gravitational field that has at least the first three of the following properties:

1. The field components that act on a body or particle should be independent of the properties of that body or particle, including the motion of that body or particle.

2. The gravitational force (including inertial force) on a body of non-zero mass in arbitrary motion should be zero if and only if all components of the gravitational field tensor are zero.

3. The gravitational field should be unique. That is, a given physical situation should lead to a unique gravitational field.

Although there is a possibility that it is not possible to find a field representation that satisfies all of the above criteria, it turns out that it is possible to satisfy the first three criteria. With regard to the strength of the inertial field, we need to recognize that inertial mass is only proportional to gravitational mass, not necessarily equal to it, and the gravitational constant G really just expresses the ratio between non-inertial gravitational forces and inertial gravitational forces. (In fact, the significance of the equivalence principle is that inertia is a gravitational force, and nothing more.)

In the frame of a body, the geodesic equation looks just like a force field, similar to the Lorentz force. In that frame, the connection $\Gamma^{\alpha}{}_{\beta\gamma}$ acts like a force field, in a similar way that $F_{\mu\nu}$ acts in the Lorentz force.

89.1.1 Ansatz ca 2012

The problem is that $\Gamma^{\alpha}{}_{\beta\gamma}$ is not a tensor, so it looks like a gravitational field only in that frame. A possible solution, would be to define a tensor $F^{\alpha}{}_{\beta\gamma}$ that equals $-\Gamma^{\alpha}{}_{\beta\gamma}$ in that frame, but has the correct tensor transformation properties, so that it differs from $-\Gamma^{\alpha}{}_{\beta\gamma}$ in any other frame.[2] [3] In general, it is not possible to pick any frame and choose $F = -\Gamma$, and have F be a tensor. But, in a frame that is a confluence of trajectories, it is possible. I think the proof for that is implicit in the development in section 89.2 because there I construct a tensor F that has that property, but I would still have to develop an explicit proof.

It is often argued that curvature is gravitation. That we have gravitational (inertial) force in flat spacetime, where there is no curvature, suggests that there is something wrong with that point of view. In addition, it may be easier to reconcile quantum theory with gravitation in a gravitational field representation than with gravitation in a geometric representation.

The goal here is to find a tensor gravitational field that satisfies all three of the above properties.

As it turns out, it is not possible to find a tensor gravitational field that satisfies all three of the above properties. Maybe it is still possible to find a tensor gravitational field by requiring only that such a field equals (the negative of) the connection along the trajectory of a body or particle.

89.1.2 Addition, 11 March 2015

It is possible to find a gravitational field tensor, but it is not unique. Similarly to the gravitational vector potential, it is possible to find a gravitational field.

Maybe I can find something to add to the affine connection so that the sum is a tensor.

[2] There should be no confusion with the electromagnetic field tensor $F_{\mu\nu}$ because the gravitational field tensor $F^{\alpha}{}_{\beta\gamma}$ has three indexes whereas the electromagnetic field tensor has only two indexes.

[3] The reason for the minus sign will be seen later.

89.1.3 Addition, 21 January 2016

Because the gravitational field in General Relativity is represented in terms of geometry, it is not possibility to find a "gravitational field" expressed within that geometry. It would be like trying to pull yourself up by your own bootstraps.

Instead, we have a situation in General Relativity where the "gravitational field" is partly expressed in the geometry and partly expressed in the affine connection.

The discussion here assumes there to be an electromagnetic field, so that a freely falling frame is not the same as a frame fixed to the body in question.

We also note, that the electromagnetic field can never be "transformed away." Instead, the electric part can be transformed away, but not both electric and magnetic parts at the same time. Notice that in a frame where the electric part of the electromagnetic field is zero, a charged particle at rest in such a frame would experience no electromagnetic forces.

Similarly, in the frame of a particle, there might be no gravitational forces on that particle even if some components of the affine connection are non-zero.

In a frame for which all components of the affine connection are zero (also called freely falling frames or inertial frames), the "gravitational field" is completely (and implicitly) expressed by the geometry. The "gravitational field" has not been "transformed away."

On the other hand, in a frame fixed to the body, the "electric part" of the gravitational field is zero. The "magnetic part" of the gravitational field would be partly in the affine connection, and partly (and implicitly) in the geometry.

89.1.4 Ansatz December 2014

I have just (December 2014) come across a preprint [253, Stephen M. Barnett] that generates a gravitational field tensor by subtracting two versions of an affine connection.

I have been thinking carefully about that preprint. It looks like what I was trying to do. The only problem is his equation (11), which is the geodesic equation. In particular, I would like to get rid of the term on the left (which is not a tensor) and the last term on the right in the second line of the equation (the part with Lambda tilde, which is also not a tensor).

If those two terms would cancel each other, then we would have something good. Just as in the usual geodesic equation, even though those two terms are not separately tensors, the combination of those two terms (if brought to the same side of the equation) is a tensor. Now, since that equation is a tensor equation, it is valid in any frame. Let us write the equation in the frame that moves with the body in question. In that frame, the term on the left side of the equation is zero. Since the equation is still a tensor equation, and the first term on the right is a tensor, then the last term on the right must also be a tensor (in that frame). (I had hoped to be able to say that the second term on the right is zero, but I can't say that.)

I thought I was going to be able to prove that the second term on the right had to be zero in the frame moving with the body. All that would be left would be the first term on the right in that frame, and therefore, since the only term left was a tensor, that had to be a correct tensor equation in all frames. However, I caught the error.

There is still some freedom in choosing Lambda tilde. Maybe it is possible to choose it in such a way that that term is zero in the frame of the body. Then I would still have what I need.

89.2 Geodesic equation using the ansatz ca 2012

Let us consider further the idea of defining a frame in which the tensor $F^{\alpha}{}_{\beta\gamma} = -\Gamma^{\alpha}{}_{\beta\gamma}$. Let us choose that frame to be the frame fixed to a particular body in question, and let the mass of that body be m. Let us choose the location in spacetime of that body in some arbitrary frame to be

x^μ. Let $\Gamma^{\rho'}_{\mu'\nu'}$ be the connection in the frame of the body in question. The transformation of the connection from this x' frame to an arbitrary x frame is given by[20, MTW, p. 262]

$$\Gamma^\alpha{}_{\beta\gamma} = L^\alpha_{\rho'} L^{\mu'}_\beta L^{\nu'}_\gamma \Gamma^{\rho'}{}_{\mu'\nu'} - L^\alpha_{\mu'} L^{\mu'}_{\beta,\gamma} \ , \tag{89.5}$$

where the negative sign is because we are transforming the frame and where

$$L^{\mu'}_\beta \equiv \frac{\partial x^{\mu'}}{\partial x^\beta} \tag{89.6}$$

is the appropriate transformation. The transform for the tensor $F^\alpha{}_{\beta\gamma}$ is

$$F^\alpha{}_{\beta\gamma} = L^\alpha_{\rho'} L^{\mu'}_\beta L^{\nu'}_\gamma F^{\rho'}{}_{\mu'\nu'} \ . \tag{89.7}$$

We define F to be equal to $-\Gamma$ in the frame of the body. That is, we define $F^{\rho'}{}_{\mu'\nu'} = -\Gamma^{\rho'}{}_{\mu'\nu'}$. Therefore (89.5) is equivalent to

$$\Gamma^\alpha{}_{\beta\gamma} = -L^\alpha_{\rho'} L^{\mu'}_\beta L^{\nu'}_\gamma F^{\rho'}{}_{\mu'\nu'} - L^\alpha_{\mu'} L^{\mu'}_{\beta,\gamma} \ . \tag{89.8}$$

Therefore, combining (89.7) and (89.8) gives

$$F^\alpha{}_{\beta\gamma} = -\Gamma^\alpha{}_{\beta\gamma} - L^\alpha_{\mu'} L^{\mu'}_{\beta,\gamma} \ . \tag{89.9}$$

From (89.6), we have

$$L^{\mu'}_{\beta,\gamma} = \frac{\partial}{\partial x^\gamma} L^{\mu'}_\beta = \frac{\partial}{\partial x^\gamma} \frac{\partial x^{\mu'}}{\partial x^\beta} = \frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\gamma} \ . \tag{89.10}$$

Substituting (89.10) into (89.9) gives

$$F^\alpha{}_{\beta\gamma} = -\Gamma^\alpha{}_{\beta\gamma} - L^\alpha_{\mu'} \frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\gamma} \ . \tag{89.11}$$

Therefore, (89.11) becomes

$$F^\alpha{}_{\beta\gamma} = -\Gamma^\alpha{}_{\beta\gamma} - L^\alpha_{\mu'} \frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\gamma} \ . \tag{89.12}$$

Consider the geodesic equation from (52.1) and [20, Misner, Thorne, and Wheeler (8.26) p. 211, (10.27) p. 263]

$$m[g_{\mu\beta}\ddot{x}^\beta + \frac{1}{2}(g_{\mu\nu,\beta} + g_{\mu\beta,\nu} - g_{\beta\nu,\mu})\dot{x}^\beta \dot{x}^\nu] + e(A_{\mu,\nu} - A_{\nu,\mu})\dot{x}^\nu = 0 \ . \tag{89.13}$$

Equation (89.13) can be written [20, Misner, Thorne, and Wheeler, Box 3.1, p. 72]

$$m[g_{\mu\beta}\ddot{x}^\beta + \Gamma_{\mu\nu\beta}\dot{x}^\beta \dot{x}^\nu] - eF_{\mu\nu}\dot{x}^\nu = 0 \ . \tag{89.14}$$

Raising an index and changing some indexes allows us to write (89.14) as

$$m[\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma] - eF^\alpha{}_\gamma \dot{x}^\gamma = 0 \ , \tag{89.15}$$

which is the usual form of the geodesic equation when combined with the Lorentz force term. [e.g., [18, Weinberg, 1972 eq. (3.2.3) p. 71] & [20, Misner, Thorne, and Wheeler, 1973, Box 3.1, p. 72 and (3.54′) p. 88]]

Let us now consider the following Ansatz[4] for a gravitational field and electromagnetic force equation.

$$-mF^{\alpha}{}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} - eF^{\alpha}{}_{\gamma}\dot{x}^{\gamma} = 0 \ . \tag{89.16}$$

Notice that (89.16) looks just like what we would expect for a tensor generalization of the Lorentz force. We see now that the reason for defining $F^{\alpha}{}_{\beta\gamma}$ with a minus sign relative to $\Gamma^{\alpha}{}_{\beta\gamma}$ is so that both terms in (89.16) have the same sign.

To show that (89.16) is valid, substitute (89.12) into (89.16). That gives

$$m[\Gamma^{\alpha}_{\beta\gamma} + L^{\alpha}_{\mu'}\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}}]\dot{x}^{\beta}\dot{x}^{\gamma} - eF^{\alpha}{}_{\gamma}\dot{x}^{\gamma} = 0 \ . \tag{89.17}$$

Rearranging terms gives

$$m[L^{\alpha}_{\mu'}\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}}\dot{x}^{\beta}\dot{x}^{\gamma} + \Gamma^{\alpha}{}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma}] - eF^{\alpha}{}_{\gamma}\dot{x}^{\gamma} = 0 \ . \tag{89.18}$$

The first term in (89.18) can be rewritten to give

$$m[L^{\alpha}_{\mu'}\ddot{x}^{\mu'} + \Gamma^{\alpha}{}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma}] - eF^{\alpha}{}_{\gamma}\dot{x}^{\gamma} = 0 \ . \tag{89.19}$$

The first term in (89.19) is multiplied by the correct factor to transform from the primed to the unprimed frame. Therefore, we can write (89.19) as

$$m[\ddot{x}^{\alpha} + \Gamma^{\alpha}{}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma}] - eF^{\alpha}{}_{\gamma}\dot{x}^{\gamma} = 0 \ . \tag{89.20}$$

Equation (89.20) is equivalent to the geodesic equation including the electromagnetic Lorentz force (89.15). Thus, we have shown that defining a tensor $F^{\alpha}{}_{\beta\gamma}$ equal to the negative of the connection $\Gamma^{\alpha}{}_{\beta\gamma}$ in the frame of a body will lead to the correct geodesic equation (including the Lorentz force) for that body by using the Ansatz in (89.16). Although we have not shown that the same $F^{\alpha}{}_{\beta\gamma}$ will give the correct geodesic equation for all bodies, notice that we will have $F^{\alpha}{}_{\beta\gamma} = -\Gamma^{\alpha}{}_{\beta\gamma}$ whenever the second term on the right of (89.12) is zero.

It might appear, from (89.12) or (89.17), that the "gravitational field" is proportional to the acceleration of the body, and therefore violates the rule that the gravitational field must be independent of the motion of the body. However, in the gravitational field representation, $F_{\alpha\beta\gamma}$ is simply a gravitational field, and nothing more. Equations (89.12) and (89.17) are not part of the gravitational field representation. They are merely to show the connection between the gravitational field representation and the geometric representation. It might appear that what I call a gravitational field is really an acceleration. However, in the gravitational field representation, it is simply a gravitational field. One response might be that this is just a mathematical trick, that it is "really" an acceleration. I respond that all of mathematics applied to physics is a "trick," but some of these tricks are very useful, including this one.

89.3 Riemann tensor

We start with the definition of the Riemann tensor [20, Misner, Thorne, and Wheeler, Box 14.2, p. 340].

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} \ . \tag{89.21}$$

[4] Because we now know that all forms of energy, not only mass, are sources of gravitation and respond to gravitation, we would expect $F^{\alpha}{}_{\beta\gamma}T^{\beta\gamma} + F^{\alpha}{}_{\gamma}J^{\gamma} = 0$ to be a probable generalization of (89.16), where $T^{\beta\gamma}$ is the stress-energy tensor.

The Riemann tensor normally satisfies certain symmetries. These are [18, Weinberg, (6.6.3)-(6.6.5), p. 141].

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \ , \tag{89.22}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \ , \tag{89.23}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \ , \tag{89.24}$$

$$R_{\alpha\beta\gamma\delta} = +R_{\beta\alpha\delta\gamma} \ , \tag{89.25}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \ . \tag{89.26}$$

89.3.1 Ansatz ca 2012

In a frame in which $F^{\alpha}{}_{\beta\gamma} = -\Gamma^{\alpha}{}_{\beta\gamma}$, we have

$$R^{\alpha}{}_{\beta\gamma\delta} = -F^{\alpha}{}_{\beta\delta,\gamma} + F^{\alpha}{}_{\beta\gamma,\delta} + F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \ . \tag{89.27}$$

Consider a confluence of trajectories. That confluence defines a frame in space-time. Consider the value of $\Gamma_{\alpha\beta\gamma}$ on that frame. Let $F_{\alpha\beta\gamma} = -\Gamma_{\alpha\beta\gamma}$ on that space-time. Then the derivatives are also equal and opposite. Now, I want to make $F_{\alpha\beta\gamma}$ transform like a tensor when changing to other frames.

In general, it is not possible to pick any frame and choose $F_{\alpha\beta\gamma} = -\Gamma_{\alpha\beta\gamma}$, and have $F_{\alpha\beta\gamma}$ be a tensor. But, in a frame that is confluence of trajectories, it is possible. I think the proof for that is implicit in the development in section 89.2 because there I construct a tensor $F_{\alpha\beta\gamma}$ that has that property, but I would still have to develop an explicit proof.

So, that means that my field equations are probably correct, but I have to be more careful in applying them. When I try to choose a solution in a frame in which $F_{\alpha\beta\gamma} = -\Gamma_{\alpha\beta\gamma}$, I need to make sure that that frame is also a confluence of trajectories. If there are no other forces, that means that it has to be a freely falling frame.

That means that the gravitational forces must be zero. So, in the exact solutions I have tried so far, we must have $g = 0$ in the first case, $\omega = 0$ in the centrifugal force case, and $M = 0$ in the Schwarzschild case. In that case, my solutions are OK, but uninteresting. To be a real test, I have to bring in a charge and an electromagnetic field. Then I should have a solution.

To help calculate (89.27), we use the fact that because $F^{\alpha}{}_{\beta\gamma}$ is a tensor, its covariant derivative satisfies

$$F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\beta\delta;\gamma} = F^{\alpha}{}_{\beta\gamma,\delta} - F^{\alpha}{}_{\beta\delta,\gamma} + F^{\epsilon}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\epsilon\delta} + F^{\alpha}{}_{\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} - F^{\alpha}{}_{\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - F^{\epsilon}{}_{\beta\delta}\Gamma^{\alpha}{}_{\epsilon\gamma} \ . \tag{89.28}$$

Equation (89.28) is valid in any frame for any tensor $F^{\alpha}{}_{\beta\gamma}$. Notice that one term has canceled. In a frame in which $F^{\alpha}{}_{\beta\gamma} = -\Gamma^{\alpha}{}_{\beta\gamma}$, we have

$$F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\beta\delta;\gamma} = F^{\alpha}{}_{\beta\gamma,\delta} - F^{\alpha}{}_{\beta\delta,\gamma} - F^{\epsilon}{}_{\beta\gamma}F^{\alpha}{}_{\epsilon\delta} - F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} + F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\epsilon}{}_{\beta\delta}F^{\alpha}{}_{\epsilon\gamma} \ . \tag{89.29}$$

or,

$$F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\beta\delta;\gamma} = F^{\alpha}{}_{\beta\gamma,\delta} - F^{\alpha}{}_{\beta\delta,\gamma} - 2F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} + 2F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} \ . \tag{89.30}$$

or, solving for the part we need,

$$F^{\alpha}{}_{\beta\gamma,\delta} - F^{\alpha}{}_{\beta\delta,\gamma} = F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\beta\delta;\gamma} + 2F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} - 2F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} \ . \tag{89.31}$$

Substituting (89.31) into (89.27) gives

$$R^{\alpha}{}_{\beta\gamma\delta} = -F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \ . \tag{89.32}$$

Equation (89.32) gives a covariant formula for the Riemann tensor.

Equation (89.32) can be rewritten with lowered indexes as

$$R_{\alpha\beta\gamma\delta} = -F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \ . \tag{89.33}$$

The symmetry on the connection leads to the symmetry

$$F_{\mu\alpha\beta} = F_{\mu\beta\alpha} \ , \tag{89.34}$$

at least in the frame where $F_{\mu\alpha\beta}$ is defined to be the negative of the connection. In other frames, we can require that symmetry because the force equation depends only on the symmetric part of $F_{\mu\alpha\beta}$ on the last two indexes.

The Riemann tensor normally satisfies certain symmetries [18, Weinberg, (6.6.3)-(6.6.5), p. 141]. The symmetry

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{89.35}$$

is identically satisfied by (89.33). The symmetry

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \tag{89.36}$$

is satisfied by (89.33) if $F_{\mu\alpha\beta}$ also satisfies the symmetry (89.34).

Imposing the symmetries

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{89.37}$$

and

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{89.38}$$

leads to additional conditions. There are two choices for those additional conditions. One choice for those additional conditions leads to a second symmetry

$$F_{\alpha\beta\gamma} = F_{\beta\alpha\gamma} = F_{\gamma\beta\alpha} \ , \tag{89.39}$$

which, combined with (89.34), implies that $F_{\alpha\beta\gamma}$ is completely symmetric on interchange of any two indexes. However, I now know that the symmetry (89.39) is too restrictive.

Imposing the symmetries (89.37) and (89.38) of the Riemann tensor combined with the symmetry (89.39) would lead to

$$F_{\alpha\beta\gamma;\delta} - F_{\alpha\beta\delta;\gamma} = 0 \ , \tag{89.40}$$

but I now know that is incorrect.

Equations (89.405) and (89.406) can be rewritten as

$$F_{\gamma\delta\beta;\alpha} - F_{\gamma\delta\alpha;\beta} - F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} = e_{\alpha\beta\gamma\delta} \tag{89.41}$$

and

$$-F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\beta\alpha\delta;\gamma} + F_{\beta\alpha\gamma;\delta} = f_{\alpha\beta\gamma\delta} \ . \tag{89.42}$$

where

$$e_{\alpha\beta\gamma\delta} \equiv F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} - F_{\gamma\epsilon\alpha}F^{\epsilon}{}_{\delta\beta} + F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta\alpha} \tag{89.43}$$

and

$$f_{\alpha\beta\gamma\delta} \equiv F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} + F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha\delta} - F_{\beta\epsilon\delta}F^{\epsilon}{}_{\alpha\gamma} \ . \tag{89.44}$$

Equations (89.41) and (89.42) are the homogeneous field equations for the gravitational field tensor. They are analogous to the homogeneous Maxwell equation

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \ . \tag{89.45}$$

However, the terms $e_{\alpha\beta\gamma\delta}$ and $f_{\alpha\beta\gamma\delta}$ on the right-hand sides of (89.41) and (89.42), which are proportional to products of components of the gravitational field tensor, represent nonlinearities in the gravitational field that are not present in the electromagnetic case. Equations (89.41) and (89.42) each have 256 individual equations. However, because of symmetries, the number of independent equations is less than that. When I try to estimate the number of independent equations in (89.41), I get 66, but that is too many, because $F_{\beta\epsilon\delta}$ has only 34 independent components, so I must have made an error in calculating. I have started counting the number of independent equations in (89.42), but I so far also get too many.

Not using (89.40) allows us to write the Riemann tensor (89.32) as

$$R^{\alpha}{}_{\beta\gamma\delta} = -F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \ , \tag{89.46}$$

and now it is right. The contracted Riemann tensor is then

$$R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta} = -F^{\alpha}{}_{\beta\delta;\alpha} + F^{\alpha}{}_{\beta\alpha;\delta} - F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\alpha} \ , \tag{89.47}$$

and that is right, now. Or,

$$R^{\beta}{}_{\delta} = R^{\alpha\beta}{}_{\alpha\delta} = -F^{\alpha\beta}{}_{\delta;\alpha} + F^{\alpha\beta}{}_{\alpha;\delta} - F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon\beta}{}_{\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon\beta}{}_{\alpha} \ , \tag{89.48}$$

We can write the Ricci scalar as

$$R = g^{\theta\phi}\left(-F^{\alpha}{}_{\theta\phi;\alpha} + F^{\alpha}{}_{\theta\alpha;\phi} - F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\theta\phi} + F^{\alpha}{}_{\epsilon\phi}F^{\epsilon}{}_{\theta\alpha}\right) \ , \tag{89.49}$$

and correct now. Or,

$$R = -F^{\alpha\phi}{}_{\phi;\alpha} + F^{\alpha\phi}{}_{\alpha;\phi} - F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon\phi}{}_{\phi} + F^{\alpha}{}_{\epsilon\phi}F^{\epsilon\phi}{}_{\alpha} \ , \tag{89.50}$$

and correct now. We can rewrite (89.49) as

$$R = g^{\theta\phi}g^{\alpha\delta}\left[-F_{\delta\theta\phi;\alpha} + F_{\delta\theta\alpha;\phi} + g^{\mu\epsilon}\left(-F_{\delta\epsilon\alpha}F_{\mu\theta\phi} + F_{\delta\epsilon\phi}F_{\mu\theta\alpha}\right)\right] \ , \tag{89.51}$$

and correct now.

We notice also, that if F is completely symmetric and satisfies the homogeneous gravitational field equation (89.40), then the second Bianchi identity

$$R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\delta\epsilon;\gamma} + R_{\alpha\beta\epsilon\gamma;\delta} = 0 \tag{89.52}$$

is also satisfied. However, I now know that F is not completely symmetric and does not satisfy (89.40). Therefore, (89.52) is an additional condition for F to satisfy. However, as it turns out, (89.52) is identically satisfied by (89.33), although it is tedious but straightforward to show it. Part of showing it uses the identity

$$F_{\alpha\beta\gamma;\mu;\nu} - F_{\alpha\beta\gamma;\nu;\mu} = F_{\tau\beta\gamma}R^{\tau}{}_{\alpha\mu\nu} + F_{\alpha\tau\gamma}R^{\tau}{}_{\beta\mu\nu} + F_{\alpha\beta\tau}R^{\tau}{}_{\gamma\mu\nu} \ , \tag{89.53}$$

which is valid for any tensor $F_{\alpha\beta\gamma}$. Equation (89.53) is the generalization of equation (16.6) on page 389 of [20, Misner, Thorne, and Wheeler].

If we define

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{1}{2}g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} + \frac{1}{2}g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\gamma}g^{\beta\delta}R_{\alpha\beta\gamma\delta} \ , \tag{89.54}$$

where I have split the first term into two equal parts. Then

$$G^{\mu\nu}{}_{;\nu} = \frac{1}{2}g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma}R_{\alpha\beta\gamma\delta;\nu} + \frac{1}{2}g^{\mu\beta}g^{\nu\delta}g^{\alpha\gamma}R_{\alpha\beta\gamma\delta;\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\gamma}g^{\beta\delta}R_{\alpha\beta\gamma\delta;\nu} \ . \tag{89.55}$$

Changing dummy indexes gives

$$G^{\mu\nu}{}_{;\nu} = \frac{1}{2}g^{\mu\gamma}g^{\nu\alpha}g^{\delta\beta}R_{\delta\gamma\beta\alpha;\nu} + \frac{1}{2}g^{\mu\gamma}g^{\delta\beta}g^{\nu\alpha}R_{\nu\gamma\alpha\beta;\delta} - \frac{1}{2}g^{\mu\gamma}g^{\alpha\nu}g^{\beta\delta}R_{\alpha\beta\nu\delta;\gamma} \ . \tag{89.56}$$

Factoring gives

$$G^{\mu\nu}{}_{;\nu} = \frac{1}{2}g^{\mu\gamma}g^{\nu\alpha}g^{\delta\beta}\left(R_{\delta\gamma\beta\alpha;\nu} + R_{\nu\gamma\alpha\beta;\delta} - R_{\alpha\beta\nu\delta;\gamma}\right) \ . \tag{89.57}$$

Using the symmetries (89.35), (89.37), and (89.38) in (89.57) gives

$$G^{\mu\nu}{}_{;\nu} = \frac{1}{2}g^{\mu\gamma}g^{\nu\alpha}g^{\delta\beta}\left(R_{\alpha\beta\gamma\delta;\nu} + R_{\alpha\beta\nu\gamma;\delta} + R_{\alpha\beta\delta\nu;\gamma}\right) = 0 \tag{89.58}$$

because of (89.52).

89.3.2 Ansatz December 2014

Barnett [253, Stephen M. Barnett] uses the following ansatz.

$$\Delta^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} - \tilde{\Gamma}^{\mu}_{\alpha\beta} \ , \tag{89.59}$$

where the connection $\tilde{\Gamma}^{\mu}_{\alpha\beta}$ is chosen such that the resulting Ricci tensor $\tilde{R}_{\alpha\beta} = 0$. Because the difference of two connections transforms as a tensor even though a connection does not transform as a tensor, $\Delta^{\mu}{}_{\alpha\beta}$ is a tensor.

Here, I define

$$F^{\mu}{}_{\alpha\beta} = -\Gamma^{\mu}{}_{\alpha\beta} + \tilde{\Gamma}^{\mu}_{\alpha\beta} \ , \tag{89.60}$$

with the same restriction on the connection $\tilde{\Gamma}^{\mu}_{\alpha\beta}$. My reason for using a different notation is first that I want the minus sign in the definition and second that I intend to put additional restrictions on $\tilde{\Gamma}^{\mu}_{\alpha\beta}$. Using a different notation will indicate those additional restrictions.

The symmetry on the connection leads to the symmetry

$$F_{\mu\alpha\beta} = F_{\mu\beta\alpha} \ . \tag{89.61}$$

Barnett [253, Stephen M. Barnett, his equation (2)] gives, in the present notation

$$R^{\alpha}{}_{\beta\gamma\delta} - \tilde{R}^{\alpha}_{\beta\gamma\delta} = -F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} + F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \ . \tag{89.62}$$

Equation (89.62) gives a covariant formula for the Riemann tensor. Equation (89.62) can be derived by using

$$F^{\alpha}{}_{\beta\gamma;\delta} = F^{\alpha}{}_{\beta\gamma,\delta} + F^{\epsilon}{}_{\beta\gamma}\hat{\Gamma}^{\alpha}_{\epsilon\delta} - F^{\alpha}{}_{\epsilon\gamma}\hat{\Gamma}^{\epsilon}_{\beta\delta} - F^{\alpha}{}_{\epsilon\beta}\hat{\Gamma}^{\epsilon}_{\gamma\delta} \tag{89.63}$$

and

$$-F^{\alpha}{}_{\beta\delta;\gamma} = -F^{\alpha}{}_{\beta\delta,\gamma} + F^{\alpha}{}_{\epsilon\delta}\hat{\Gamma}^{\epsilon}_{\beta\gamma} - F^{\epsilon}{}_{\beta\delta}\hat{\Gamma}^{\alpha}_{\epsilon\gamma} + F^{\alpha}{}_{\epsilon\beta}\hat{\Gamma}^{\epsilon}_{\gamma\delta} \ , \tag{89.64}$$

where

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} \equiv \frac{1}{2}\left(\Gamma^{\alpha}{}_{\beta\gamma} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}\right) \ . \tag{89.65}$$

Equation (89.62) can be rewritten with lowered indexes as

$$R_{\alpha\beta\gamma\delta} - \tilde{R}_{\alpha\beta\gamma\delta} = -F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} + F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \ . \tag{89.66}$$

The Riemann tensor normally satisfies certain symmetries [18, Weinberg, (6.6.3)-(6.6.5), p. 141]. The symmetry

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{89.67}$$

is identically satisfied by (89.66). The symmetry

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \tag{89.68}$$

is satisfied by (89.66) if $F_{\mu\alpha\beta}$ also satisfies the symmetry (89.61).

Imposing the symmetries

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{89.69}$$

and

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{89.70}$$

leads to additional conditions.

These are

$$F_{\gamma\delta\beta;\alpha} - F_{\gamma\delta\alpha;\beta} - F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} = e_{\alpha\beta\gamma\delta} \tag{89.71}$$

and

$$-F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\beta\alpha\delta;\gamma} + F_{\beta\alpha\gamma;\delta} = f_{\alpha\beta\gamma\delta} \ . \tag{89.72}$$

where

$$e_{\alpha\beta\gamma\delta} \equiv -F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} + F_{\gamma\epsilon\alpha}F^{\epsilon}{}_{\delta\beta} - F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta\alpha} \tag{89.73}$$

and

$$f_{\alpha\beta\gamma\delta} \equiv -F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} - F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha\delta} + F_{\beta\epsilon\delta}F^{\epsilon}{}_{\alpha\gamma} \ . \tag{89.74}$$

Equations (89.71) and (89.72) are the homogeneous field equations for the gravitational field tensor. They are analogous to the homogeneous Maxwell equation

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \ . \tag{89.75}$$

However, the terms $e_{\alpha\beta\gamma\delta}$ and $f_{\alpha\beta\gamma\delta}$ on the right-hand sides of (89.71) and (89.72), which are proportional to products of components of the gravitational field tensor, represent nonlinearities in the gravitational field that are not present in the electromagnetic case.

The contracted Riemann tensor is then

$$R_{\beta\delta} = R_{\beta\delta} - \tilde{R}_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta} - \tilde{R}^{\alpha}_{\beta\alpha\delta} = -F^{\alpha}{}_{\beta\delta;\alpha} + F^{\alpha}{}_{\beta\alpha;\delta} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\delta} - F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\alpha} \ . \tag{89.76}$$

Notice the disagreement with [253, Stephen M. Barnett, his equation (3)] because he has contracted on a different pair of indexes. Or,

$$R^{\beta}{}_{\delta} = R^{\alpha\beta}{}_{\alpha\delta} = -F^{\alpha\beta}{}_{\delta;\alpha} + F^{\alpha\beta}{}_{\alpha;\delta} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon\beta}{}_{\delta} - F^{\alpha}{}_{\epsilon\delta}F^{\epsilon\beta}{}_{\alpha} \ . \tag{89.77}$$

We can write the Ricci scalar as

$$R = g^{\theta\phi}\left(-F^{\alpha}{}_{\theta\phi;\alpha} + F^{\alpha}{}_{\theta\alpha;\phi} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\theta\phi} - F^{\alpha}{}_{\epsilon\phi}F^{\epsilon}{}_{\theta\alpha}\right) \ , \tag{89.78}$$

Or,

$$R = -F^{\alpha\phi}{}_{\phi;\alpha} + F^{\alpha\phi}{}_{\alpha;\phi} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon\phi}{}_{\phi} - F^{\alpha}{}_{\epsilon\phi}F^{\epsilon\phi}{}_{\alpha} \ . \tag{89.79}$$

We can rewrite (89.78) as

$$R = g^{\theta\phi}g^{\alpha\delta}\left[-F_{\delta\theta\phi;\alpha} + F_{\delta\theta\alpha;\phi} + g^{\mu\epsilon}\left(F_{\delta\epsilon\alpha}F_{\mu\theta\phi} - F_{\delta\epsilon\phi}F_{\mu\theta\alpha}\right)\right] \ . \tag{89.80}$$

89.3.3 Ansatz January 2015

Everything in this sub section is based on the assumption that

$$F^{\alpha}{}_{\beta\gamma} = -\Gamma^{\alpha}{}_{\beta\gamma} - \frac{\partial x^{\alpha}}{\partial x^{\mu'}}\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}} \tag{89.81}$$

is a tensor. However, direct calculation shows that (89.81) does not transform as a tensor. Therefore, nearly everything to follow in this subsection is wrong.

We start with (89.12), which we write as

$$\Gamma^{\alpha}{}_{\beta\gamma} = -F^{\alpha}{}_{\beta\gamma} - \frac{\partial x^{\alpha}}{\partial x^{\mu'}}\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}} \ . \tag{89.82}$$

Substituting (89.82) into (89.21) gives

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = &-F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \\ &+\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}}\left(-\frac{\partial x^{\epsilon}}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\epsilon}\partial x^{\delta}} + \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\delta}}\right) \\ &-\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\delta}}\left(-\frac{\partial x^{\epsilon}}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\epsilon}\partial x^{\gamma}} + \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\gamma}}\right) \ . \end{aligned} \tag{89.83}$$

Or,

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = &-F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \\ &+\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial x^{\epsilon}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\epsilon}\partial x^{\delta}} + \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\delta}}\right) \\ &-\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\delta}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial x^{\epsilon}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\epsilon}\partial x^{\gamma}} + \frac{\partial^2 x^{\alpha}}{\partial x^{\mu'}\partial x^{\gamma}}\right) \ . \end{aligned} \tag{89.84}$$

Or,

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = &-F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \\ &+\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\delta}} + \frac{\partial}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\delta}}\right) \\ &-\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\delta}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\gamma}} + \frac{\partial}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\gamma}}\right) \ . \end{aligned} \tag{89.85}$$

Or,

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = &-F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \\ &+\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\delta}}\frac{\partial x^{\nu'}}{\partial x^{\mu'}} + \frac{\partial}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\delta}}\right) \\ &-\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\delta}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\gamma}}\frac{\partial x^{\nu'}}{\partial x^{\mu'}} + \frac{\partial}{\partial x^{\mu'}}\frac{\partial x^{\alpha}}{\partial x^{\gamma}}\right) \ . \end{aligned} \tag{89.86}$$

Or,

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = &-F^{\alpha}{}_{\beta\delta;\gamma} + F^{\alpha}{}_{\beta\gamma;\delta} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F^{\alpha}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \\ &+\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\gamma}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\delta}}\delta^{\nu'}_{\mu'} + \frac{\partial}{\partial x^{\mu'}}\delta^{\alpha}_{\delta}\right) \\ &-\frac{\partial^2 x^{\mu'}}{\partial x^{\beta}\partial x^{\delta}}\left(-\frac{\partial x^{\alpha}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\gamma}}\delta^{\nu'}_{\mu'} + \frac{\partial}{\partial x^{\mu'}}\delta^{\alpha}_{\gamma}\right) \ . \end{aligned} \tag{89.87}$$

Or,

$$R^\alpha{}_{\beta\gamma\delta} = -F^\alpha{}_{\beta\delta;\gamma} + F^\alpha{}_{\beta\gamma;\delta} - F^\alpha{}_{\epsilon\gamma}F^\epsilon{}_{\beta\delta} + F^\alpha{}_{\epsilon\delta}F^\epsilon{}_{\beta\gamma} + \frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\gamma}(0+0) - \frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\delta}(0+0) \ . \tag{89.88}$$

Or,

$$R^\alpha{}_{\beta\gamma\delta} = -F^\alpha{}_{\beta\delta;\gamma} + F^\alpha{}_{\beta\gamma;\delta} - F^\alpha{}_{\epsilon\gamma}F^\epsilon{}_{\beta\delta} + F^\alpha{}_{\epsilon\delta}F^\epsilon{}_{\beta\gamma} \ . \tag{89.89}$$

Equation (89.83) can be written with lowered indexes as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\alpha\epsilon\gamma}F^\epsilon{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^\epsilon{}_{\beta\gamma} \\ &+ g_{\alpha\phi}\frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\gamma}\left(-\frac{\partial x^\epsilon}{\partial x^{\mu'}}\frac{\partial x^\phi}{\partial x^{\nu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\epsilon \partial x^\delta} + \frac{\partial^2 x^\phi}{\partial x^{\mu'}\partial x^\delta}\right) \\ &- g_{\alpha\phi}\frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\delta}\left(-\frac{\partial x^\epsilon}{\partial x^{\mu'}}\frac{\partial x^\phi}{\partial x^{\nu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\epsilon \partial x^\gamma} + \frac{\partial^2 x^\phi}{\partial x^{\mu'}\partial x^\gamma}\right) \ . \end{aligned} \tag{89.90}$$

Equation (89.89) can be written with lowered indexes as

$$R_{\alpha\beta\gamma\delta} = -F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\alpha\epsilon\gamma}F^\epsilon{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^\epsilon{}_{\beta\gamma} \ . \tag{89.91}$$

As before, we get equations for $F_{\alpha\beta\gamma}$ by imposing the symmetries on the Riemann tensor. Equation (89.90) satisfies (89.24) identically. The symmetry (89.26) or (89.36) is satisfied by (89.90) if $F_{\mu\alpha\beta}$ also satisfies the symmetry (89.34). Substituting (89.91) into (89.23) gives

$$F_{\gamma\delta\beta;\alpha} - F_{\gamma\delta\alpha;\beta} - F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} = e_{\alpha\beta\gamma\delta} \ , \tag{89.92}$$

where

$$e_{\alpha\beta\gamma\delta} \equiv F_{\alpha\epsilon\gamma}F^\epsilon{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^\epsilon{}_{\beta\gamma} - F_{\gamma\epsilon\alpha}F^\epsilon{}_{\delta\beta} + F_{\gamma\epsilon\beta}F^\epsilon{}_{\delta\alpha} \ . \tag{89.93}$$

Substituting (89.91) into (89.25) gives

$$-F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\beta\alpha\delta;\gamma} + F_{\beta\alpha\gamma;\delta} = f_{\alpha\beta\gamma\delta} \ , \tag{89.94}$$

where

$$f_{\alpha\beta\gamma\delta} \equiv F_{\alpha\epsilon\gamma}F^\epsilon{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^\epsilon{}_{\beta\gamma} + F_{\beta\epsilon\gamma}F^\epsilon{}_{\alpha\delta} - F_{\beta\epsilon\delta}F^\epsilon{}_{\alpha\gamma} \ . \tag{89.95}$$

89.4 Validity of the ansatz ca 2012

The validity of the development of a gravitational field representation of General Relativity seems to hinge on being able to define a tensor $F_{\alpha\beta\gamma}$ that is the negative of the connection $\Gamma_{\alpha\beta\gamma}$ in some particular coordinate frame. In other frames, the two would be different. However, we have shown that $F_{\alpha\beta\gamma}$ is symmetric under interchange of any two indexes. It would seem that $\Gamma_{\alpha\beta\gamma}$ would have to have that same symmetry in that particular frame. In general, $\Gamma_{\alpha\beta\gamma}$ does not have that symmetry. What would make it have that symmetry in some particular frame? On the other hand, if it does not have that symmetry in some frame, how does that affect the gravitational field representation of General Relativity? I now know that the answer is that neither F nor Γ have that symmetry.

The formula for $R_{\alpha\beta\gamma\delta}$ in terms of $\Gamma_{\alpha\beta\gamma}$ is very similar to its formula in terms of $F_{\alpha\beta\gamma}$. Therefore, it might be reasonable for $\Gamma_{\alpha\beta\gamma}$ to have the same symmetries as $F_{\alpha\beta\gamma}$, since they derive from the symmetries of $R_{\alpha\beta\gamma\delta}$. However, because $F_{\alpha\beta\gamma}$ is a tensor, it must satisfy conditions that $\Gamma_{\alpha\beta\gamma}$ does not have to satisfy. In particular, $F_{\alpha\beta\gamma;\delta}$ is a tensor, whereas $\Gamma_{\alpha\beta\gamma,\delta}$ is not a tensor. In deriving the symmetry property for $F_{\alpha\beta\gamma}$, I made contractions on $F_{\alpha\beta\gamma;\delta}$, and raised and lowered its indexes. I don't think I could do that with $\Gamma_{\alpha\beta\gamma,\delta}$.

Still, let us see how far we can get in deriving symmetries for $\Gamma_{\alpha\beta\gamma}$. First of all, let us write (89.21) with lower indexes. This is

$$R_{\alpha\beta\gamma\delta} = \Gamma_{\alpha\beta\delta,\gamma} - \Gamma_{\alpha\beta\gamma,\delta} + \Gamma_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} \ . \tag{89.96}$$

We see that (89.96) identically satisfies (89.36) and (89.35).

Substituting (89.96) into (89.37) gives

$$\Gamma_{\gamma\delta\beta,\alpha} - \Gamma_{\gamma\delta\alpha,\beta} - \Gamma_{\alpha\beta\delta,\gamma} + \Gamma_{\alpha\beta\gamma,\delta} = e'{}_{\alpha\beta\gamma\delta} \ , \tag{89.97}$$

where

$$e'{}_{\alpha\beta\gamma\delta} \equiv \Gamma_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} - \Gamma_{\gamma\epsilon\alpha}\Gamma^{\epsilon}{}_{\delta\beta} + \Gamma_{\gamma\epsilon\beta}\Gamma^{\epsilon}{}_{\delta\alpha} \tag{89.98}$$

We can rewrite (89.38) as

$$R_{(\alpha\beta)\gamma\delta} = 0 \ , \tag{89.99}$$

where parentheses denote symmetrization. Substituting (89.96) into (89.99) gives

$$-\Gamma_{\alpha\beta\delta,\gamma} + \Gamma_{\alpha\beta\gamma,\delta} - \Gamma_{\beta\alpha\delta,\gamma} + \Gamma_{\beta\alpha\gamma,\delta} = f'{}_{\alpha\beta\gamma\delta} \ . \tag{89.100}$$

where

$$f'{}_{\alpha\beta\gamma\delta} \equiv \Gamma_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} + \Gamma_{\beta\epsilon\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} - \Gamma_{\beta\epsilon\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} \ . \tag{89.101}$$

In an inertial frame, $\Gamma_{\alpha\epsilon\gamma}$ is zero, so in an inertial frame, $e'{}_{\alpha\beta\gamma\delta} = 0$ and $f'{}_{\alpha\beta\gamma\delta} = 0$. Also, $e'{}_{\alpha\beta\gamma\delta} = 0$ and $f'{}_{\alpha\beta\gamma\delta} = 0$ in any frame for which $\Gamma_{\alpha\beta\gamma} = \Gamma_{\beta\alpha\gamma}$. So, how do we find such a frame?

89.5 Lagrangian

Conceptually, defining a tensor $F^{\alpha}{}_{\beta\gamma}$ that equals $-\Gamma^{\alpha}{}_{\beta\gamma}$ in a particular frame does not seem very difficult. However, to make this a real theory, we need to derive field equations for this new gravitational tensor field, similar to Maxwell's equations for the electromagnetic field tensor.

We start with the Einstein-Hilbert Lagrangian

$$L = \frac{R - 2\Lambda}{16\pi} - \frac{1}{2}\rho g_{\mu\nu}U^{\mu}U^{\nu} + A_{\mu}J^{\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \ , \tag{89.102}$$

where the first term (geometry) is from [20, Misner, Thorne, and Wheeler, (21.13) p. 491], the second term is the cosmological term, the third term is the matter Lagrangian, the fourth term (electromagnetic coupling) comes from [239, Sakurai (1.70) p. 13], and the last term is from [20, Misner, Thorne, and Wheeler, Box 21.1, p. 495] and [239, Sakurai (1.70) p. 13].

To represent gravitation as a gravitational field rather than as geometry, we replace the matter term in the Lagrangian by a gravitational-coupling term[5] $+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}$, where $Y^{\mu\nu}$ is a second-rank tensor and $A_{\mu\nu}$ is a gravitational potential tensor, analogous to the electromagnetic potential A_{μ}. The lagrangian in (89.102) then becomes

$$L = \frac{R - 2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_{\mu}J^{\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \ , \tag{89.103}$$

[5]The mathematics to follow would be valid using any second-rank tensor for $Y^{\mu\nu}$. Choosing $Y^{\mu\nu}$ determines the physical content of the theory.

89.5.1 Ansatz ca 2012

The next major step is to substitute (89.50) into (89.103) to get the Lagrangian in a form that can be varied to give field equations. That gives

$$L = \frac{1}{16\pi} g^{\beta\gamma}(-F^{\alpha}{}_{\beta\gamma;\alpha} + F^{\alpha}{}_{\beta\alpha;\gamma}) + \frac{1}{16\pi}(-F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon\gamma}{}_{\gamma} + F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon\gamma}{}_{\alpha}) - \frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_{\mu}J^{\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \ . \tag{89.104}$$

Equation (89.105) can be rewritten

$$L = \frac{1}{16\pi} g^{\beta\gamma}(-F^{\alpha}{}_{\beta\gamma;\alpha} + F^{\alpha}{}_{\beta\alpha;\gamma} - F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma} + F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\alpha}) - \frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_{\mu}J^{\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \ . \tag{89.105}$$

89.5.2 Ansatz December 2014

The next major step is to substitute (89.79) into (89.103) to get the Lagrangian in a form that can be varied to give field equations. That gives

$$L = \frac{1}{16\pi} g^{\beta\gamma}(-F^{\alpha}{}_{\beta\gamma;\alpha} + F^{\alpha}{}_{\beta\alpha;\gamma}) + \frac{1}{16\pi}(F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon\gamma}{}_{\gamma} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon\gamma}{}_{\alpha}) - \frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_{\mu}J^{\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \ . \tag{89.106}$$

Equation (89.107) can be rewritten

$$L = \frac{1}{16\pi} g^{\beta\gamma}(-F^{\alpha}{}_{\beta\gamma;\alpha} + F^{\alpha}{}_{\beta\alpha;\gamma} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\alpha}) - \frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_{\mu}J^{\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \ . \tag{89.107}$$

89.6 Action

The action [20, MTW, eq. (21.2), p. 485]

$$I = \int L(-g)^{\frac{1}{2}}\, \mathrm{d}^4x \tag{89.108}$$

must be an extremum.

89.6.1 Ansatz ca 2012

Substituting (89.105) into (89.108) gives

$$\begin{aligned} I = \frac{1}{16\pi}\int g^{\beta\gamma}\left(-F^{\alpha}{}_{\beta\gamma;\alpha} + F^{\alpha}{}_{\beta\alpha;\gamma} - F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma} + F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\alpha}\right)(-g)^{\frac{1}{2}}\, \mathrm{d}^4x \\ + \int\left(-\frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_{\mu}J^{\mu} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\right)(-g)^{\frac{1}{2}}\, \mathrm{d}^4x \ . \end{aligned} \tag{89.109}$$

The most difficult part of correctly formulating a variation procedure is correctly choosing which variables to vary. In the case of tensors, we must choose which indexes should be covariant and which contravariant. Choosing to vary $g^{\alpha\beta}$, $F^{\epsilon}{}_{\beta\alpha}$, and $F_{\mu\nu}$ gives the correct contribution of $F_{\mu\nu}$ to $T_{\mu\nu}$, but gives no contribution of $F^{\epsilon}{}_{\beta\alpha}$ to $T_{\mu\nu}$. These considerations suggest we vary $g^{\alpha\beta}$, $F_{\epsilon\beta\alpha}$, and $F_{\mu\nu}$. There is a question as to what form to use for the two coupling terms $+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}$ and $A_{\mu}J^{\mu}$. If we use $g^{\alpha\beta}$ to write those terms with lowered indexes, then that will introduce corresponding terms into the final stress-energy tensor. It is not clear what the correct choice is, but the same choice should probably be made for both terms. For now, I shall leave those terms as they are, but

I can change that choice later if that is the correct choice. With that in mind, it is useful to rewrite (89.109) as

$$I = \frac{1}{16\pi}\int g^{\beta\gamma}g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} - F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} + F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_\mu J^\mu - \frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ , \quad (89.110)$$

where I have used the symmetry (89.34), and ( ) in indexes denotes symmetrization.

89.6.2 Ansatz December 2014

Substituting (89.107) into (89.108) gives

$$I = \frac{1}{16\pi}\int g^{\beta\gamma}\left(-F^{\alpha}{}_{\beta\gamma;\alpha} + F^{\alpha}{}_{\beta\alpha;\gamma} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma} - F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\alpha}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_\mu J^\mu - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \quad (89.111)$$

The most difficult part of correctly formulating a variation procedure is correctly choosing which variables to vary. In the case of tensors, we must choose which indexes should be covariant and which contravariant. Choosing to vary $g^{\alpha\beta}$, $F^{\epsilon}{}_{\beta\alpha}$, and $F_{\mu\nu}$ gives the correct contribution of $F_{\mu\nu}$ to $T_{\mu\nu}$, but gives no contribution of $F^{\epsilon}{}_{\beta\alpha}$ to $T_{\mu\nu}$. These considerations suggest we vary $g^{\alpha\beta}$, $F_{\epsilon\beta\alpha}$, and $F_{\mu\nu}$. There is a question as to what form to use for the two coupling terms $+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}$ and $A_\mu J^\mu$. If we use $g^{\alpha\beta}$ to write those terms with lowered indexes, then that will introduce corresponding terms into the final stress-energy tensor. It is not clear what the correct choice is, but the same choice should probably be made for both terms. For now, I shall leave those terms as they are, but I can change that choice later if that is the correct choice. With that in mind, it is useful to rewrite (89.111) as

$$I = \frac{1}{16\pi}\int g^{\beta\gamma}g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_\mu J^\mu - \frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ , \quad (89.112)$$

where I have used the symmetry (89.61), and ( ) in indexes denotes symmetrization.

89.7 Variational Procedure

89.7.1 Ansatz ca 2012

Taking the variation of (89.110) gives

$$\delta I = \frac{1}{16\pi}\int \delta g^{\beta\gamma}g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} - F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} + F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\beta\gamma}g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} - F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} + F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)\left(-\frac{1}{2}\right)g_{\theta\phi}\delta g^{\theta\phi}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\beta\gamma}\delta g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} - F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} + F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\beta\gamma}g^{\alpha\delta}\left(-F_{(\delta\alpha)(\epsilon}F_{\mu)\beta\gamma} + F_{(\delta(\gamma(\epsilon}F_{\mu)\beta)\alpha)}\right)\delta g^{\mu\epsilon}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_\mu J^\mu - \frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$

$$+\int\left[-\frac{1}{16\pi}F_{\mu\nu}F_{\epsilon\delta}\left(g^{\mu\epsilon}\delta g^{\nu\delta}+\delta g^{\mu\epsilon}g^{\nu\delta}\right)\right](-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\beta\gamma}g^{\alpha\delta}\left(-\delta F_{\delta\beta\gamma;\alpha}+\delta F_{\delta\beta\alpha;\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}\left(-F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma}-\delta F_{\delta\epsilon\alpha}F_{\mu\beta\gamma}+F_{\delta\epsilon\gamma}\delta F_{\mu\beta\alpha}+\delta F_{\delta\epsilon\gamma}F_{\mu\beta\alpha}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(+\frac{1}{6}\delta(A_{\beta\gamma}Y^{\beta\gamma})\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(g^{\mu\nu}\delta(A_\mu J_\nu)\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{1}{16\pi}g^{\mu\epsilon}g^{\nu\delta}(F_{\mu\nu}\delta F_{\epsilon\delta}+\delta F_{\mu\nu}F_{\epsilon\delta})\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \tag{89.113}$$

We can write the seventh integral in (89.113) as

$$\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\gamma;\alpha}+g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\alpha;\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.114}$$

Changing indexes in (89.114) gives

$$\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\gamma;\alpha}+g^{\beta\alpha}g^{\gamma\delta}\delta F_{\delta\beta\gamma;\alpha}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.115}$$

We can rewrite (89.115) as

$$\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\gamma}+g^{\beta\alpha}g^{\gamma\delta}\delta F_{\delta\beta\gamma}\right)_{;\alpha}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.116}$$

Integrating (89.116) by parts gives zero.

We can rewrite (89.113) by changing some indexes to give

$$\delta I=\frac{1}{16\pi}\int\delta g^{\beta\gamma}g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha}+F_{\delta\beta\alpha;\gamma}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma}+F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\theta\phi}g^{\alpha\delta}\left(-F_{\delta\theta\phi;\alpha}+F_{\delta\theta\alpha;\phi}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\theta\phi}+F_{(\delta(\phi(\epsilon}g^{\mu\epsilon}F_{\mu)\theta)\alpha)}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\alpha\delta}\delta g^{\beta\gamma}\left(-F_{\gamma\alpha\delta;\beta}+F_{\gamma\alpha\beta;\delta}-F_{(\gamma\beta)(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha\delta}+F_{(\gamma(\delta(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha)\beta)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int g^{\mu\epsilon}g^{\alpha\delta}\left(-F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon}+F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{2\Lambda}{16\pi}+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+A_\mu J^\mu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{2}{16\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\delta g^{\beta\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int\Big(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}\delta F_{\mu\beta\gamma}F_{\delta\epsilon\alpha}$$
$$+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\delta F_{\mu\beta\gamma}F_{\delta\epsilon\alpha}\Big)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{6}\int\left(\delta(A_{\beta\gamma}Y^{\beta\gamma})\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(g^{\mu\nu}(J_\nu\delta A_\mu+A_\mu\delta J_\nu)\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{1}{8\pi}\int\left(g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\delta F_{\mu\nu}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \tag{89.117}$$

89.7.2 Ansatz December 2014

Taking the variation of (89.112) gives

$$
\begin{aligned}
\delta I = \frac{1}{16\pi}\int \delta g^{\beta\gamma} g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\beta\gamma} g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)\left(-\frac{1}{2}\right) g_{\theta\phi}\delta g^{\theta\phi}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\beta\gamma} \delta g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\beta\gamma} g^{\alpha\delta}\left(+F_{(\delta\alpha)(\epsilon}F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon}F_{\mu)\beta)\alpha)}\right)\delta g^{\mu\epsilon}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\int\left(-\frac{2\Lambda}{16\pi} + \frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + A_\mu J^\mu - \frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\left(-\frac{1}{2}\right) g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\int\left[-\frac{1}{16\pi}F_{\mu\nu}F_{\epsilon\delta}\left(g^{\mu\epsilon}\delta g^{\nu\delta} + \delta g^{\mu\epsilon}g^{\nu\delta}\right)\right](-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\beta\gamma} g^{\alpha\delta}\left(-\delta F_{\delta\beta\gamma;\alpha} + \delta F_{\delta\beta\alpha;\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\beta\gamma} g^{\alpha\delta}g^{\mu\epsilon}\left(+F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma} + \delta F_{\delta\epsilon\alpha}F_{\mu\beta\gamma} - F_{\delta\epsilon\gamma}\delta F_{\mu\beta\alpha} - \delta F_{\delta\epsilon\gamma}F_{\mu\beta\alpha}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\int\left(+\frac{1}{6}\delta(A_{\beta\gamma}Y^{\beta\gamma})\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\int\left(g^{\mu\nu}\delta(A_\mu J_\nu)\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\int\left(-\frac{1}{16\pi}g^{\mu\epsilon}g^{\nu\delta}(F_{\mu\nu}\delta F_{\epsilon\delta} + \delta F_{\mu\nu}F_{\epsilon\delta})\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x
\end{aligned}
\tag{89.118}
$$

We can write the seventh integral in (89.118) as

$$
\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\gamma;\alpha} + g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\alpha;\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.119}
$$

Changing indexes in (89.119) gives

$$
\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\gamma;\alpha} + g^{\beta\alpha}g^{\gamma\delta}\delta F_{\delta\beta\gamma;\alpha}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.120}
$$

We can rewrite (89.120) as

$$
\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}\delta F_{\delta\beta\gamma} + g^{\beta\alpha}g^{\gamma\delta}\delta F_{\delta\beta\gamma}\right)_{;\alpha}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.121}
$$

Integrating (89.121) by parts gives zero.

We can rewrite (89.118) by changing some indexes to give

$$
\begin{aligned}
\delta I = \frac{1}{16\pi}\int \delta g^{\beta\gamma} g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\theta\phi} g^{\alpha\delta}\left(-F_{\delta\theta\phi;\alpha} + F_{\delta\theta\alpha;\phi} + F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\theta\phi} - F_{(\delta(\phi(\epsilon}g^{\mu\epsilon}F_{\mu)\theta)\alpha)}\right)\left(-\frac{1}{2}\right) g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\alpha\delta} \delta g^{\beta\gamma}\left(-F_{\gamma\alpha\delta;\beta} + F_{\gamma\alpha\beta;\delta} + F_{(\gamma\beta)(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha\delta} - F_{(\gamma(\delta(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha)\beta)}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \\
+\frac{1}{16\pi}\int g^{\mu\epsilon} g^{\alpha\delta}\left(+F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon} - F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x
\end{aligned}
$$

$$+\int\left(-\frac{2\Lambda}{16\pi}+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+A_\mu J^\mu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(-\frac{2}{16\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\delta g^{\beta\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{16\pi}\int\Big(g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma}+g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}\delta F_{\mu\beta\gamma}F_{\delta\epsilon\alpha}$$
$$-g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma}-g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\delta F_{\mu\beta\gamma}F_{\delta\epsilon\alpha}\Big)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\frac{1}{6}\int\left(\delta(A_{\beta\gamma}Y^{\beta\gamma})\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$+\int\left(g^{\mu\nu}(J_\nu\delta A_\mu+A_\mu\delta J_\nu)\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{1}{8\pi}\int\left(g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\delta F_{\mu\nu}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x \qquad (89.122)$$

89.7.3 Electromagnetic terms

Maxwell's equations are given by [20, Misner, Thorne, and Wheeler, (22.17) p. 568].

To correctly calculate field equations from a variational procedure, we must transform the last integral in (89.117). To do that, we use [20, Misner, Thorne, and Wheeler, (22.19a) p. 569]

$$F_{\mu\nu}=A_{\nu;\mu}-A_{\mu;\nu}=A_{\nu,\mu}-A_{\mu,\nu}\ . \qquad (89.123)$$

Using (89.123) allows us to write the last integral in (89.117) as

$$-\frac{1}{8\pi}\int g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\delta\left(A_{\nu;\mu}-A_{\mu;\nu}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \qquad (89.124)$$

We can write (89.124) as

$$-\frac{1}{8\pi}\int g^{\mu\epsilon}g^{\nu\delta}\left(F_{\epsilon\delta}\delta A_{\nu;\mu}-F_{\epsilon\delta}\delta A_{\mu;\nu}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \qquad (89.125)$$

We can write (89.125) as

$$-\frac{1}{8\pi}\int g^{\mu\epsilon}g^{\nu\delta}\left[(F_{\epsilon\delta}\delta A_\nu)_{;\mu}-(F_{\epsilon\delta}\delta A_\mu)_{;\nu}\right](-g)^{\frac{1}{2}}\,\mathrm{d}^4x+\frac{1}{8\pi}\int g^{\mu\epsilon}g^{\nu\delta}\left(F_{\epsilon\delta;\mu}\delta A_\nu-F_{\epsilon\delta;\nu}\delta A_\mu\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \qquad (89.126)$$

The first integral in (89.126) is zero, so that we can rewrite (89.126) as

$$+\frac{1}{8\pi}\int g^{\mu\epsilon}g^{\nu\delta}\left(F_{\epsilon\delta;\mu}\delta A_\nu-F_{\epsilon\delta;\nu}\delta A_\mu\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \qquad (89.127)$$

We can rewrite (89.127) as

$$+\frac{1}{8\pi}\int\left(g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta;\mu}\delta A_\nu-g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta;\nu}\delta A_\mu\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \qquad (89.128)$$

Changing indexes in the first term gives

$$+\frac{1}{8\pi}\int\left(g^{\nu\delta}g^{\mu\epsilon}F_{\delta\epsilon;\nu}\delta A_\mu-g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta;\nu}\delta A_\mu\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \qquad (89.129)$$

We can combine terms and rewrite (89.129) as

$$-\frac{1}{4\pi}\int g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta;\nu}\delta A_\mu(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \qquad (89.130)$$

This has the opposite sign from [20, Misner, Thorne, and Wheeler, Box 21.1, p. 496], but I now get the correct sign for the inhomogeneous Maxwell equation.

89.7.4 Gravitational terms

89.7.5 Ansatz ca 2012

To correctly calculate field equations from a variational procedure, we must transform the seventh integral (lines 7 and 8) in (89.117). We can write the seventh integral in (89.117) as

$$\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.131}$$

We rewrite (89.131) as

$$\frac{1}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}F_{\delta\epsilon\alpha}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}F_{\delta\epsilon\alpha}\right)\delta F_{\mu\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.132}$$

Or,

$$\frac{1}{16\pi}\int\left(-g^{\beta\gamma}F^{\alpha\mu}{}_{\alpha}-g^{\gamma\mu}F^{\beta\alpha}{}_{\alpha}+F^{\gamma\mu\beta}+F^{\beta\gamma\mu}\right)\delta F_{\mu\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.133}$$

89.7.6 Ansatz December 2014

To correctly calculate field equations from a variational procedure, we must transform the seventh integral (lines 7 and 8) in (89.122). We can write the seventh integral in (89.117) as

$$\frac{1}{16\pi}\int\left(g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}+g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}-g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}-g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta F_{\mu\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.134}$$

We rewrite (89.134) as

$$\frac{1}{16\pi}\int\left(g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}+g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}F_{\delta\epsilon\alpha}-g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}F_{\delta\epsilon\alpha}-g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}F_{\delta\epsilon\alpha}\right)\delta F_{\mu\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.135}$$

Or,

$$\frac{1}{16\pi}\int\left(g^{\beta\gamma}F^{\alpha\mu}{}_{\alpha}+g^{\gamma\mu}F^{\beta\alpha}{}_{\alpha}-F^{\gamma\mu\beta}-F^{\beta\gamma\mu}\right)\delta F_{\mu\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.136}$$

89.7.7 Partially symmetric combination of potentials

Using the combination of terms

$$F_{\mu\beta\gamma}=\left(aA_{\mu\beta;\gamma}+bA_{\beta\gamma;\mu}+cA_{\gamma\mu;\beta}+bA_{\gamma\beta;\mu}+aA_{\mu\gamma;\beta}+cA_{\beta\mu;\gamma}\right)\ , \tag{89.137}$$

where $A_{\mu\beta}$ is a tensor potential, makes $F_{\mu\beta\gamma}$ satisfy the symmetry (89.34) or (89.61), which makes $F_{\mu\beta\gamma}$ symmetric on the last two indexes.

We need to make sure that $F_{\mu\beta\gamma}$ also satisfies the symmetry (89.439). Substituting (89.137) into (89.439) gives (after much tedious algebra)

$$\begin{aligned}&(b-c)A_{\alpha[\gamma;\phi}\left(A^{\phi\alpha}+A^{\alpha\phi}\right)_{;\beta]}+(a-c)A_{\alpha[\gamma;\phi}\left(A_{\beta]}{}^{\phi;\alpha}+A_{\beta]}{}^{\alpha;\phi}\right)+(b-a)A_{[\gamma\alpha;\phi}\left(A^{\phi\alpha}{}_{;\beta]}+A^{\alpha\phi}{}_{;\beta]}\right)\\ =\ &-4F_{\phi}{}^{\nu}{}_{\nu}\frac{(b-c)A^{\phi}{}_{[\beta;\gamma]}+(a-c)A_{[\beta\gamma]}{}^{;\phi}+(b-a)A_{[\beta}{}^{\phi}{}_{;\gamma]}}{a+b+c}\ ,\end{aligned} \tag{89.138}$$

where brackets [ ] denote anti symmetrization. We shall use (89.138) later.

For the 2012 Ansatz, we substitute (89.137) into (89.131) to give

$$\begin{aligned}
&\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta A_{\mu\beta;\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta A_{\beta\gamma;\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta A_{\gamma\mu;\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta A_{\gamma\beta;\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta A_{\mu\gamma;\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha}\delta A_{\beta\mu;\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ .
\end{aligned}\tag{89.139}$$

For the 2014 Ansatz, we substitute (89.137) into (89.134) to give the negative of the above. Interchanging δ and covariant derivative and integrating by parts in (89.139) gives

$$\begin{aligned}
&\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)\left(F_{\delta\epsilon\alpha}\delta A_{\mu\beta}\right)_{;\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)\left(F_{\delta\epsilon\alpha}\delta A_{\beta\gamma}\right)_{;\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)\left(F_{\delta\epsilon\alpha}\delta A_{\gamma\mu}\right)_{;\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)\left(F_{\delta\epsilon\alpha}\delta A_{\gamma\beta}\right)_{;\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)\left(F_{\delta\epsilon\alpha}\delta A_{\mu\gamma}\right)_{;\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)\left(F_{\delta\epsilon\alpha}\delta A_{\beta\mu}\right)_{;\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\gamma}\delta A_{\mu\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\beta}\delta A_{\gamma\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\gamma\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\beta}\delta A_{\mu\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\gamma}\delta A_{\beta\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ .
\end{aligned}\tag{89.140}$$

The first six integrals in (89.140) are zero. That gives

$$\begin{aligned}
&-\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\gamma}\delta A_{\mu\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\beta}\delta A_{\gamma\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\gamma\beta}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x
\end{aligned}$$

$$-\frac{a}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\beta}\delta A_{\mu\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{c}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\gamma}\delta A_{\beta\mu}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \quad (89.141)$$

Changing dummy indexes gives

$$-\frac{a}{16\pi}\int\left(-g^{\gamma\mu}g^{\alpha\delta}g^{\beta\epsilon}-g^{\epsilon\alpha}g^{\mu\beta}g^{\delta\gamma}+g^{\gamma\alpha}g^{\mu\delta}g^{\beta\epsilon}+g^{\epsilon\mu}g^{\alpha\beta}g^{\delta\gamma}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{b}{16\pi}\int\left(-g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}+g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}+g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{c}{16\pi}\int\left(-g^{\mu\beta}g^{\alpha\delta}g^{\gamma\epsilon}-g^{\epsilon\alpha}g^{\beta\gamma}g^{\delta\mu}+g^{\mu\alpha}g^{\beta\delta}g^{\gamma\epsilon}+g^{\epsilon\beta}g^{\alpha\gamma}g^{\delta\mu}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{b}{16\pi}\int\left(-g^{\gamma\beta}g^{\alpha\delta}g^{\mu\epsilon}-g^{\epsilon\alpha}g^{\beta\mu}g^{\delta\gamma}+g^{\gamma\alpha}g^{\beta\delta}g^{\mu\epsilon}+g^{\epsilon\beta}g^{\alpha\mu}g^{\delta\gamma}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{a}{16\pi}\int\left(-g^{\mu\gamma}g^{\alpha\delta}g^{\beta\epsilon}-g^{\epsilon\alpha}g^{\gamma\beta}g^{\delta\mu}+g^{\mu\alpha}g^{\gamma\delta}g^{\beta\epsilon}+g^{\epsilon\gamma}g^{\alpha\beta}g^{\delta\mu}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x$$
$$-\frac{c}{16\pi}\int\left(-g^{\beta\mu}g^{\alpha\delta}g^{\gamma\epsilon}-g^{\epsilon\alpha}g^{\mu\gamma}g^{\delta\beta}+g^{\beta\alpha}g^{\mu\delta}g^{\gamma\epsilon}+g^{\epsilon\mu}g^{\alpha\gamma}g^{\delta\beta}\right)F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \quad (89.142)$$

Combining and factoring gives

$$\begin{aligned}\frac{1}{16\pi}\int\quad&\Big[a\left(g^{\gamma\mu}g^{\alpha\delta}g^{\beta\epsilon}+g^{\epsilon\alpha}g^{\mu\beta}g^{\delta\gamma}-g^{\gamma\alpha}g^{\mu\delta}g^{\beta\epsilon}-g^{\epsilon\mu}g^{\alpha\beta}g^{\delta\gamma}\right)\\&+b\left(g^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}+g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}-g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}-g^{\epsilon\gamma}g^{\alpha\mu}g^{\delta\beta}\right)\\&+c\left(g^{\mu\beta}g^{\alpha\delta}g^{\gamma\epsilon}+g^{\epsilon\alpha}g^{\beta\gamma}g^{\delta\mu}-g^{\mu\alpha}g^{\beta\delta}g^{\gamma\epsilon}-g^{\epsilon\beta}g^{\alpha\gamma}g^{\delta\mu}\right)\\&+b\left(g^{\gamma\beta}g^{\alpha\delta}g^{\mu\epsilon}+g^{\epsilon\alpha}g^{\beta\mu}g^{\delta\gamma}-g^{\gamma\alpha}g^{\beta\delta}g^{\mu\epsilon}-g^{\epsilon\beta}g^{\alpha\mu}g^{\delta\gamma}\right)\\&+a\left(g^{\mu\gamma}g^{\alpha\delta}g^{\beta\epsilon}+g^{\epsilon\alpha}g^{\gamma\beta}g^{\delta\mu}-g^{\mu\alpha}g^{\gamma\delta}g^{\beta\epsilon}-g^{\epsilon\gamma}g^{\alpha\beta}g^{\delta\mu}\right)\\&+c\left(g^{\beta\mu}g^{\alpha\delta}g^{\gamma\epsilon}+g^{\epsilon\alpha}g^{\mu\gamma}g^{\delta\beta}-g^{\beta\alpha}g^{\mu\delta}g^{\gamma\epsilon}-g^{\epsilon\mu}g^{\alpha\gamma}g^{\delta\beta}\right)\Big]\quad F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\end{aligned} \quad (89.143)$$

Combining terms gives

$$\begin{aligned}\frac{1}{16\pi}\int\quad&\Big[2ag^{\gamma\mu}g^{\alpha\delta}g^{\beta\epsilon}+(a+b)g^{\epsilon\alpha}g^{\mu\beta}g^{\delta\gamma}-(a+c)g^{\gamma\alpha}g^{\mu\delta}g^{\beta\epsilon}\\&+2bg^{\beta\gamma}g^{\alpha\delta}g^{\mu\epsilon}+(b+c)g^{\epsilon\alpha}g^{\gamma\mu}g^{\delta\beta}-(a+b)g^{\beta\alpha}g^{\gamma\delta}g^{\mu\epsilon}\\&+2cg^{\mu\beta}g^{\alpha\delta}g^{\gamma\epsilon}+(a+c)g^{\epsilon\alpha}g^{\beta\gamma}g^{\delta\mu}-(b+c)g^{\mu\alpha}g^{\beta\delta}g^{\gamma\epsilon}\\&-(b+c)g^{\gamma\alpha}g^{\beta\delta}g^{\mu\epsilon}-(a+b)g^{\mu\alpha}g^{\gamma\delta}g^{\beta\epsilon}-(a+c)g^{\beta\alpha}g^{\mu\delta}g^{\gamma\epsilon}\Big]\quad F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\end{aligned} \quad (89.144)$$

Rearranging terms gives

$$\begin{aligned}\frac{1}{16\pi}\int\quad&\Big[2ag^{\alpha\delta}g^{\gamma\mu}g^{\beta\epsilon}+2bg^{\alpha\delta}g^{\beta\gamma}g^{\mu\epsilon}+2cg^{\alpha\delta}g^{\mu\beta}g^{\gamma\epsilon}\\&+(b+c)g^{\delta\beta}g^{\epsilon\alpha}g^{\gamma\mu}-(b+c)g^{\beta\delta}g^{\mu\alpha}g^{\gamma\epsilon}-(b+c)g^{\beta\delta}g^{\gamma\alpha}g^{\mu\epsilon}\\&+(a+c)g^{\delta\mu}g^{\epsilon\alpha}g^{\beta\gamma}-(a+c)g^{\mu\delta}g^{\gamma\alpha}g^{\beta\epsilon}-(a+c)g^{\mu\delta}g^{\beta\alpha}g^{\gamma\epsilon}\\&+(a+b)g^{\delta\gamma}g^{\epsilon\alpha}g^{\mu\beta}-(a+b)g^{\gamma\delta}g^{\beta\alpha}g^{\mu\epsilon}-(a+b)g^{\gamma\delta}g^{\mu\alpha}g^{\beta\epsilon}\ \Big]\quad F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x.\end{aligned} \quad (89.145)$$

Factoring terms gives

$$\frac{1}{16\pi}\int\qquad\Big[2g^{\alpha\delta}\left(ag^{\gamma\mu}g^{\beta\epsilon}+bg^{\beta\gamma}g^{\mu\epsilon}+cg^{\mu\beta}g^{\gamma\epsilon}\right)$$

$$+(b+c)g^{\delta\beta}\left(g^{\epsilon\alpha}g^{\gamma\mu}-g^{\mu\alpha}g^{\gamma\epsilon}-g^{\gamma\alpha}g^{\mu\epsilon}\right)$$
$$+(a+c)g^{\delta\mu}\left(g^{\epsilon\alpha}g^{\beta\gamma}-g^{\gamma\alpha}g^{\beta\epsilon}-g^{\beta\alpha}g^{\gamma\epsilon}\right)$$
$$+(a+b)g^{\delta\gamma}\left(g^{\epsilon\alpha}g^{\mu\beta}-g^{\beta\alpha}g^{\mu\epsilon}-g^{\mu\alpha}g^{\beta\epsilon}\right)\Big]\quad F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\ . \tag{89.146}$$

For the 2014 Ansatz, we get the negative of the above.

Substituting (89.130) for the last integral in (89.117) and (89.146) for the seventh integral in (89.117) gives

$$\begin{aligned}
\delta I &= \frac{1}{16\pi}\int \delta g^{\beta\gamma}g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha}+F_{\delta\beta\alpha;\gamma}-F_{\delta\epsilon\alpha}g^{\mu\epsilon}F_{\mu\beta\gamma}+F_{\delta\epsilon\gamma}g^{\mu\epsilon}F_{\mu\beta\alpha}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{16\pi}\int g^{\theta\phi}g^{\alpha\delta}\left(-F_{\delta\theta\phi;\alpha}+F_{\delta\theta\alpha;\phi}-F_{\delta\epsilon\alpha}g^{\mu\epsilon}F_{\mu\theta\phi}+F_{\delta\epsilon\phi}g^{\mu\epsilon}F_{\mu\theta\alpha}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{16\pi}\int g^{\alpha\delta}\delta g^{\beta\gamma}\left(-F_{\gamma\alpha\delta;\beta}+F_{\gamma\alpha\beta;\delta}-F_{\gamma\epsilon\beta}g^{\mu\epsilon}F_{\mu\alpha\delta}+F_{\gamma\epsilon\delta}g^{\mu\epsilon}F_{\mu\alpha\beta}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{16\pi}\int g^{\mu\epsilon}g^{\alpha\delta}\left(-F_{\delta\gamma\alpha}F_{\beta\mu\epsilon}+F_{\delta\gamma\epsilon}F_{\beta\mu\alpha}\right)\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\int\left(-\frac{2\Lambda}{16\pi}+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\int\left(-\frac{2}{16\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\delta g^{\beta\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{16\pi}\int\Big[2g^{\alpha\delta}\left(ag^{\gamma\mu}g^{\beta\epsilon}+bg^{\beta\gamma}g^{\mu\epsilon}+cg^{\mu\beta}g^{\gamma\epsilon}\right)\\
&+(b+c)g^{\delta\beta}\left(g^{\epsilon\alpha}g^{\gamma\mu}-g^{\mu\alpha}g^{\gamma\epsilon}-g^{\gamma\alpha}g^{\mu\epsilon}\right)\\
&+(a+c)g^{\delta\mu}\left(g^{\epsilon\alpha}g^{\beta\gamma}-g^{\gamma\alpha}g^{\beta\epsilon}-g^{\beta\alpha}g^{\gamma\epsilon}\right)\\
&+(a+b)g^{\delta\gamma}\left(g^{\epsilon\alpha}g^{\mu\beta}-g^{\beta\alpha}g^{\mu\epsilon}-g^{\mu\alpha}g^{\beta\epsilon}\right)\Big]F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{6}\int\left(Y^{\beta\gamma}\delta A_{\beta\gamma}+A_{\beta\gamma}\delta Y^{\beta\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\int\left(g^{\mu\nu}(J_\nu\delta A_\mu+A_\mu\delta J_\nu)\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{1}{4\pi}\int g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta;\nu}\delta A_\mu(-g)^{\frac{1}{2}}\,\mathrm{d}^4x
\end{aligned}\tag{89.147}$$

for the 2012 ansatz.

For the 2014 ansatz, we get

$$\begin{aligned}
\delta I &= \frac{1}{16\pi}\int \delta g^{\beta\gamma}g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha}+F_{\delta\beta\alpha;\gamma}+F_{\delta\epsilon\alpha}g^{\mu\epsilon}F_{\mu\beta\gamma}-F_{\delta\epsilon\gamma}g^{\mu\epsilon}F_{\mu\beta\alpha}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{16\pi}\int g^{\theta\phi}g^{\alpha\delta}\left(-F_{\delta\theta\phi;\alpha}+F_{\delta\theta\alpha;\phi}+F_{\delta\epsilon\alpha}g^{\mu\epsilon}F_{\mu\theta\phi}-F_{\delta\epsilon\phi}g^{\mu\epsilon}F_{\mu\theta\alpha}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{16\pi}\int g^{\alpha\delta}\delta g^{\beta\gamma}\left(-F_{\gamma\alpha\delta;\beta}+F_{\gamma\alpha\beta;\delta}+F_{\gamma\epsilon\beta}g^{\mu\epsilon}F_{\mu\alpha\delta}-F_{\gamma\epsilon\delta}g^{\mu\epsilon}F_{\mu\alpha\beta}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{1}{16\pi}\int g^{\mu\epsilon}g^{\alpha\delta}\left(-F_{\delta\gamma\alpha}F_{\beta\mu\epsilon}+F_{\delta\gamma\epsilon}F_{\beta\mu\alpha}\right)\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\int\left(-\frac{2\Lambda}{16\pi}+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\delta g^{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\int\left(-\frac{2}{16\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\delta g^{\beta\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{1}{16\pi}\int\Big[2g^{\alpha\delta}\left(ag^{\gamma\mu}g^{\beta\epsilon}+bg^{\beta\gamma}g^{\mu\epsilon}+cg^{\mu\beta}g^{\gamma\epsilon}\right)
\end{aligned}$$

$$
\begin{aligned}
&+(b+c)g^{\delta\beta}\left(g^{\epsilon\alpha}g^{\gamma\mu}-g^{\mu\alpha}g^{\gamma\epsilon}-g^{\gamma\alpha}g^{\mu\epsilon}\right)\\
&+(a+c)g^{\delta\mu}\left(g^{\epsilon\alpha}g^{\beta\gamma}-g^{\gamma\alpha}g^{\beta\epsilon}-g^{\beta\alpha}g^{\gamma\epsilon}\right)\\
&+(a+b)g^{\delta\gamma}\left(g^{\epsilon\alpha}g^{\mu\beta}-g^{\beta\alpha}g^{\mu\epsilon}-g^{\mu\alpha}g^{\beta\epsilon}\right)\Big]F_{\delta\epsilon\alpha;\mu}\delta A_{\beta\gamma}(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\frac{1}{6}\int\left(Y^{\beta\gamma}\delta A_{\beta\gamma}+A_{\beta\gamma}\delta Y^{\beta\gamma}\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&+\int\left(g^{\mu\nu}(J_\nu\delta A_\mu+A_\mu\delta J_\nu)\right)(-g)^{\frac{1}{2}}\,\mathrm{d}^4x\\
&-\frac{1}{4\pi}\int g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta;\nu}\delta A_\mu(-g)^{\frac{1}{2}}\,\mathrm{d}^4x
\end{aligned}
\tag{89.148}
$$

89.8 Field Equations From a Variational Procedure

89.8.1 Maxwell's equations

Setting the coefficient of δA_μ in (89.147) to zero gives

$$
-\frac{1}{4\pi}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta;\nu}+g^{\mu\nu}J_\nu=0\ . \tag{89.149}
$$

This gives

$$
F^{\mu\nu}{}_{;\nu}=4\pi J^\mu\ , \tag{89.150}
$$

which is the inhomogeneous Maxwell equation with the correct sign.

89.8.2 Gravitational field equations from a partially symmetric combination of potentials

89.8.3 Ansatz ca 2012

Setting the coefficient of $\delta A_{\beta\gamma}$ in (89.147) to zero gives

$$
\begin{aligned}
&\frac{1}{16\pi}\Big[2g^{\alpha\delta}\left(ag^{\gamma\mu}g^{\beta\epsilon}+bg^{\beta\gamma}g^{\mu\epsilon}+cg^{\mu\beta}g^{\gamma\epsilon}\right)\\
&+(b+c)g^{\delta\beta}\left(g^{\epsilon\alpha}g^{\gamma\mu}-g^{\mu\alpha}g^{\gamma\epsilon}-g^{\gamma\alpha}g^{\mu\epsilon}\right)\\
&+(a+c)g^{\delta\mu}\left(g^{\epsilon\alpha}g^{\beta\gamma}-g^{\gamma\alpha}g^{\beta\epsilon}-g^{\beta\alpha}g^{\gamma\epsilon}\right)\\
&+(a+b)g^{\delta\gamma}\left(g^{\epsilon\alpha}g^{\mu\beta}-g^{\beta\alpha}g^{\mu\epsilon}-g^{\mu\alpha}g^{\beta\epsilon}\right)\Big]F_{\delta\epsilon\alpha;\mu}+\frac{1}{6}Y^{\beta\gamma}=0\ .
\end{aligned}
\tag{89.151}
$$

Equation (89.151) can be rewritten

$$
\begin{aligned}
&\frac{1}{16\pi}\Big[2\left(ag^{\alpha\delta}g^{\gamma\mu}g^{\beta\epsilon}F_{\delta\epsilon\alpha;\mu}+bg^{\alpha\delta}g^{\beta\gamma}g^{\mu\epsilon}F_{\delta\epsilon\alpha;\mu}+cg^{\alpha\delta}g^{\mu\beta}g^{\gamma\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\\
&+(b+c)\left(g^{\delta\beta}g^{\epsilon\alpha}g^{\gamma\mu}F_{\delta\epsilon\alpha;\mu}-g^{\delta\beta}g^{\mu\alpha}g^{\gamma\epsilon}F_{\delta\epsilon\alpha;\mu}-g^{\delta\beta}g^{\gamma\alpha}g^{\mu\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\\
&+(a+c)\left(g^{\delta\mu}g^{\epsilon\alpha}g^{\beta\gamma}F_{\delta\epsilon\alpha;\mu}-g^{\delta\mu}g^{\gamma\alpha}g^{\beta\epsilon}F_{\delta\epsilon\alpha;\mu}-g^{\delta\mu}g^{\beta\alpha}g^{\gamma\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\\
&+(a+b)\left(g^{\delta\gamma}g^{\epsilon\alpha}g^{\mu\beta}F_{\delta\epsilon\alpha;\mu}-g^{\delta\gamma}g^{\beta\alpha}g^{\mu\epsilon}F_{\delta\epsilon\alpha;\mu}-g^{\delta\gamma}g^{\mu\alpha}g^{\beta\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\Big]+\frac{1}{6}Y^{\beta\gamma}=0\ .
\end{aligned}
\tag{89.152}
$$

Or,

$$
\frac{1}{16\pi}\Big[2\left(ag^{\gamma\mu}F^{\alpha\beta}{}_{\alpha;\mu}+bg^{\beta\gamma}F^{\alpha\mu}{}_{\alpha;\mu}+cg^{\mu\beta}F^{\alpha\gamma}{}_{\alpha;\mu}\right)
$$

$$
\begin{aligned}
&+(b+c)\left(g^{\gamma\mu}F^{\beta\alpha}{}_{\alpha;\mu}-F^{\beta\gamma\mu}{}_{;\mu}-F^{\beta\mu\gamma}{}_{;\mu}\right)\\
&+(a+c)\left(g^{\beta\gamma}F^{\mu\alpha}{}_{\alpha;\mu}-F^{\mu\beta\gamma}{}_{;\mu}-F^{\mu\gamma\beta}{}_{;\mu}\right)\\
&+(a+b)\left(g^{\mu\beta}F^{\gamma\alpha}{}_{\alpha;\mu}-F^{\gamma\mu\beta}{}_{;\mu}-F^{\gamma\beta\mu}{}_{;\mu}\right)\Big]+\frac{1}{6}Y^{\beta\gamma}=0\ . \qquad (89.153)
\end{aligned}
$$

89.8.4 Ansatz December 2014

Setting the coefficient of $\delta A_{\beta\gamma}$ in (89.148) to zero gives

$$
\begin{aligned}
&-\frac{1}{16\pi}\Big[2g^{\alpha\delta}\left(ag^{\gamma\mu}g^{\beta\epsilon}+bg^{\beta\gamma}g^{\mu\epsilon}+cg^{\mu\beta}g^{\gamma\epsilon}\right)\\
&+(b+c)g^{\delta\beta}\left(g^{\epsilon\alpha}g^{\gamma\mu}-g^{\mu\alpha}g^{\gamma\epsilon}-g^{\gamma\alpha}g^{\mu\epsilon}\right)\\
&+(a+c)g^{\delta\mu}\left(g^{\epsilon\alpha}g^{\beta\gamma}-g^{\gamma\alpha}g^{\beta\epsilon}-g^{\beta\alpha}g^{\gamma\epsilon}\right)\\
&+(a+b)g^{\delta\gamma}\left(g^{\epsilon\alpha}g^{\mu\beta}-g^{\beta\alpha}g^{\mu\epsilon}-g^{\mu\alpha}g^{\beta\epsilon}\right)\Big]F_{\delta\epsilon\alpha;\mu}+\frac{1}{6}Y^{\beta\gamma}=0\ . \qquad (89.154)
\end{aligned}
$$

Equation (89.154) can be rewritten

$$
\begin{aligned}
&-\frac{1}{16\pi}\Big[2\left(ag^{\alpha\delta}g^{\gamma\mu}g^{\beta\epsilon}F_{\delta\epsilon\alpha;\mu}+bg^{\alpha\delta}g^{\beta\gamma}g^{\mu\epsilon}F_{\delta\epsilon\alpha;\mu}+cg^{\alpha\delta}g^{\mu\beta}g^{\gamma\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\\
&+(b+c)\left(g^{\delta\beta}g^{\epsilon\alpha}g^{\gamma\mu}F_{\delta\epsilon\alpha;\mu}-g^{\delta\beta}g^{\mu\alpha}g^{\gamma\epsilon}F_{\delta\epsilon\alpha;\mu}-g^{\delta\beta}g^{\gamma\alpha}g^{\mu\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\\
&+(a+c)\left(g^{\delta\mu}g^{\epsilon\alpha}g^{\beta\gamma}F_{\delta\epsilon\alpha;\mu}-g^{\delta\mu}g^{\gamma\alpha}g^{\beta\epsilon}F_{\delta\epsilon\alpha;\mu}-g^{\delta\mu}g^{\beta\alpha}g^{\gamma\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\\
&+(a+b)\left(g^{\delta\gamma}g^{\epsilon\alpha}g^{\mu\beta}F_{\delta\epsilon\alpha;\mu}-g^{\delta\gamma}g^{\beta\alpha}g^{\mu\epsilon}F_{\delta\epsilon\alpha;\mu}-g^{\delta\gamma}g^{\mu\alpha}g^{\beta\epsilon}F_{\delta\epsilon\alpha;\mu}\right)\Big]+\frac{1}{6}Y^{\beta\gamma}=0\ . \qquad (89.155)
\end{aligned}
$$

Or,

$$
\begin{aligned}
&\frac{1}{16\pi}\Big[2\left(ag^{\gamma\mu}F^{\alpha\beta}{}_{\alpha;\mu}+bg^{\beta\gamma}F^{\alpha\mu}{}_{\alpha;\mu}+cg^{\mu\beta}F^{\alpha\gamma}{}_{\alpha;\mu}\right)\\
&+(b+c)\left(g^{\gamma\mu}F^{\beta\alpha}{}_{\alpha;\mu}-F^{\beta\gamma\mu}{}_{;\mu}-F^{\beta\mu\gamma}{}_{;\mu}\right)\\
&+(a+c)\left(g^{\beta\gamma}F^{\mu\alpha}{}_{\alpha;\mu}-F^{\mu\beta\gamma}{}_{;\mu}-F^{\mu\gamma\beta}{}_{;\mu}\right)\\
&+(a+b)\left(g^{\mu\beta}F^{\gamma\alpha}{}_{\alpha;\mu}-F^{\gamma\mu\beta}{}_{;\mu}-F^{\gamma\beta\mu}{}_{;\mu}\right)\Big]-\frac{1}{6}Y^{\beta\gamma}=0\ . \qquad (89.156)
\end{aligned}
$$

89.8.5 Antisymmetric part

Equations (89.153) and (89.156) are also valid with β and γ reversed. We can subtract to get the antisymmetric part of (89.153) and (89.156) and add to get the symmetric part. The antisymmetric parts are the same for both (89.153) and (89.156).

The antisymmetric part is (after changing some dummy indexes)

$$
\frac{a-c}{8\pi}\left[\left(F^{\alpha}{}_{\beta\alpha}-\frac{1}{2}F_{\beta}{}^{\alpha}{}_{\alpha}\right)_{;\gamma}-\left(F^{\alpha}{}_{\gamma\alpha}-\frac{1}{2}F_{\gamma}{}^{\alpha}{}_{\alpha}\right)_{;\beta}+\left(F_{\beta\gamma}{}^{\alpha}-F_{\gamma\beta}{}^{\alpha}\right)_{;\alpha}\right]=0\ . \qquad (89.157)
$$

To simplify writing, we use the shorthand notation of (89.412) through (89.415). This gives

$$
\frac{a-c}{8\pi}\left(b_{\beta\gamma}-\frac{1}{2}c_{\beta\gamma}-b_{\gamma\beta}+\frac{1}{2}c_{\gamma\beta}+d_{\beta\gamma}-d_{\gamma\beta}\right)=0\ . \qquad (89.158)
$$

Using (89.417) and (89.418) gives

$$\frac{a-c}{16\pi}\left(c_{\beta\gamma}-c_{\gamma\beta}\right)=0\ . \tag{89.159}$$

and

$$\frac{a-c}{16\pi}\left(d_{\beta\gamma}-d_{\gamma\beta}\right)=0\ . \tag{89.160}$$

Writing these back out gives

$$\frac{a-c}{16\pi}\left(F_{\beta}{}^{\alpha}{}_{\alpha;\gamma}-F_{\gamma}{}^{\alpha}{}_{\alpha;\beta}\right)=0\ . \tag{89.161}$$

and

$$\frac{a-c}{16\pi}\left(F_{\beta\gamma}{}^{\alpha}-F_{\gamma\beta}{}^{\alpha}\right)_{;\alpha}=0\ . \tag{89.162}$$

89.8.6 Ansatz ca 2012

The symmetric part of (89.153) is (after lowering indexes and changing some dummy indexes)

$$2d\left(F^{\alpha}{}_{\beta\alpha;\gamma}+F^{\alpha}{}_{\gamma\alpha;\beta}\right)+2dg_{\beta\gamma}F^{\mu\alpha}{}_{\alpha;\mu}+(b+d)\left(F_{\beta}{}^{\alpha}{}_{\alpha;\gamma}+F_{\gamma}{}^{\alpha}{}_{\alpha;\beta}-2F_{\beta\gamma}{}^{\alpha}{}_{;\alpha}-2F_{\gamma\beta}{}^{\alpha}{}_{;\alpha}\right)$$
$$+2bg_{\beta\gamma}F^{\alpha\mu}{}_{\alpha;\mu}-4dF^{\alpha}{}_{(\beta\gamma);\alpha}+\frac{8\pi}{3}Y_{\beta\gamma}=0\ , \tag{89.163}$$

where $d\equiv(a+c)/2$ and ( ) denote symmetrization. To simplify writing, we use the shorthand notation of (89.412) through (89.415). This gives

$$2d\left(b_{\beta\gamma}+b_{\gamma\beta}\right)+2dg_{\beta\gamma}a^{\alpha}{}_{\alpha}+(b+d)\left(c_{\beta\gamma}+c_{\gamma\beta}-2d_{\beta\gamma}-2d_{\gamma\beta}\right)+2bg_{\beta\gamma}b^{\mu}{}_{\mu}-4da_{(\beta\gamma)}+\frac{8\pi}{3}Y_{\beta\gamma}=0\ . \tag{89.164}$$

We can write (89.164) as

$$2d\left(-a_{\beta\gamma}+b_{\beta\gamma}+c_{\beta\gamma}-d_{\beta\gamma}\right)+2d\left(-a_{\gamma\beta}+b_{\gamma\beta}+c_{\gamma\beta}-d_{\gamma\beta}\right)$$
$$+(b-d)\left(c_{\beta\gamma}+c_{\gamma\beta}\right)-2b\left(d_{\beta\gamma}+d_{\gamma\beta}\right)+2dg_{\beta\gamma}a^{\alpha}{}_{\alpha}+2bg_{\beta\gamma}b^{\mu}{}_{\mu}+\frac{8\pi}{3}Y_{\beta\gamma}=0\ . \tag{89.165}$$

We can use (89.419) to give

$$2df^{\alpha}{}_{\beta\alpha\gamma}+2df^{\alpha}{}_{\gamma\alpha\beta}+(b-d)\left(c_{\beta\gamma}+c_{\gamma\beta}\right)-2b\left(d_{\beta\gamma}+d_{\gamma\beta}\right)+2dg_{\beta\gamma}a^{\alpha}{}_{\alpha}+2bg_{\beta\gamma}b^{\mu}{}_{\mu}+\frac{8\pi}{3}Y_{\beta\gamma}=0\ , \tag{89.166}$$

where $f^{\alpha}{}_{\beta\alpha\gamma}$ is defined in (89.44). Using $a^{\beta}{}_{\beta}=c^{\beta}{}_{\beta}$ and $b^{\gamma}{}_{\gamma}=d^{\gamma}{}_{\gamma}$ gives

$$4df^{\alpha}{}_{(\beta\alpha\gamma)}+2(b-d)c_{(\beta\gamma)}-4bd_{(\beta\gamma)}+2dg_{\beta\gamma}c^{\alpha}{}_{\alpha}+2bg_{\beta\gamma}d^{\mu}{}_{\mu}+\frac{8\pi}{3}Y_{\beta\gamma}=0\ , \tag{89.167}$$

where ( ) denotes symmetrization. Or,

$$2(d-b)c_{(\beta\gamma)}-2dg_{\beta\gamma}c^{\alpha}{}_{\alpha}+4bd_{(\beta\gamma)}-2bg_{\beta\gamma}d^{\mu}{}_{\mu}=4df^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{8\pi}{3}Y_{\beta\gamma}\ . \tag{89.168}$$

Or,

$$2(d-b)\left(c_{(\beta\gamma)}-\frac{d}{d-b}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right)+4b\left(d_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right)=4df^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{8\pi}{3}Y_{\beta\gamma}\ . \tag{89.169}$$

Dividing by $2(d-b)$ gives

$$\left(c_{(\beta\gamma)}-\frac{d}{d-b}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right)+\frac{2b}{d-b}\left(d_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right)=\frac{2d}{d-b}f^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{4\pi}{3(d-b)}Y_{\beta\gamma}\ . \tag{89.170}$$

Taking $d/(d-b)=1/2$ gives

$$\left(c_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right)-\left(d_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right)=f^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{2\pi}{3d}Y_{\beta\gamma}\ . \tag{89.171}$$

Taking $d=1/4$ gives

$$\left(c_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right)-\left(d_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right)=f^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{8\pi}{3}Y_{\beta\gamma}\ . \tag{89.172}$$

Writing out the shorthand gives

$$\left(F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}-\frac{1}{2}g_{\beta\gamma}F^{\alpha\mu}{}_{\mu;\alpha)}\right)-\left(F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}-\frac{1}{2}g_{\beta\gamma}F^{\mu\alpha}{}_{\mu;\alpha}\right)=f^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{8\pi}{3}Y_{\beta\gamma}\ . \tag{89.173}$$

Equation (89.173) can be considered to be one form of the inhomogeneous field equations for the gravitational field tensor.

Contracting (taking the trace of) (89.166) gives

$$2df^{\alpha\gamma}{}_{\alpha\gamma}+2df^{\alpha\beta}{}_{\alpha\beta}+(b-d)\left(c^{\gamma}{}_{\gamma}+c^{\beta}{}_{\beta}\right)-2b\left(d^{\gamma}{}_{\gamma}+d^{\beta}{}_{\beta}\right)+2d\delta^{\gamma}_{\gamma}a^{\alpha}{}_{\alpha}+2b\delta^{\gamma}_{\gamma}b^{\mu}{}_{\mu}+\frac{8\pi}{3}Y^{\gamma}_{\gamma}=0\ . \tag{89.174}$$

Using $c^{\beta}{}_{\beta}=a^{\beta}{}_{\beta}$ and $d^{\gamma}{}_{\gamma}=b^{\gamma}{}_{\gamma}$ gives $f^{\alpha\gamma}{}_{\alpha\gamma}=0$ and gives

$$2(b+3d)a^{\beta}{}_{\beta}+4bb^{\mu}{}_{\mu}+\frac{8\pi}{3}Y=0\ . \tag{89.175}$$

Substituting (89.175) into (89.166) gives

$$+2(d-b)c_{(\beta\gamma)}+4bd_{(\beta\gamma)}+(b+d)g_{\beta\gamma}a^{\alpha}{}_{\alpha}=4df^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ , \tag{89.176}$$

where ( ) denotes symmetrization. Writing out the shorthand gives

$$+2(d-b)F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}+4bF_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}+(b+d)g_{\beta\gamma}F^{\mu\alpha}{}_{\alpha;\mu}=4df^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ . \tag{89.177}$$

The subscripts don't seem lined up correctly. I have to fix that. Equation (89.177) is the inhomogeneous gravitational field equation, analogous to the inhomogeneous Maxwell equation (89.150). The term on the right-hand side of (89.177) proportional to $f^{\alpha}{}_{(\beta\alpha\gamma)}$, which is proportional to the products of some components of the gravitational field tensor, shows the explicit nonlinearity of gravitation. The constants b and d are still to be determined.

Using the above values for b and d gives

$$F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}-F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}=f^{\alpha}{}_{(\beta\alpha\gamma)}+\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ . \tag{89.178}$$

Using (89.44) gives

$$F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}-F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}=-F^{\alpha}{}_{\epsilon(\gamma}F^{\epsilon}{}_{\beta)\alpha}+F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma}-F_{(\beta\epsilon\gamma)}F^{\epsilon\alpha}{}_{\alpha}+F_{(\beta\epsilon\alpha}F^{\epsilon\alpha}{}_{\gamma)}+\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ . \tag{89.179}$$

Probably $Y_{\beta\gamma} = T_{\beta\gamma}$, so[6]

$$F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)} - F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha} = -F^{\alpha}{}_{\epsilon(\gamma}F^{\epsilon}{}_{\beta)\alpha} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma} - F_{(\beta\epsilon\gamma)}F^{\epsilon\alpha}{}_{\alpha} + F_{(\beta\epsilon\alpha}F^{\epsilon\alpha}{}_{\gamma)} + \frac{8\pi}{3}\left(T_{\beta\gamma} - \frac{1}{2}g_{\beta\gamma}T\right) . \tag{89.180}$$

Equation (89.179) is one form of the inhomogeneous field equation for the gravitational field tensor.

89.8.7 Ansatz December 2014

The symmetric part of (89.156) is (after lowering indexes and changing some dummy indexes)

$$2d\left(F^{\alpha}{}_{\beta\alpha;\gamma} + F^{\alpha}{}_{\gamma\alpha;\beta}\right) + 2dg_{\beta\gamma}F^{\mu\alpha}{}_{\alpha;\mu} + (b+d)\left(F_{\beta}{}^{\alpha}{}_{\alpha;\gamma} + F_{\gamma}{}^{\alpha}{}_{\alpha;\beta} - 2F_{\beta\gamma}{}^{\alpha}{}_{;\alpha} - 2F_{\gamma\beta}{}^{\alpha}{}_{;\alpha}\right)$$
$$+2bg_{\beta\gamma}F^{\alpha\mu}{}_{\alpha;\mu} - 4dF^{\alpha}{}_{(\beta\gamma);\alpha} - \frac{8\pi}{3}Y_{\beta\gamma} = 0 , \tag{89.181}$$

where $d \equiv (a+c)/2$ and ( ) denote symmetrization. To simplify writing, we use the shorthand notation of (89.412) through (89.415). This gives

$$2d\left(b_{\beta\gamma} + b_{\gamma\beta}\right) + 2dg_{\beta\gamma}a^{\alpha}{}_{\alpha} + (b+d)\left(c_{\beta\gamma} + c_{\gamma\beta} - 2d_{\beta\gamma} - 2d_{\gamma\beta}\right) + 2bg_{\beta\gamma}b^{\mu}{}_{\mu} - 4da_{(\beta\gamma)} - \frac{8\pi}{3}Y_{\beta\gamma} = 0 . \tag{89.182}$$

We can write (89.182) as

$$2d\left(-a_{\beta\gamma} + b_{\beta\gamma} + c_{\beta\gamma} - d_{\beta\gamma}\right) + 2d\left(-a_{\gamma\beta} + b_{\gamma\beta} + c_{\gamma\beta} - d_{\gamma\beta}\right)$$
$$+(b-d)\left(c_{\beta\gamma} + c_{\gamma\beta}\right) - 2b\left(d_{\beta\gamma} + d_{\gamma\beta}\right) + 2dg_{\beta\gamma}a^{\alpha}{}_{\alpha} + 2bg_{\beta\gamma}b^{\mu}{}_{\mu} - \frac{8\pi}{3}Y_{\beta\gamma} = 0 . \tag{89.183}$$

For the ca 2012 ansatz, we could use (89.419), but for the December 2014 ansatz, $f^{\alpha}{}_{\beta\alpha\gamma}$ has the opposite sign. Therefore, for the December 2014 ansatz, we use the opposite sign from (89.419) to give

$$-2df^{\alpha}{}_{\beta\alpha\gamma} - 2df^{\alpha}{}_{\gamma\alpha\beta} + (b-d)\left(c_{\beta\gamma} + c_{\gamma\beta}\right) - 2b\left(d_{\beta\gamma} + d_{\gamma\beta}\right) + 2dg_{\beta\gamma}a^{\alpha}{}_{\alpha} + 2bg_{\beta\gamma}b^{\mu}{}_{\mu} - \frac{8\pi}{3}Y_{\beta\gamma} = 0 , \tag{89.184}$$

where $f^{\alpha}{}_{\beta\alpha\gamma}$ is defined in (89.74). Using $a^{\beta}{}_{\beta} = c^{\beta}{}_{\beta}$ and $b^{\gamma}{}_{\gamma} = d^{\gamma}{}_{\gamma}$ gives

$$-4df^{\alpha}{}_{(\beta\alpha\gamma)} + 2(b-d)c_{(\beta\gamma)} - 4bd_{(\beta\gamma)} + 2dg_{\beta\gamma}c^{\alpha}{}_{\alpha} + 2bg_{\beta\gamma}d^{\mu}{}_{\mu} - \frac{8\pi}{3}Y_{\beta\gamma} = 0 , \tag{89.185}$$

where ( ) denotes symmetrization. Or,

$$2(d-b)c_{(\beta\gamma)} - 2dg_{\beta\gamma}c^{\alpha}{}_{\alpha} + 4bd_{(\beta\gamma)} - 2bg_{\beta\gamma}d^{\mu}{}_{\mu} = -4df^{\alpha}{}_{(\beta\alpha\gamma)} - \frac{8\pi}{3}Y_{\beta\gamma} . \tag{89.186}$$

Or,

$$2(d-b)\left(c_{(\beta\gamma)} - \frac{d}{d-b}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right) + 4b\left(d_{(\beta\gamma)} - \frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right) = -4df^{\alpha}{}_{(\beta\alpha\gamma)} - \frac{8\pi}{3}Y_{\beta\gamma} . \tag{89.187}$$

Dividing by $2(d-b)$ gives

$$\left(c_{(\beta\gamma)} - \frac{d}{d-b}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right) + \frac{2b}{d-b}\left(d_{(\beta\gamma)} - \frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right) = -\frac{2d}{d-b}f^{\alpha}{}_{(\beta\alpha\gamma)} - \frac{4\pi}{3(d-b)}Y_{\beta\gamma} . \tag{89.188}$$

[6] Actually, probably not, because $Y_{\beta\gamma}$ and $T_{\beta\gamma}$ have different units. Actually, it is probably OK, but I need to put in a factor of G when we put in real units.

Taking $d/(d-b)=1/2$ gives

$$\left(c_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right)-\left(d_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right)=-f^{\alpha}{}_{(\beta\alpha\gamma)}-\frac{2\pi}{3d}Y_{\beta\gamma}\ . \tag{89.189}$$

Taking $d=1/4$ gives

$$\left(c_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}c^{\alpha}{}_{\alpha}\right)-\left(d_{(\beta\gamma)}-\frac{1}{2}g_{\beta\gamma}d^{\mu}{}_{\mu}\right)=-f^{\alpha}{}_{(\beta\alpha\gamma)}-\frac{8\pi}{3}Y_{\beta\gamma}\ . \tag{89.190}$$

Writing out the shorthand gives

$$\left(F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}-\frac{1}{2}g_{\beta\gamma}F^{\alpha\mu}{}_{\mu;\alpha)}\right)-\left(F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}-\frac{1}{2}g_{\beta\gamma}F^{\mu\alpha}{}_{\mu;\alpha}\right)=-f^{\alpha}{}_{(\beta\alpha\gamma)}-\frac{8\pi}{3}Y_{\beta\gamma}\ . \tag{89.191}$$

Equation (89.173) can be considered to be one form of the inhomogeneous field equations for the gravitational field tensor.

Contracting (taking the trace of) (89.184) gives

$$-2df^{\alpha\gamma}{}_{\alpha\gamma}-2df^{\alpha\beta}{}_{\alpha\beta}+(b-d)\left(c^{\gamma}{}_{\gamma}+c^{\beta}{}_{\beta}\right)-2b\left(d^{\gamma}{}_{\gamma}+d^{\beta}{}_{\beta}\right)+2d\delta^{\gamma}_{\gamma}a^{\alpha}{}_{\alpha}+2b\delta^{\gamma}_{\gamma}b^{\mu}{}_{\mu}-\frac{8\pi}{3}Y^{\gamma}_{\gamma}=0\ . \tag{89.192}$$

Using $c^{\beta}{}_{\beta}=a^{\beta}{}_{\beta}$ and $d^{\gamma}{}_{\gamma}=b^{\gamma}{}_{\gamma}$ gives $f^{\alpha\gamma}{}_{\alpha\gamma}=0$ and gives

$$2(b+3d)a^{\beta}{}_{\beta}+4bb^{\mu}{}_{\mu}-\frac{8\pi}{3}Y=0\ . \tag{89.193}$$

Substituting (89.193) into (89.184) gives

$$+2(d-b)c_{(\beta\gamma)}+4bd_{(\beta\gamma)}+(b+d)g_{\beta\gamma}a^{\alpha}{}_{\alpha}=-4df^{\alpha}{}_{(\beta\alpha\gamma)}-\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ , \tag{89.194}$$

where ( ) denotes symmetrization. Writing out the shorthand gives

$$+2(d-b)F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}+4bF_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}+(b+d)g_{\beta\gamma}F^{\mu\alpha}{}_{\alpha;\mu}=-4df^{\alpha}{}_{(\beta\alpha\gamma)}-\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ . \tag{89.195}$$

The subscripts don't seem lined up correctly. I have to fix that. Equation (89.195) is the inhomogeneous gravitational field equation, analogous to the inhomogeneous Maxwell equation (89.150). The term on the right-hand side of (89.195) proportional to $f^{\alpha}{}_{(\beta\alpha\gamma)}$, which is proportional to the products of some components of the gravitational field tensor, shows the explicit nonlinearity of gravitation. The constants b and d are still to be determined.

Using the above values for b and d gives

$$F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}-F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}=-f^{\alpha}{}_{(\beta\alpha\gamma)}-\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ . \tag{89.196}$$

Using (89.74) gives

$$F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}-F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}=-F^{\alpha}{}_{\epsilon(\gamma}F^{\epsilon}{}_{\beta)\alpha}+F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma}-F_{(\beta\epsilon\gamma)}F^{\epsilon\alpha}{}_{\alpha}+F_{(\beta\epsilon\alpha}F^{\epsilon\alpha}{}_{\gamma)}-\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)\ . \tag{89.197}$$

Probably $Y_{\beta\gamma}=T_{\beta\gamma}$, so[7]

$$F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)}-F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha}=-F^{\alpha}{}_{\epsilon(\gamma}F^{\epsilon}{}_{\beta)\alpha}+F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma}-F_{(\beta\epsilon\gamma)}F^{\epsilon\alpha}{}_{\alpha}+F_{(\beta\epsilon\alpha}F^{\epsilon\alpha}{}_{\gamma)}-\frac{8\pi}{3}\left(T_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}T\right)\ . \tag{89.198}$$

Equation (89.197) is one form of the inhomogeneous field equation for the gravitational field tensor.

[7] Actually, probably not, because $Y_{\beta\gamma}$ and $T_{\beta\gamma}$ have different units. Actually, it is probably OK, but I need to put in a factor of G when we put in real units.

89.8.8 Einstein's field equations from variation of $g^{\beta\gamma}$

89.8.9 Ansatz ca 2012

We can set the coefficient of $\delta g^{\beta\gamma}$ in (89.147) to zero to give

$$\begin{aligned}
&g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha}+F_{\delta\beta\alpha;\gamma}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma}+F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)\\
&+g^{\theta\phi}g^{\alpha\delta}\left(-F_{\delta\theta\phi;\alpha}+F_{\delta\theta\alpha;\phi}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\theta\phi}+F_{(\delta(\phi(\epsilon}g^{\mu\epsilon}F_{\mu)\theta)\alpha)}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\\
&+g^{\alpha\delta}\left(-F_{\gamma\alpha\delta;\beta}+F_{\gamma\alpha\beta;\delta}-F_{(\gamma\beta)(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha\delta}+F_{(\gamma(\delta(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha)\beta)}\right)\\
&+g^{\mu\epsilon}g^{\alpha\delta}\left(-F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon}+F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\\
&+16\pi\left(-\frac{2\Lambda}{16\pi}+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\left(-\frac{1}{2}\right)g_{\beta\gamma}\\
&+16\pi\left(-\frac{2}{16\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\right)=0\ . \qquad (89.199)
\end{aligned}$$

Equation (89.199) can be written as

$$\begin{aligned}
&g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha}+F_{\delta\beta\alpha;\gamma}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma}+F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)\\
&-\frac{1}{2}g_{\beta\gamma}g^{\theta\phi}g^{\alpha\delta}\left(-F_{\delta\theta\phi;\alpha}+F_{\delta\theta\alpha;\phi}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\theta\phi}+F_{(\delta(\phi(\epsilon}g^{\mu\epsilon}F_{\mu)\theta)\alpha)}\right)+g_{\beta\gamma}\Lambda\\
&+g^{\alpha\delta}\left(-F_{\gamma\alpha\delta;\beta}+F_{\gamma\alpha\beta;\delta}-F_{(\gamma\beta)(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha\delta}+F_{(\gamma(\delta(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha)\beta)}\right)\\
&+g^{\mu\epsilon}g^{\alpha\delta}\left(-F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon}+F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\\
&-\frac{1}{2}g_{\beta\gamma}16\pi\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\\
&+16\pi\left(-\frac{2}{16\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\right)=0\ . \qquad (89.200)
\end{aligned}$$

Equation (89.200) can be written as

$$\begin{aligned}
&g^{\alpha\delta}\left(-F_{\delta\beta\gamma;\alpha}+F_{\delta\beta\alpha;\gamma}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\beta\gamma}+F_{(\delta(\gamma(\epsilon}g^{\mu\epsilon}F_{\mu)\beta)\alpha)}\right)\\
&-\frac{1}{2}g_{\beta\gamma}g^{\theta\phi}g^{\alpha\delta}\left(-F_{\delta\theta\phi;\alpha}+F_{\delta\theta\alpha;\phi}-F_{(\delta\alpha)(\epsilon}g^{\mu\epsilon}F_{\mu)\theta\phi}+F_{(\delta(\phi(\epsilon}g^{\mu\epsilon}F_{\mu)\theta)\alpha)}\right)+g_{\beta\gamma}\Lambda\\
&+g^{\alpha\delta}\left(-F_{\gamma\alpha\delta;\beta}+F_{\gamma\alpha\beta;\delta}-F_{(\gamma\beta)(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha\delta}+F_{(\gamma(\delta(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha)\beta)}\right)\\
&+g^{\mu\epsilon}g^{\alpha\delta}\left(-F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon}+F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\\
&-g_{\beta\gamma}8\pi\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\\
&+8\pi\left(-\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\right)=0\ . \qquad (89.201)
\end{aligned}$$

Comparison with (89.47) and (89.50) shows that (89.201) can be rewritten

$$R_{\beta\gamma}-1/2g_{\beta\gamma}R+\Lambda g_{\beta\gamma}=8\pi T_{\beta\gamma}\ , \qquad (89.202)$$

where

$$T_{\beta\gamma}=-\frac{1}{8\pi}g^{\alpha\delta}\left(-F_{\gamma\alpha\delta;\beta}+F_{\gamma\alpha\beta;\delta}-F_{(\gamma\beta)(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha\delta}+F_{(\gamma(\delta(\epsilon}g^{\mu\epsilon}F_{\mu)\alpha)\beta)}\right)$$

$$-\frac{1}{8\pi} g^{\mu\epsilon} g^{\alpha\delta} \left(-F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} + F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \right)$$
$$+ g_{\beta\gamma} \left(+\frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \right)$$
$$+ \frac{1}{4\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \ . \tag{89.203}$$

89.8.10 Ansatz December 2014

We can set the coefficient of $\delta g^{\beta\gamma}$ in (89.148) to zero to give

$$g^{\alpha\delta} \left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon} g^{\mu\epsilon} F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon} g^{\mu\epsilon} F_{\mu)\beta)\alpha)} \right)$$
$$+ g^{\theta\phi} g^{\alpha\delta} \left(-F_{\delta\theta\phi;\alpha} + F_{\delta\theta\alpha;\phi} + F_{(\delta\alpha)(\epsilon} g^{\mu\epsilon} F_{\mu)\theta\phi} - F_{(\delta(\phi(\epsilon} g^{\mu\epsilon} F_{\mu)\theta)\alpha)} \right) \left(-\frac{1}{2} \right) g_{\beta\gamma}$$
$$+ g^{\alpha\delta} \left(-F_{\gamma\alpha\delta;\beta} + F_{\gamma\alpha\beta;\delta} + F_{(\gamma\beta)(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha\delta} - F_{(\gamma(\delta(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha)\beta)} \right)$$
$$+ g^{\mu\epsilon} g^{\alpha\delta} \left(+F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} - F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \right)$$
$$+16\pi \left(-\frac{2\Lambda}{16\pi} + \frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \right) \left(-\frac{1}{2} \right) g_{\beta\gamma}$$
$$+16\pi \left(-\frac{2}{16\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \right) = 0 \ . \tag{89.204}$$

Equation (89.204) can be written as

$$g^{\alpha\delta} \left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon} g^{\mu\epsilon} F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon} g^{\mu\epsilon} F_{\mu)\beta)\alpha)} \right)$$
$$-\frac{1}{2} g_{\beta\gamma} g^{\theta\phi} g^{\alpha\delta} \left(-F_{\delta\theta\phi;\alpha} + F_{\delta\theta\alpha;\phi} + F_{(\delta\alpha)(\epsilon} g^{\mu\epsilon} F_{\mu)\theta\phi} - F_{(\delta(\phi(\epsilon} g^{\mu\epsilon} F_{\mu)\theta)\alpha)} \right) + g_{\beta\gamma} \Lambda$$
$$+ g^{\alpha\delta} \left(-F_{\gamma\alpha\delta;\beta} + F_{\gamma\alpha\beta;\delta} + F_{(\gamma\beta)(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha\delta} - F_{(\gamma(\delta(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha)\beta)} \right)$$
$$+ g^{\mu\epsilon} g^{\alpha\delta} \left(+F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} - F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \right)$$
$$-\frac{1}{2} g_{\beta\gamma} 16\pi \left(+\frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \right)$$
$$+16\pi \left(-\frac{2}{16\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \right) = 0 \ . \tag{89.205}$$

Equation (89.205) can be written as

$$g^{\alpha\delta} \left(-F_{\delta\beta\gamma;\alpha} + F_{\delta\beta\alpha;\gamma} + F_{(\delta\alpha)(\epsilon} g^{\mu\epsilon} F_{\mu)\beta\gamma} - F_{(\delta(\gamma(\epsilon} g^{\mu\epsilon} F_{\mu)\beta)\alpha)} \right)$$
$$-\frac{1}{2} g_{\beta\gamma} g^{\theta\phi} g^{\alpha\delta} \left(-F_{\delta\theta\phi;\alpha} + F_{\delta\theta\alpha;\phi} + F_{(\delta\alpha)(\epsilon} g^{\mu\epsilon} F_{\mu)\theta\phi} - F_{(\delta(\phi(\epsilon} g^{\mu\epsilon} F_{\mu)\theta)\alpha)} \right) + g_{\beta\gamma} \Lambda$$
$$+ g^{\alpha\delta} \left(-F_{\gamma\alpha\delta;\beta} + F_{\gamma\alpha\beta;\delta} + F_{(\gamma\beta)(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha\delta} - F_{(\gamma(\delta(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha)\beta)} \right)$$
$$+ g^{\mu\epsilon} g^{\alpha\delta} \left(+F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} - F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \right)$$
$$- g_{\beta\gamma} 8\pi \left(+\frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \right)$$
$$+8\pi \left(-\frac{1}{4\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \right) = 0 \ . \tag{89.206}$$

Comparison with (89.76) and (89.79) shows that (89.206) can be rewritten

$$R_{\beta\gamma} - 1/2 g_{\beta\gamma} R + \Lambda g_{\beta\gamma} = 8\pi T_{\beta\gamma} \ , \tag{89.207}$$

where

$$\begin{aligned} T_{\beta\gamma} = -\frac{1}{8\pi} g^{\alpha\delta} \Big(-F_{\gamma\alpha\delta;\beta} + F_{\gamma\alpha\beta;\delta} + F_{(\gamma\beta)(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha\delta} - F_{(\gamma(\delta(\epsilon} g^{\mu\epsilon} F_{\mu)\alpha)\beta)} \Big) \\ -\frac{1}{8\pi} g^{\mu\epsilon} g^{\alpha\delta} \Big(+F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} - F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \Big) \\ +g_{\beta\gamma} \Big(+\frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \Big) \\ +\frac{1}{4\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \ . \end{aligned} \tag{89.208}$$

89.9 Stress-energy tensor

89.9.1 Ansatz ca 2012

We rewrite (89.203) as

$$\begin{aligned} T_{\beta\gamma} = -\frac{1}{8\pi} g^{\alpha\delta} R_{\gamma\alpha\beta\delta} \\ -\frac{1}{8\pi} g^{\mu\epsilon} g^{\alpha\delta} \Big(-F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} + F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \Big) \\ +g_{\beta\gamma} \Big(+\frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \Big) \\ +\frac{1}{4\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \ . \end{aligned} \tag{89.209}$$

We can use the symmetry condition (89.22) and interchanging α and δ in the first term to write (89.209) as

$$\begin{aligned} T_{\beta\gamma} = -\frac{1}{8\pi} g^{\alpha\delta} R_{\alpha\beta\delta\gamma} \\ -\frac{1}{8\pi} g^{\mu\epsilon} g^{\alpha\delta} \Big(-F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} + F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \Big) \\ +g_{\beta\gamma} \Big(+\frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \Big) \\ +\frac{1}{4\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \ . \end{aligned} \tag{89.210}$$

Substituting (89.33) into (89.210) gives

$$\begin{aligned} T_{\beta\gamma} = -\frac{1}{8\pi} g^{\alpha\delta} \left(-F_{\alpha\beta\gamma;\delta} + F_{\alpha\beta\delta;\gamma} - F_{\alpha\epsilon\delta} F^{\epsilon}{}_{\beta\gamma} + F_{\alpha\epsilon\gamma} F^{\epsilon}{}_{\beta\delta} \right) \\ -\frac{1}{8\pi} g^{\mu\epsilon} g^{\alpha\delta} \Big(-F_{(\delta\alpha)(\gamma} F_{\beta)\mu\epsilon} + F_{(\delta(\epsilon(\gamma} F_{\beta)\mu)\alpha)} \Big) \\ +g_{\beta\gamma} \Big(+\frac{1}{6} A_{\mu\nu} Y^{\mu\nu} + g^{\mu\nu} A_\mu J_\nu - \frac{1}{16\pi} F_{\mu\nu} g^{\mu\epsilon} g^{\nu\delta} F_{\epsilon\delta} \Big) \\ +\frac{1}{4\pi} F_{\mu\beta} F_{\epsilon\gamma} g^{\mu\epsilon} \ . \end{aligned} \tag{89.211}$$

We can rewrite (89.211) as

$$\begin{aligned} T_{\beta\gamma} = -\frac{1}{8\pi} \Big(-g^{\alpha\delta} F_{\alpha\beta\gamma;\delta} + g^{\alpha\delta} F_{\alpha\beta\delta;\gamma} - F^{\delta}{}_{\epsilon\delta} F^{\epsilon}{}_{\beta\gamma} + F^{\delta}{}_{\epsilon\gamma} F^{\epsilon}{}_{\beta\delta} \\ -F^{\alpha}{}_{\alpha(\gamma} F_{\beta)\mu}{}^{\mu} + F^{\alpha\mu}{}_{(\gamma} F_{\beta)\mu\alpha} \Big) \end{aligned}$$

$$+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)$$
$$+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ . \tag{89.212}$$

We can write (89.212 as

$$T_{\beta\gamma}=-\frac{1}{8\pi}\left(-a_{\beta\gamma}+b_{\beta\gamma}-F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}+F^{\delta}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}\right.$$
$$\left.-F^{\alpha}{}_{\alpha(\gamma}F_{\beta)\mu}{}^{\mu}+F^{\alpha\mu}{}_{(\gamma}F_{\beta)\mu\alpha}\right)$$
$$+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)$$
$$+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ . \tag{89.213}$$

We can rewrite (89.213) as

$$T_{\beta\gamma}=\frac{1}{8\pi}\left(a_{\beta\gamma}-b_{\beta\gamma}+F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F^{\delta}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}+F^{\alpha}{}_{\alpha(\gamma}F_{\beta)\mu}{}^{\mu}-F^{\alpha\mu}{}_{(\gamma}F_{\beta)\mu\alpha}\right)$$
$$+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)$$
$$+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ . \tag{89.214}$$

To get a force equation, we need to raise indexes in (89.214). This gives

$$T^{\beta\gamma}=\frac{1}{8\pi}\left(a^{\beta\gamma}-b^{\beta\gamma}+F^{\delta}{}_{\epsilon\delta}F^{\epsilon\beta\gamma}-F^{\delta\gamma}{}_{\epsilon}F^{\epsilon\beta}{}_{\delta}+F^{\alpha}{}_{\alpha}{}^{(\gamma}F^{\beta)}{}_{\mu}{}^{\mu}-F^{\alpha\mu(\gamma}F^{\beta)}{}_{\mu\alpha}\right)$$
$$+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)$$
$$+\frac{1}{4\pi}F_{\mu}{}^{\beta}F_{\epsilon}{}^{\gamma}g^{\mu\epsilon}\ . \tag{89.215}$$

Or, changing dummy indexes gives

$$T^{\beta\gamma}=-\frac{1}{8\pi}\left(-a^{\beta\gamma}+b^{\beta\gamma}-F^{\delta}{}_{\epsilon\delta}F^{\epsilon\beta\gamma}+F^{\alpha\mu(\gamma}F^{\beta)}{}_{\mu\alpha}+F^{\alpha\gamma\mu}F_{\mu}{}^{\beta}{}_{\alpha}-F^{\alpha}{}_{\alpha}{}^{(\gamma}F^{\beta)}{}_{\mu}{}^{\mu}\right)$$
$$+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)$$
$$+\frac{1}{4\pi}F_{\mu}{}^{\beta}F_{\epsilon}{}^{\gamma}g^{\mu\epsilon}\ . \tag{89.216}$$

Equation (89.216) can be written as

$$T^{\beta\gamma}=-\frac{1}{8\pi}g^{\beta\mu}g^{\nu\gamma}\left[-a_{\mu\nu}+b_{\mu\nu}+g^{\delta\theta}g^{\epsilon\phi}\left(F_{\phi\delta\nu}F_{\mu\epsilon\theta}+F_{\delta\phi\nu}F_{\epsilon\mu\theta}-F_{\theta\delta\phi}F_{\epsilon\mu\nu}-F_{\phi\nu\epsilon}F_{\mu\delta\theta}\right)\right]$$
$$+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)$$
$$+\frac{1}{4\pi}F_{\mu}{}^{\beta}F_{\epsilon}{}^{\gamma}g^{\mu\epsilon}\ . \tag{89.217}$$

89.9.2 Ansatz December 2014

We rewrite (89.208) as

$$T_{\beta\gamma}=-\frac{1}{8\pi}g^{\alpha\delta}\left(R_{\gamma\alpha\beta\delta}-\tilde{R}_{\gamma\alpha\beta\delta}\right)$$

$$
\begin{aligned}
&-\frac{1}{8\pi}g^{\mu\epsilon}g^{\alpha\delta}\left(+F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon}-F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\\
&+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\\
&+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ .
\end{aligned}
\tag{89.218}
$$

We can use the symmetry condition (89.22) and interchanging α and δ in the first term to write (89.218) as

$$
\begin{aligned}
T_{\beta\gamma}=&-\frac{1}{8\pi}g^{\alpha\delta}\left(R_{\alpha\beta\delta\gamma}-\tilde{R}_{\alpha\beta\delta\gamma}\right)\\
&-\frac{1}{8\pi}g^{\mu\epsilon}g^{\alpha\delta}\left(+F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon}-F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\\
&+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\\
&+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ .
\end{aligned}
\tag{89.219}
$$

Substituting (89.66) into (89.219) gives

$$
\begin{aligned}
T_{\beta\gamma}=&-\frac{1}{8\pi}g^{\alpha\delta}\left(-F_{\alpha\beta\gamma;\delta}+F_{\alpha\beta\delta;\gamma}+F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}\right)\\
&-\frac{1}{8\pi}g^{\mu\epsilon}g^{\alpha\delta}\left(+F_{(\delta\alpha)(\gamma}F_{\beta)\mu\epsilon}-F_{(\delta(\epsilon(\gamma}F_{\beta)\mu)\alpha)}\right)\\
&+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\\
&+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ .
\end{aligned}
\tag{89.220}
$$

We can rewrite (89.220) as

$$
\begin{aligned}
T_{\beta\gamma}=&-\frac{1}{8\pi}\left(-g^{\alpha\delta}F_{\alpha\beta\gamma;\delta}+g^{\alpha\delta}F_{\alpha\beta\delta;\gamma}+F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F^{\delta}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}\right.\\
&\left.+F^{\alpha}{}_{\alpha(\gamma}F_{\beta)\mu}{}^{\mu}-F^{\alpha\mu}{}_{(\gamma}F_{\beta)\mu\alpha}\right)\\
&+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\\
&+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ .
\end{aligned}
\tag{89.221}
$$

We can write (89.221 as

$$
\begin{aligned}
T_{\beta\gamma}=&-\frac{1}{8\pi}\left(-a_{\beta\gamma}+b_{\beta\gamma}+F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F^{\delta}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}\right.\\
&\left.+F^{\alpha}{}_{\alpha(\gamma}F_{\beta)\mu}{}^{\mu}-F^{\alpha\mu}{}_{(\gamma}F_{\beta)\mu\alpha}\right)\\
&+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right)\\
&+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ .
\end{aligned}
\tag{89.222}
$$

We can rewrite (89.222) as

$$
T_{\beta\gamma}=\frac{1}{8\pi}\left(a_{\beta\gamma}-b_{\beta\gamma}-F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}+F^{\delta}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}-F^{\alpha}{}_{\alpha(\gamma}F_{\beta)\mu}{}^{\mu}+F^{\alpha\mu}{}_{(\gamma}F_{\beta)\mu\alpha}\right)
$$

$$+g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right) + \frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\ . \tag{89.223}$$

To get a force equation, we need to raise indexes in (89.223). This gives

$$T^{\beta\gamma}=\frac{1}{8\pi}\left(a^{\beta\gamma}-b^{\beta\gamma}-F^{\delta}{}_{\epsilon\delta}F^{\epsilon\beta\gamma}+F^{\delta\gamma}{}_{\epsilon}F^{\epsilon\beta}{}_{\delta}-F^{\alpha}{}_{\alpha}{}^{(\gamma}F^{\beta)}{}_{\mu}{}^{\mu}+F^{\alpha\mu(\gamma}F^{\beta)}{}_{\mu\alpha}\right) +g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right) +\frac{1}{4\pi}F_{\mu}{}^{\beta}F_{\epsilon}{}^{\gamma}g^{\mu\epsilon}\ . \tag{89.224}$$

Or, changing dummy indexes gives

$$T^{\beta\gamma}=-\frac{1}{8\pi}\left(-a^{\beta\gamma}+b^{\beta\gamma}+F^{\delta}{}_{\epsilon\delta}F^{\epsilon\beta\gamma}-F^{\alpha\mu(\gamma}F^{\beta)}{}_{\mu\alpha}-F^{\alpha\gamma\mu}F_{\mu}{}^{\beta}{}_{\alpha}+F^{\alpha}{}_{\alpha}{}^{(\gamma}F^{\beta)}{}_{\mu}{}^{\mu}\right) +g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right) +\frac{1}{4\pi}F_{\mu}{}^{\beta}F_{\epsilon}{}^{\gamma}g^{\mu\epsilon}\ . \tag{89.225}$$

Equation (89.225) can be written as

$$T^{\beta\gamma}=-\frac{1}{8\pi}g^{\beta\mu}g^{\nu\gamma}\left[-a_{\mu\nu}+b_{\mu\nu}+g^{\delta\theta}g^{\epsilon\phi}\left(-F_{\phi\delta\nu}F_{\mu\epsilon\theta}-F_{\delta\phi\nu}F_{\epsilon\mu\theta}+F_{\theta\delta\phi}F_{\epsilon\mu\nu}+F_{\phi\nu\epsilon}F_{\mu\delta\theta}\right)\right] +g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu-\frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right) +\frac{1}{4\pi}F_{\mu}{}^{\beta}F_{\epsilon}{}^{\gamma}g^{\mu\epsilon}\ . \tag{89.226}$$

89.10 Force equation

89.10.1 Ansatz ca 2012

Taking the covariant derivative of (89.217) gives

$$\begin{aligned} T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}g^{\beta\mu}g^{\nu\gamma}\left[-a_{\mu\nu;\gamma}+b_{\mu\nu;\gamma}+g^{\delta\theta}g^{\epsilon\phi}\left(F_{\mu\epsilon\theta}F_{\phi\delta\nu;\gamma}+F_{\phi\delta\nu}F_{\mu\epsilon\theta;\gamma}\right.\right. \\ & \left.\left.+F_{\epsilon\mu\theta}F_{\delta\phi\nu;\gamma}+F_{\delta\phi\nu}F_{\epsilon\mu\theta;\gamma}-F_{\epsilon\mu\nu}F_{\theta\delta\phi;\gamma}-F_{\theta\delta\phi}F_{\epsilon\mu\nu;\gamma}-F_{\phi\nu\epsilon}F_{\mu\delta\theta;\gamma}-F_{\mu\delta\theta}F_{\phi\nu\epsilon;\gamma}\right)\right] \\ & +g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu\right)_{;\gamma} \\ & +\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}+\frac{1}{4\pi}g^{\alpha\beta}F^{\epsilon\gamma}F_{\epsilon\alpha;\gamma} \\ & -\frac{1}{16\pi}g^{\beta\gamma}g^{\mu\epsilon}g^{\nu\delta}F_{\mu\nu}F_{\epsilon\delta;\gamma}-\frac{1}{16\pi}g^{\beta\gamma}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}F_{\mu\nu;\gamma}\ . \end{aligned} \tag{89.227}$$

Equation (89.227) can be written (including changing dummy indexes in the second- and third-to-last terms)

$$\begin{aligned} T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}g^{\beta\mu}\left(-g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\nu;\theta;\gamma}+g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\theta;\nu;\gamma}\right. \\ & +g^{\delta\theta}g^{\epsilon\phi}F_{\mu\epsilon\theta}g^{\nu\gamma}F_{\phi\delta\nu;\gamma}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\phi\delta\nu}F_{\mu\epsilon\theta;\gamma}+g^{\delta\theta}g^{\epsilon\phi}F_{\epsilon\mu\theta}g^{\nu\gamma}F_{\delta\phi\nu;\gamma}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\delta\phi\nu}F_{\epsilon\mu\theta;\gamma} \end{aligned}$$

$$
\begin{aligned}
&-g^{\nu\gamma}g^{\epsilon\phi}F_{\epsilon\mu\nu}g^{\delta\theta}F_{\theta\delta\phi;\gamma}-g^{\delta\theta}g^{\epsilon\phi}F_{\theta\delta\phi}g^{\nu\gamma}F_{\epsilon\mu\nu;\gamma}-g^{\nu\gamma}g^{\epsilon\phi}F_{\phi\nu\epsilon}g^{\delta\theta}F_{\mu\delta\theta;\gamma}-g^{\delta\theta}F_{\mu\delta\theta}g^{\nu\gamma}g^{\epsilon\phi}F_{\phi\nu\epsilon;\gamma}\Big)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}+\frac{1}{4\pi}g^{\alpha\beta}F^{\mu\nu}F_{\mu\alpha;\nu}\\
&-\frac{1}{16\pi}g^{\beta\gamma}g^{\epsilon\mu}g^{\delta\nu}F_{\epsilon\delta}F_{\mu\nu;\gamma}-\frac{1}{16\pi}g^{\beta\gamma}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}F_{\mu\nu;\gamma}\ . \qquad (89.228)
\end{aligned}
$$

Combining the last two equal terms and splitting the third-to-last term into two equal parts allows us to write (89.228) as

$$
\begin{aligned}
T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}g^{\beta\mu}\Big(-g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\nu;\theta;\gamma}+g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\theta;\nu;\gamma}\\
&+g^{\delta\theta}g^{\epsilon\phi}F_{\mu\epsilon\theta}d_{\phi\delta}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\phi\delta\nu}F_{\mu\epsilon\theta;\gamma}+g^{\delta\theta}g^{\epsilon\phi}F_{\epsilon\mu\theta}d_{\delta\phi}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\delta\phi\nu}F_{\epsilon\mu\theta;\gamma}\\
&-g^{\nu\gamma}g^{\epsilon\phi}F_{\epsilon\mu\nu}b_{\phi\gamma}-g^{\delta\theta}g^{\epsilon\phi}F_{\theta\delta\phi}d_{\epsilon\mu}-g^{\nu\gamma}g^{\epsilon\phi}F_{\phi\nu\epsilon}c_{\mu\gamma}-g^{\delta\theta}F_{\mu\delta\theta}g^{\nu\gamma}b_{\nu\gamma}\Big)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}-\frac{1}{8\pi}g^{\alpha\beta}F^{\nu\mu}F_{\mu\alpha;\nu}-\frac{1}{8\pi}g^{\alpha\beta}F^{\mu\nu}F_{\alpha\mu;\nu}\\
&-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}F_{\mu\nu;\gamma}\ . \qquad (89.229)
\end{aligned}
$$

Changing some indexes and simplifying (89.229) gives

$$
\begin{aligned}
T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}g^{\beta\mu}\Big(-g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\nu;\theta;\gamma}+g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\theta;\nu;\gamma}\\
&+g^{\delta\theta}g^{\epsilon\phi}F_{\mu\epsilon\theta}d_{\phi\delta}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\phi\delta\nu}F_{\mu\epsilon\theta;\gamma}+g^{\delta\theta}g^{\epsilon\phi}F_{\epsilon\mu\theta}d_{\delta\phi}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\delta\phi\nu}F_{\epsilon\mu\theta;\gamma}\\
&-g^{\theta\delta}g^{\epsilon\phi}F_{\epsilon\mu\theta}b_{\phi\delta}-g^{\delta\theta}g^{\epsilon\phi}F_{\theta\delta\phi}d_{\epsilon\mu}-g^{\phi\epsilon}g^{\theta\delta}F_{\delta\phi\theta}c_{\mu\epsilon}-g^{\delta\theta}F_{\mu\delta\theta}b^{\gamma}{}_{\gamma}\Big)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}\\
&-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu}+F_{\gamma\mu;\nu}+F^{\mu\nu}F_{\mu\nu;\gamma}\right)\ . \qquad (89.230)
\end{aligned}
$$

We can write (89.230) as

$$
\begin{aligned}
T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}g^{\beta\mu}\Big(-g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\nu;\theta;\gamma}+g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\theta;\nu;\gamma}\\
&+g^{\delta\theta}g^{\epsilon\phi}F_{\mu\epsilon\theta}d_{\phi\delta}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\phi\delta\nu}F_{\mu\epsilon\theta;\gamma}+g^{\nu\gamma}g^{\delta\theta}g^{\epsilon\phi}F_{\delta\phi\nu}F_{\epsilon\mu\theta;\gamma}+g^{\theta\delta}g^{\epsilon\phi}F_{\epsilon\mu\theta}\left(d_{\delta\phi}-b_{\phi\delta}\right)\\
&-g^{\delta\theta}g^{\epsilon\phi}F_{\theta\delta\phi}d_{\epsilon\mu}+g^{\phi\epsilon}g^{\theta\delta}F_{\delta\phi\theta}\left(-c_{\mu\epsilon}\right)+g^{\delta\theta}F_{\mu\delta\theta}\left(-b^{\gamma}{}_{\gamma}\right)\Big)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_\mu J_\nu\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}\\
&-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu}+F_{\gamma\mu;\nu}+F^{\mu\nu}F_{\mu\nu;\gamma}\right)\ . \qquad (89.231)
\end{aligned}
$$

Or,

$$
T^{\beta\gamma}{}_{;\gamma} \;=\; -\frac{1}{8\pi}\Big(-g^{\beta\mu}g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\nu;\theta;\gamma}+g^{\beta\mu}g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\theta;\nu;\gamma}
$$

$$+F^{\beta\phi\delta}d_{\phi\delta} + F^{\epsilon\theta\gamma}F^{\beta}{}_{\epsilon\theta;\gamma} + F^{\theta\epsilon\gamma}F_{\epsilon}{}^{\beta}{}_{\theta;\gamma} + F^{\phi\beta\delta}\left(d_{\delta\phi} - b_{\phi\delta}\right)$$
$$-F_{\theta}{}^{\theta\epsilon}d_{\epsilon}{}^{\beta} + F^{\theta\epsilon}{}_{\theta}\left(-c^{\beta}{}_{\epsilon}\right) + F^{\beta\theta}{}_{\theta}\left(-b^{\gamma}{}_{\gamma}\right)\Big)$$
$$+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma}$$
$$+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}$$
$$-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu} + F_{\gamma\mu;\nu} + F^{\mu\nu}F_{\mu\nu;\gamma}\right) \ . \tag{89.232}$$

Or,

$$\begin{aligned}T^{\beta\gamma}{}_{;\gamma} \;=\; & -\frac{1}{8\pi}\Big(-g^{\beta\mu}g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\nu;\theta;\gamma} + g^{\beta\mu}g^{\nu\gamma}g^{\delta\theta}F_{\delta\mu\theta;\nu;\gamma} \\ & +F^{\beta\phi\delta}d_{\phi\delta} + F^{\epsilon\theta\gamma}F^{\beta}{}_{\epsilon\theta;\gamma} + F^{\theta\epsilon\gamma}F_{\epsilon}{}^{\beta}{}_{\theta;\gamma} - F_{\theta}{}^{\theta\epsilon}d_{\epsilon}{}^{\beta} \\ & +F^{\phi\beta\delta}\left(d_{\delta\phi} - b_{\phi\delta}\right) + F^{\theta\epsilon}{}_{\theta}\left(-c^{\beta}{}_{\epsilon}\right) + F^{\beta\theta}{}_{\theta}\left(-b^{\gamma}{}_{\gamma}\right)\Big) \\ & +g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma} \\ & +\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma} \\ & -\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu} + F_{\gamma\mu;\nu} + F^{\mu\nu}F_{\mu\nu;\gamma}\right) \ . \end{aligned} \tag{89.233}$$

MODIFIED TO HERE - THIS IS AS FAR AS I HAVE GOTTEN ON THE REVISION.

Or, using (89.412) and changing some indexes gives

$$\begin{aligned}T^{\beta\gamma}{}_{;\gamma} \;=\; & -\frac{1}{8\pi}\left[F^{\beta}{}_{\phi\delta}a^{\phi\delta} - g^{\phi\delta}F^{\beta}{}_{\phi\delta}a^{\gamma}{}_{\gamma} + 2g^{\beta\alpha}F^{\epsilon\theta\gamma}F_{\alpha\epsilon\theta;\gamma} - 2F^{\theta}{}_{\epsilon\theta}F^{\gamma\beta\epsilon}{}_{;\gamma}\right] \\ & -\frac{1}{8\pi}\left[F^{\beta\phi\delta}\left(a_{\phi\delta} - b_{\phi\delta}\right) + g^{\beta\phi}F^{\theta\delta}{}_{\theta}\left(a_{\phi\delta} - b_{\phi\delta}\right) + g^{\phi\delta}F^{\beta\theta}{}_{\theta}\left(a_{\phi\delta} - b_{\phi\delta}\right)\right] \\ & +g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma} \\ & +\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma} \\ & -\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu} + F_{\gamma\mu;\nu} + F^{\mu\nu}F_{\mu\nu;\gamma}\right) \ . \end{aligned} \tag{89.234}$$

Or, changing indexes and factoring gives

$$\begin{aligned}T^{\beta\gamma}{}_{;\gamma} \;=\; & -\frac{1}{8\pi}\left[F^{\beta}{}_{\phi\delta}\left(a^{\phi\delta} - g^{\phi\delta}a^{\gamma}{}_{\gamma}\right) + 2g^{\beta\alpha}F^{\epsilon\theta\gamma}F_{\alpha\epsilon\theta;\gamma} - 2g^{\mu\beta}g^{\nu\gamma}g^{\epsilon\delta}g^{\phi\theta}F_{\phi\epsilon\theta}F_{\delta\nu\mu;\gamma}\right] \\ & -\frac{1}{8\pi}\left[F^{\beta\phi\delta} + g^{\beta\phi}F^{\theta\delta}{}_{\theta} + g^{\phi\delta}F^{\beta\theta}{}_{\theta}\right]\left(a_{\phi\delta} - b_{\phi\delta}\right) \\ & +g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma} \\ & +\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma} \\ & -\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu} + F_{\gamma\mu;\nu} + F^{\mu\nu}F_{\mu\nu;\gamma}\right) \ . \end{aligned} \tag{89.235}$$

Using (89.412) and (89.413) and rearranging some terms gives

$$T^{\beta\gamma}{}_{;\gamma} \;=\; -\frac{1}{8\pi}\left[F^{\beta}{}_{\phi\delta}\left(F^{\mu\phi\delta}{}_{;\mu} - g^{\phi\delta}F^{\gamma\mu}{}_{\gamma;\mu}\right) + 2g^{\beta\alpha}F^{\epsilon\theta\gamma}F_{\gamma\epsilon\theta;\alpha} - 2g^{\mu\beta}g^{\nu\gamma}g^{\epsilon\delta}g^{\phi\theta}F_{\phi\epsilon\theta}F_{\delta\nu\gamma;\mu}\right]$$

$$
\begin{aligned}
&-\frac{1}{8\pi}\left[2g^{\beta\alpha}F^{\epsilon\theta\gamma}\left(F_{\alpha\epsilon\theta;\gamma}-F_{\gamma\epsilon\theta;\alpha}\right)-2g^{\mu\beta}g^{\nu\gamma}g^{\epsilon\delta}g^{\phi\theta}F_{\phi\epsilon\theta}\left(F_{\delta\nu\mu;\gamma}-F_{\delta\nu\gamma;\mu}\right)\right]\\
&-\frac{1}{8\pi}\left[F^{\beta\phi\delta}+g^{\beta\phi}F^{\theta\delta}{}_{\theta}+g^{\phi\delta}F^{\beta\theta}{}_{\theta}\right]g^{\mu\gamma}\left(F_{\mu\phi\delta;\gamma}-F_{\mu\phi\gamma;\delta}\right)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}\\
&-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu}+F_{\gamma\mu;\nu}+F^{\mu\nu}F_{\mu\nu;\gamma}\right)\ . \qquad (89.236)
\end{aligned}
$$

Changing some indexes and using symmetries gives

$$
\begin{aligned}
T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}\left[F^{\beta}{}_{\phi\delta}\left(F^{\mu\phi\delta}{}_{;\mu}-g^{\phi\delta}F^{\gamma\mu}{}_{\gamma;\mu}\right)+2g^{\beta\gamma}F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta;\gamma}-2g^{\beta\gamma}g^{\nu\mu}g^{\epsilon\delta}g^{\phi\theta}F_{\phi\epsilon\theta}F_{\delta\nu\mu;\gamma}\right]\\
&-\frac{1}{8\pi}\left[2g^{\beta\alpha}F^{\epsilon\theta\gamma}\left(F_{\epsilon\theta\alpha;\gamma}-F_{\epsilon\theta\gamma;\alpha}\right)-2g^{\mu\beta}g^{\nu\gamma}g^{\epsilon\delta}g^{\phi\theta}F_{\phi\epsilon\theta}\left(F_{\delta\nu\mu;\gamma}-F_{\delta\nu\gamma;\mu}\right)\right]\\
&-\frac{1}{8\pi}\left[F^{\beta\phi\delta}+g^{\beta\phi}F^{\theta\delta}{}_{\theta}+g^{\phi\delta}F^{\beta\theta}{}_{\theta}\right]g^{\mu\gamma}\left(F_{\mu\phi\delta;\gamma}-F_{\mu\phi\gamma;\delta}\right)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}\\
&-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu}+F_{\gamma\mu;\nu}+F^{\mu\nu}F_{\mu\nu;\gamma}\right)\ . \qquad (89.237)
\end{aligned}
$$

Or,

$$
\begin{aligned}
T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}\left[F^{\beta}{}_{\phi\delta}\left(F^{\mu\phi\delta}{}_{;\mu}-g^{\phi\delta}F^{\gamma\mu}{}_{\gamma;\mu}\right)+g^{\beta\gamma}\left(F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta}\right)_{;\gamma}-g^{\beta\gamma}g^{\nu\mu}g^{\epsilon\delta}g^{\phi\theta}\left(F_{\phi\epsilon\theta}F_{\delta\nu\mu}\right)_{;\gamma}\right]\\
&-\frac{1}{8\pi}\left[2g^{\beta\alpha}F^{\epsilon\theta\gamma}\left(F_{\epsilon\theta\alpha;\gamma}-F_{\epsilon\theta\gamma;\alpha}\right)-2g^{\mu\beta}g^{\nu\gamma}g^{\epsilon\delta}g^{\phi\theta}F_{\phi\epsilon\theta}\left(F_{\delta\nu\mu;\gamma}-F_{\delta\nu\gamma;\mu}\right)\right]\\
&-\frac{1}{8\pi}\left[F^{\beta\phi\delta}+g^{\beta\phi}F^{\theta\delta}{}_{\theta}+g^{\phi\delta}F^{\beta\theta}{}_{\theta}\right]g^{\mu\gamma}\left(F_{\mu\phi\delta;\gamma}-F_{\mu\phi\gamma;\delta}\right)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}\\
&-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu}+F_{\gamma\mu;\nu}+F^{\mu\nu}F_{\mu\nu;\gamma}\right)\ . \qquad (89.238)
\end{aligned}
$$

Or,

$$
\begin{aligned}
T^{\beta\gamma}{}_{;\gamma} \;=\;& -\frac{1}{8\pi}\left[F^{\beta}{}_{\phi\delta}\left(F^{\mu\phi\delta}{}_{;\mu}-g^{\phi\delta}F^{\gamma\mu}{}_{\gamma;\mu}\right)+g^{\beta\gamma}\left(F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta}\right)_{;\gamma}-g^{\beta\gamma}\left(F^{\theta}{}_{\epsilon\theta}F^{\epsilon\mu}{}_{\mu}\right)_{;\gamma}\right]\\
&-\frac{1}{8\pi}\left[2g^{\beta\alpha}F^{\epsilon\theta\gamma}\left(F_{\epsilon\theta\alpha;\gamma}-F_{\epsilon\theta\gamma;\alpha}\right)-2g^{\mu\beta}g^{\nu\gamma}g^{\epsilon\delta}g^{\phi\theta}F_{\phi\epsilon\theta}\left(F_{\delta\nu\mu;\gamma}-F_{\delta\nu\gamma;\mu}\right)\right]\\
&-\frac{1}{8\pi}\left[F^{\beta\phi\delta}+g^{\beta\phi}F^{\theta\delta}{}_{\theta}+g^{\phi\delta}F^{\beta\theta}{}_{\theta}\right]g^{\mu\gamma}\left(F_{\mu\phi\delta;\gamma}-F_{\mu\phi\gamma;\delta}\right)\\
&+g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu}+g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma}\\
&+\frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}F^{\epsilon\gamma}{}_{;\gamma}\\
&-\frac{1}{8\pi}g^{\beta\gamma}F^{\mu\nu}\left(F_{\nu\gamma;\mu}+F_{\gamma\mu;\nu}+F^{\mu\nu}F_{\mu\nu;\gamma}\right)\ . \qquad (89.239)
\end{aligned}
$$

Finally, using the inhomogeneous gravitational field equation (89.166) or (89.177) , the homogeneous gravitational field equation (89.40), the inhomogeneous Maxwell equation (89.150), and the homogeneous Maxwell equation (89.411) gives

$$\begin{aligned} T^{\beta\gamma}{}_{;\gamma} &= -\frac{1}{8\pi}\left[F^{\beta}{}_{\phi\delta}\left(+8\pi T^{\phi\delta}\right) + g^{\beta\gamma}\left(F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta}\right)_{;\gamma} - g^{\beta\gamma}\left(F^{\theta}{}_{\epsilon\theta}F^{\epsilon\mu}{}_{\mu}\right)_{;\gamma}\right] \\ &\quad + g^{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + g^{\mu\nu}A_{\mu}J_{\nu}\right)_{;\gamma} \\ &\quad + \frac{1}{4\pi}g^{\alpha\beta}F_{\epsilon\alpha}4\pi J^{\epsilon}\ . \end{aligned} \tag{89.240}$$

Or,

$$T^{\beta\gamma}{}_{;\gamma} = -F^{\beta}{}_{\phi\delta}T^{\phi\delta} - F^{\beta}{}_{\epsilon}J^{\epsilon} - \frac{1}{8\pi}g^{\beta\gamma}\left(\Lambda_{DE} + \Lambda_{DM} + \Lambda_{L}\right)_{;\gamma}\ , \tag{89.241}$$

where the first term is the gravitational force term, the second term is the Lorentz force,

$$\Lambda_{DE} \equiv F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta}\ , \tag{89.242}$$

$$\Lambda_{DM} \equiv -F^{\theta}{}_{\epsilon\theta}F^{\epsilon\mu}{}_{\mu}\ , \tag{89.243}$$

and

$$\Lambda_{L} \equiv -\frac{1}{8\pi}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + g^{\mu\nu}A_{\mu}J_{\nu}\right)\ . \tag{89.244}$$

The Λ_{DE} and Λ_{DM} terms might be dark energy and dark matter. The presence of the Λ_L term depends on the definition of action. It is not yet clear if it should be there. It should probably not be there [255, 256, Dirac, General Relativity, Section 29, p. 55].

89.11 Contracting the stress-energy tensor

We can contract (89.210) by multiplying it by $g^{\beta\gamma}$ to give

$$\begin{aligned} T = g^{\beta\gamma}T_{\beta\gamma} &= -\frac{1}{8\pi}g^{\beta\gamma}g^{\alpha\delta}R_{\alpha\beta\gamma\delta} \\ &\quad -\frac{1}{8\pi}g^{\beta\gamma}g^{\mu\epsilon}g^{\alpha\delta}\left(-F_{\delta\gamma\alpha}F_{\beta\mu\epsilon} + F_{\delta\gamma\epsilon}F_{\beta\mu\alpha}\right) \\ &\quad g^{\beta\gamma}g_{\beta\gamma}\left(+\frac{1}{6}A_{\mu\nu}Y^{\mu\nu} + g^{\mu\nu}A_{\mu}J_{\nu} - \frac{1}{16\pi}F_{\mu\nu}g^{\mu\epsilon}g^{\nu\delta}F_{\epsilon\delta}\right) \\ &\quad + g^{\beta\gamma}\left(+\frac{1}{4\pi}F_{\mu\beta}F_{\epsilon\gamma}g^{\mu\epsilon}\right)\ . \end{aligned} \tag{89.245}$$

Contracting (89.202) gives

$$T = -\frac{R}{8\pi} + \frac{\Lambda}{2\pi}\ . \tag{89.246}$$

89.12 Exact solutions

The real test of the gravitational field representation of General Relativity is if it gives the same solutions as General Relativity for the same cases. Here, I demonstrate a few of these cases. The main purpose here is to show the consistency of the gravitational field representation of General Relativity with the geometrical representation of General Relativity. In addition, some of these examples show how the gravitational field representation of General Relativity can separate gravitational aspects from geometrical aspects.

The field equations are:

$$\begin{aligned}
& F_{\alpha\beta\gamma;\delta} - F_{\alpha\beta\delta;\gamma} + F_{\gamma\delta\beta;\alpha} - F_{\gamma\delta\alpha;\beta} \\
= \ & F_{\alpha\beta\gamma,\delta} - F_{\alpha\beta\delta,\gamma} + F_{\gamma\delta\beta,\alpha} - F_{\gamma\delta\alpha,\beta} \\
& -F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} - F_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - F_{\alpha\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma\delta} + F_{\epsilon\beta\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} + F_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} + F_{\alpha\beta\epsilon}\Gamma^{\epsilon}{}_{\delta\gamma} \\
& -F_{\epsilon\delta\beta}\Gamma^{\epsilon}{}_{\gamma\alpha} - F_{\gamma\epsilon\beta}\Gamma^{\epsilon}{}_{\delta\alpha} - F_{\gamma\delta\epsilon}\Gamma^{\epsilon}{}_{\beta\alpha} + F_{\epsilon\delta\alpha}\Gamma^{\epsilon}{}_{\gamma\beta} + F_{\gamma\epsilon\alpha}\Gamma^{\epsilon}{}_{\delta\beta} + F_{\gamma\delta\epsilon}\Gamma^{\epsilon}{}_{\alpha\beta} \\
= \ & F_{\alpha\beta\gamma,\delta} - F_{\alpha\beta\delta,\gamma} + F_{\gamma\delta\beta,\alpha} - F_{\gamma\delta\alpha,\beta} \\
& -F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} - F_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} + F_{\epsilon\beta\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} + F_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} \\
& -F_{\epsilon\delta\beta}\Gamma^{\epsilon}{}_{\gamma\alpha} - F_{\gamma\epsilon\beta}\Gamma^{\epsilon}{}_{\delta\alpha} + F_{\epsilon\delta\alpha}\Gamma^{\epsilon}{}_{\gamma\beta} + F_{\gamma\epsilon\alpha}\Gamma^{\epsilon}{}_{\delta\beta} \\
= \ & e_{\alpha\beta\gamma\delta} \equiv F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} - F_{\gamma\epsilon\alpha}F^{\epsilon}{}_{\delta\beta} + F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta\alpha}
\end{aligned} \tag{89.247}$$

because terms cancel,

$$\begin{aligned}
& -F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\beta\alpha\delta;\gamma} + F_{\beta\alpha\gamma;\delta} \\
= \ & -F_{\alpha\beta\delta,\gamma} + F_{\alpha\beta\gamma,\delta} - F_{\beta\alpha\delta,\gamma} + F_{\beta\alpha\gamma,\delta} \\
& +F_{\epsilon\beta\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} + F_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} + F_{\alpha\beta\epsilon}\Gamma^{\epsilon}{}_{\delta\gamma} - F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} - F_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - F_{\alpha\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma\delta} \\
& +F_{\epsilon\alpha\delta}\Gamma^{\epsilon}{}_{\beta\gamma} + F_{\beta\epsilon\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} + F_{\beta\alpha\epsilon}\Gamma^{\epsilon}{}_{\delta\gamma} - F_{\epsilon\alpha\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - F_{\beta\epsilon\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} - F_{\beta\alpha\epsilon}\Gamma^{\epsilon}{}_{\gamma\delta} \\
= \ & -F_{\alpha\beta\delta,\gamma} + F_{\alpha\beta\gamma,\delta} - F_{\beta\alpha\delta,\gamma} + F_{\beta\alpha\gamma,\delta} \\
& +F_{\epsilon\beta\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} + F_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} - F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} - F_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} \\
& +F_{\epsilon\alpha\delta}\Gamma^{\epsilon}{}_{\beta\gamma} + F_{\beta\epsilon\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} - F_{\epsilon\alpha\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - F_{\beta\epsilon\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} \\
= \ & f_{\alpha\beta\gamma\delta} \equiv F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} + F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha\delta} - F_{\beta\epsilon\delta}F^{\epsilon}{}_{\alpha\gamma}
\end{aligned} \tag{89.248}$$

because some terms cancel, and

$$\begin{aligned}
& F_{(\beta}{}^{\alpha}{}_{\alpha;\gamma)} - F_{(\beta}{}^{\alpha}{}_{\gamma);\alpha} \\
= \ & F_{(\beta}{}^{\alpha}{}_{\alpha,\gamma)} - F_{(\beta}{}^{\alpha}{}_{\gamma),\alpha} - F_{\epsilon}{}^{\alpha}{}_{\alpha}\Gamma^{\epsilon}{}_{\beta\gamma} + F_{\epsilon}{}^{\alpha}{}_{(\gamma}\Gamma^{\epsilon}{}_{\beta)\alpha} \\
= \ & f^{\alpha}{}_{(\beta\alpha\gamma)} + \frac{8\pi}{3}\left(Y_{\beta\gamma} - \frac{1}{2}g_{\beta\gamma}Y\right) \\
= \ & -F^{\alpha}{}_{\epsilon(\gamma}F^{\epsilon}{}_{\beta)\alpha} + F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\beta\gamma} - F_{(\beta\epsilon\gamma)}F^{\epsilon\alpha}{}_{\alpha} + F_{(\beta\epsilon\alpha}F^{\epsilon\alpha}{}_{\gamma)} + \frac{8\pi}{3}\left(Y_{\beta\gamma} - \frac{1}{2}g_{\beta\gamma}Y\right) .
\end{aligned} \tag{89.249}$$

Equation (89.247) can be written

$$\begin{aligned}
& F_{\alpha\beta\gamma,\delta} - F_{\alpha\beta\delta,\gamma} + F_{\gamma\delta\beta,\alpha} - F_{\gamma\delta\alpha,\beta} \\
& -F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} + F_{\epsilon\beta\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} + F_{\alpha\epsilon\delta}\left(\Gamma^{\epsilon}{}_{\beta\gamma} + F^{\epsilon}{}_{\beta\gamma}\right) \\
& -F_{\epsilon\delta\beta}\Gamma^{\epsilon}{}_{\gamma\alpha} + F_{\epsilon\delta\alpha}\Gamma^{\epsilon}{}_{\gamma\beta} + F_{\gamma\epsilon\alpha}\left(\Gamma^{\epsilon}{}_{\delta\beta} + F^{\epsilon}{}_{\delta\beta}\right) \\
& -F_{\alpha\epsilon\gamma}\left(\Gamma^{\epsilon}{}_{\beta\delta} + F^{\epsilon}{}_{\beta\delta}\right) - F_{\gamma\epsilon\beta}\left(\Gamma^{\epsilon}{}_{\delta\alpha} + F^{\epsilon}{}_{\delta\alpha}\right) = 0 .
\end{aligned} \tag{89.250}$$

Or,

$$\begin{aligned}
& F_{\alpha\beta\gamma,\delta} - F_{\alpha\beta\delta,\gamma} + F_{\gamma\delta\beta,\alpha} - F_{\gamma\delta\alpha,\beta} \\
& -F_{\epsilon\beta\gamma}\left(\Gamma^{\epsilon}{}_{\alpha\delta} + F^{\epsilon}{}_{\alpha\delta}\right) + F_{\epsilon\beta\delta}\left(\Gamma^{\epsilon}{}_{\alpha\gamma} + F^{\epsilon}{}_{\gamma\alpha}\right) + F_{\alpha\epsilon\delta}\left(\Gamma^{\epsilon}{}_{\beta\gamma} + F^{\epsilon}{}_{\beta\gamma}\right) \\
& -F_{\epsilon\delta\beta}\left(\Gamma^{\epsilon}{}_{\gamma\alpha} + F^{\epsilon}{}_{\gamma\alpha}\right) + F_{\epsilon\delta\alpha}\left(\Gamma^{\epsilon}{}_{\gamma\beta} + F^{\epsilon}{}_{\beta\gamma}\right) + F_{\gamma\epsilon\alpha}\left(\Gamma^{\epsilon}{}_{\delta\beta} + F^{\epsilon}{}_{\delta\beta}\right) \\
& -F_{\alpha\epsilon\gamma}\left(\Gamma^{\epsilon}{}_{\beta\delta} + F^{\epsilon}{}_{\beta\delta}\right) - F_{\gamma\epsilon\beta}\left(\Gamma^{\epsilon}{}_{\delta\alpha} + F^{\epsilon}{}_{\delta\alpha}\right) = 0 .
\end{aligned} \tag{89.251}$$

Equation (89.248) can be written

$$\begin{aligned}
&-F_{\alpha\beta\delta,\gamma}+F_{\alpha\beta\gamma,\delta}-F_{\beta\alpha\delta,\gamma}+F_{\beta\alpha\gamma,\delta}\\
&+F_{\epsilon\beta\delta}\Gamma^{\epsilon}{}_{\alpha\gamma}-F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\alpha\delta}-F_{\alpha\epsilon\gamma}\left(\Gamma^{\epsilon}{}_{\beta\delta}+F^{\epsilon}{}_{\beta\delta}\right)\\
&+F_{\epsilon\alpha\delta}\Gamma^{\epsilon}{}_{\beta\gamma}-F_{\epsilon\alpha\gamma}\Gamma^{\epsilon}{}_{\beta\delta}-F_{\beta\epsilon\gamma}\left(\Gamma^{\epsilon}{}_{\alpha\delta}+F^{\epsilon}{}_{\alpha\delta}\right)\\
&+F_{\alpha\epsilon\delta}\left(\Gamma^{\epsilon}{}_{\beta\gamma}+F^{\epsilon}{}_{\beta\gamma}\right)+F_{\beta\epsilon\delta}\left(\Gamma^{\epsilon}{}_{\alpha\gamma}+F^{\epsilon}{}_{\alpha\gamma}\right)=0
\end{aligned}\tag{89.252}$$

Or,

$$\begin{aligned}
&-F_{\alpha\beta\delta,\gamma}+F_{\alpha\beta\gamma,\delta}-F_{\beta\alpha\delta,\gamma}+F_{\beta\alpha\gamma,\delta}\\
&+F_{\epsilon\beta\delta}\left(\Gamma^{\epsilon}{}_{\alpha\gamma}+F^{\epsilon}{}_{\alpha\gamma}\right)-F_{\epsilon\beta\gamma}\left(\Gamma^{\epsilon}{}_{\alpha\delta}+F^{\epsilon}{}_{\alpha\delta}\right)-F_{\alpha\epsilon\gamma}\left(\Gamma^{\epsilon}{}_{\beta\delta}+F^{\epsilon}{}_{\beta\delta}\right)\\
&+F_{\epsilon\alpha\delta}\left(\Gamma^{\epsilon}{}_{\beta\gamma}+F^{\epsilon}{}_{\beta\gamma}\right)-F_{\epsilon\alpha\gamma}\left(\Gamma^{\epsilon}{}_{\beta\delta}+F^{\epsilon}{}_{\beta\delta}\right)-F_{\beta\epsilon\gamma}\left(\Gamma^{\epsilon}{}_{\alpha\delta}+F^{\epsilon}{}_{\alpha\delta}\right)\\
&+F_{\alpha\epsilon\delta}\left(\Gamma^{\epsilon}{}_{\beta\gamma}+F^{\epsilon}{}_{\beta\gamma}\right)+F_{\beta\epsilon\delta}\left(\Gamma^{\epsilon}{}_{\alpha\gamma}+F^{\epsilon}{}_{\alpha\gamma}\right)=0
\end{aligned}\tag{89.253}$$

Equation (89.249) can be written

$$\begin{aligned}
&F_{(\beta}{}^{\alpha}{}_{\alpha,\gamma)}-F_{(\beta}{}^{\alpha}{}_{\gamma),\alpha}-F^{\epsilon\alpha}{}_{\alpha}\Gamma_{\epsilon\beta\gamma}+F^{\epsilon\alpha}{}_{(\gamma}\Gamma_{\epsilon\beta)\alpha}\\
&+F^{\alpha\epsilon}{}_{(\gamma}F_{\epsilon\beta)\alpha}-F^{\alpha\epsilon}{}_{\alpha}F_{\epsilon\beta\gamma}\\
&+F_{(\beta\epsilon\gamma)}F^{\epsilon\alpha}{}_{\alpha}-F_{(\beta\epsilon\alpha}F^{\epsilon\alpha}{}_{\gamma)}-\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)=0\ .
\end{aligned}\tag{89.254}$$

Or,

$$\begin{aligned}
&F_{(\beta}{}^{\alpha}{}_{\alpha,\gamma)}-F_{(\beta}{}^{\alpha}{}_{\gamma),\alpha}-F^{\epsilon\alpha}{}_{\alpha}\left(\Gamma_{\epsilon\beta\gamma}+F_{\epsilon\beta\gamma}\right)+F^{\epsilon\alpha}{}_{(\gamma}\left(\Gamma_{\epsilon\beta)\alpha}+F_{\epsilon\beta)\alpha}\right)\\
&+\left(-F^{\epsilon\alpha}{}_{(\gamma}+F^{\alpha\epsilon}{}_{(\gamma}\right)F_{\epsilon\beta)\alpha}+\left(F^{\epsilon\alpha}{}_{\alpha}-F^{\alpha\epsilon}{}_{\alpha}\right)F_{\epsilon\beta\gamma}\\
&+F_{(\beta\epsilon\gamma)}F^{\epsilon\alpha}{}_{\alpha}-F_{(\beta\epsilon\alpha}F^{\epsilon\alpha}{}_{\gamma)}-\frac{8\pi}{3}\left(Y_{\beta\gamma}-\frac{1}{2}g_{\beta\gamma}Y\right)=0\ .
\end{aligned}\tag{89.255}$$

89.12.1 Constant gravitational field and/or constant acceleration

THIS IS A BAD EXAMPLE, IN THAT THERE IS NO MATTER TO GIVE A GRAVITATIONAL FIELD (INERTIA, IN THIS CASE). INSTEAD, I NEED TO INCLUDE MATTER THAT ACCELERATES.

Even with Cartesian coordinates, the Γs are not all zero if there is gravitation or an accelerated frame.

It is useful to use a Rindler frame, in which the metric is given by

$$g_{\mu\nu}=\begin{pmatrix}-g^2x^2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\ .\tag{89.256}$$

The non-zero connections are

$$\Gamma_{100}=g^2x\ ,\tag{89.257}$$

$$\Gamma_{010}=-g^2x\ ,\tag{89.258}$$

$$\Gamma_{001}=-g^2x\ ,\tag{89.259}$$

$$\Gamma^1{}_{00} = g^2 x \ , \tag{89.260}$$

$$\Gamma^0{}_{10} = \frac{1}{x} \ , \tag{89.261}$$

and

$$\Gamma^0{}_{01} = \frac{1}{x} \ . \tag{89.262}$$

When I substitute $F = -\Gamma$ into the field equations, I get zero for the two homogeneous equations, but not for the inhomogeneous equation.

In trying to figure this out, I realized that I need to worry more about units. In fact, the Langrangian L should have the units of length to the -4 power, but I get length to the -2 power. I need to resolve that. Part of the resolution is that the Lagrangian is divided by G, and that makes the units resolution for that. However, $T_{\mu\nu}$ and $Y_{\mu\nu}$ do not have the same units. The resolution for that is to multiply any term that has $Y_{\mu\nu}$ by G in the Lagrangian.

At one point, I realized that in an accelerating frame, $T_{\mu\nu}$ is not zero because of the Unruh effect. When I went to check out the details, I should get contributions proportional to the fourth power of the Hawking temperature, which would be proportional to the fourth power of the acceleration, but the terms I get are proportional to the second power of the acceleration.

However, the Unruh temperature involves $\hbar$, but I need a classical effect, so the additional terms to $Y_{\mu\nu}$ must be something else. So, I probably now have the correct field equations, but I do not yet know the meaning of $Y_{\mu\nu}$.

$F = -\Gamma$ satisfies (89.250), (89.252), and 89.254 if we take

$$\frac{8\pi}{3}\left(Y_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} Y\right) = \begin{pmatrix} 2g^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{89.263}$$

However, any frame for which $F = -\Gamma$ must be a frame defined by a congruence of geodesics. In such a frame, $\Gamma = 0$, so that means that $g = 0$. And in that case, the equation is satisfied, although that is not an interesting case. To check further, we must take a case where $F \neq -\Gamma$.

To do that, we first look at (89.254), keeping only terms that have two zeros and one one for subscripts and superscripts. This gives

$$-F_0{}^1{}_{0,1} + F^{01}{}_0 \Gamma_{001} + F_{010} F^{10}{}_0 - F_{001} F^{01}{}_0 - \frac{8\pi}{3}\left(Y_{00} - \frac{1}{2} g_{00} Y\right) = 0 \ , \tag{89.264}$$

$$F_1{}^0{}_{0,1} + F^{00}{}_1 \Gamma_{010} + F^{00}{}_1 F_{010} - F_{100} F^{00}{}_1 - \frac{8\pi}{3}\left(Y_{11} - \frac{1}{2} g_{11} Y\right) = 0 \ , \tag{89.265}$$

and

$$-\frac{1}{2} F_0{}^0{}_{1,0} - \frac{8\pi}{3}\left(Y_{01} - \frac{1}{2} g_{01} Y\right) = 0 \ . \tag{89.266}$$

Taking $Y_{\alpha\beta} = 0$, and using (89.256) when needed gives as solutions to (89.264), (89.265), and (89.266)

$$F^1{}_{00} = F_{100} = \frac{g^2 x^2}{x_1} + \frac{x}{x_0^2} e^{-x/x_1} \ , \tag{89.267}$$

$$F^0{}_{01} = F^0{}_{10} = -\frac{1}{g^2 x_0^2 x} e^{-x/x_1} \ , \tag{89.268}$$

and

$$F_{010} = F_{001} = \frac{x}{x_0^2} e^{-x/x_1} \ , \tag{89.269}$$

where x_0 and x_1 are constants of integration.

On the other hand, we can solve (89.250) and (89.252) with α, β, γ, δ = 1,0,0,1 to give

$$F^1{}_{00} = F_{100} = -g^2 x + \text{constant of integration} \ , \tag{89.270}$$

$$F^0{}_{01} = F^0{}_{10} = -\frac{1}{x} \ , \tag{89.271}$$

and

$$F_{010} = F_{001} = g^2 x \ . \tag{89.272}$$

Not only do (89.270), (89.271), and (89.272) not agree with (89.267), (89.268), and (89.269), but neither set of solutions gives the correct force equation, which would require

$$F^1{}_{00} = F_{100} = -g^2 x + x\frac{d\ln x}{dt}\frac{d\ln \dot{x}}{dt} \tag{89.273}$$

and

$$F^0{}_{01} = F^0{}_{10} = -\frac{1}{x} - \frac{1}{2}\frac{d\ln \dot{t}}{dx} \ . \tag{89.274}$$

It appears that this whole procedure is a failure. I can think of no way around it. It is not possible to get a gravitational field representation of General Relativity. Maybe it is still possible to find a tensor gravitational field by requiring only that such a field equals (the negative of) the connection along the trajectory of a body or particle.

The solution for a homogeneous medium in Cartesian coordinates is a constant. The appropriate non-zero elements of the gravitational tensor field are:

$$F_{100} = F_{010} = F_{001} = g_x \ . \tag{89.275}$$

$$F_{200} = F_{020} = F_{002} = g_y \ . \tag{89.276}$$

$$F_{300} = F_{030} = F_{003} = g_z \ . \tag{89.277}$$

The Riemann tensor is:

$$R_{\alpha\beta\gamma\delta} = -F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} = 0 \ . \tag{89.278}$$

We also have:

$$\Lambda_{DE} \equiv F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta} = 3F^{100}F_{100} + 3F^{200}F_{200} + 3F^{300}F_{300} \ . \tag{89.279}$$

$$\Lambda_{DM} \equiv -F^{\theta}{}_{\epsilon\theta}F^{\epsilon\mu}{}_{\mu} = -F^0{}_{10}F^{10}{}_0 - F^0{}_{20}F^{20}{}_0 - F^0{}_{30}F^{30}{}_0 \ . \tag{89.280}$$

89.12.2 Centrifugal force

THIS IS A BAD EXAMPLE, IN THAT THERE IS NO MATTER TO GIVE A GRAVITATIONAL FIELD (INERTIA, IN THIS CASE). INSTEAD, I NEED TO INCLUDE MATTER THAT ROTATES.

It is easy to calculate centrifugal force in Newtonian Mechanics. It is more difficult in General Relativity, as is well known. The purpose of the following calculations is to first show consistency between the gravitational field representation and the geometrical representation, and second to exhibit the specific components of the gravitational field tensor that give rise to centrifugal force.

The field equations are:

$$F_{\alpha\beta\gamma;\delta} - F_{\alpha\beta\delta;\gamma} = F_{\alpha\beta\gamma,\delta} - F_{\alpha\beta\delta,\gamma} = 0 \tag{89.281}$$

and

$$
\begin{aligned}
&F^{\mu\beta\gamma}{}_{;\mu} \\
&\quad = F^{\mu\beta\gamma}{}_{,\mu} + F^{\epsilon\beta\gamma}\Gamma^{\mu}{}_{\epsilon\mu} + F^{\mu\epsilon\gamma}\Gamma^{\beta}{}_{\epsilon\mu} + F^{\mu\beta\epsilon}\Gamma^{\gamma}{}_{\epsilon\mu} \\
&\quad = +8\pi\left(T^{\beta\gamma} - \frac{1}{3}g^{\beta\gamma}T\right) = 0\ . \qquad (89.282)
\end{aligned}
$$

With lowered indexes, we have

$$
\begin{aligned}
&g^{\mu\nu}F_{\mu\beta\gamma;\nu} \\
&\quad = g^{\mu\nu}\left(F_{\mu\beta\gamma,\nu} - F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\mu\nu} - F_{\mu\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\nu} - F_{\mu\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma\nu}\right) \\
&\quad = g^{\mu\nu}F_{\mu\beta\gamma,\nu} - F_{\epsilon\beta\gamma}\Gamma^{\epsilon}{}_{\mu}{}^{\mu} - F_{\mu\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta}{}^{\mu} - F_{\mu\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma}{}^{\mu} \\
&\quad = +8\pi\left(T_{\beta\gamma} - \frac{1}{3}g_{\beta\gamma}T\right) = 0\ . \qquad (89.283)
\end{aligned}
$$

Explicitly, in the (t, r, ϕ, z) coordinate system we will be using,

$$
\begin{aligned}
g^{tt}F_{t\beta\gamma,t} + g^{\phi t}F_{\phi\beta\gamma,t} + g^{rr}F_{r\beta\gamma,r} + g^{t\phi}F_{t\beta\gamma,\phi} + g^{\phi\phi}F_{\phi\beta\gamma,\phi} + g^{zz}F_{z\beta\gamma,z}& \\
-F_{t\beta\gamma}\Gamma^{t}{}_{\mu}{}^{\mu} - F_{r\beta\gamma}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\beta\gamma}\Gamma^{\phi}{}_{\mu}{}^{\mu} - F_{z\beta\gamma}\Gamma^{z}{}_{\mu}{}^{\mu}& \\
-F_{tt\gamma}\Gamma^{t}{}_{\beta}{}^{t} - F_{rt\gamma}\Gamma^{t}{}_{\beta}{}^{r} - F_{\phi t\gamma}\Gamma^{t}{}_{\beta}{}^{\phi} - F_{zt\gamma}\Gamma^{t}{}_{\beta}{}^{z}& \\
-F_{tr\gamma}\Gamma^{r}{}_{\beta}{}^{t} - F_{rr\gamma}\Gamma^{r}{}_{\beta}{}^{r} - F_{\phi r\gamma}\Gamma^{r}{}_{\beta}{}^{\phi} - F_{zr\gamma}\Gamma^{r}{}_{\beta}{}^{z}& \\
-F_{t\phi\gamma}\Gamma^{\phi}{}_{\beta}{}^{t} - F_{r\phi\gamma}\Gamma^{\phi}{}_{\beta}{}^{r} - F_{\phi\phi\gamma}\Gamma^{\phi}{}_{\beta}{}^{\phi} - F_{z\phi\gamma}\Gamma^{\phi}{}_{\beta}{}^{z}& \\
-F_{tz\gamma}\Gamma^{z}{}_{\beta}{}^{t} - F_{rz\gamma}\Gamma^{z}{}_{\beta}{}^{r} - F_{\phi z\gamma}\Gamma^{z}{}_{\beta}{}^{\phi} - F_{zz\gamma}\Gamma^{z}{}_{\beta}{}^{z}& \\
-F_{t\beta t}\Gamma^{t}{}_{\gamma}{}^{t} - F_{r\beta t}\Gamma^{t}{}_{\gamma}{}^{r} - F_{\phi\beta t}\Gamma^{t}{}_{\gamma}{}^{\phi} - F_{z\beta t}\Gamma^{t}{}_{\gamma}{}^{z}& \\
-F_{t\beta r}\Gamma^{r}{}_{\gamma}{}^{t} - F_{r\beta r}\Gamma^{r}{}_{\gamma}{}^{r} - F_{\phi\beta r}\Gamma^{r}{}_{\gamma}{}^{\phi} - F_{z\beta r}\Gamma^{r}{}_{\gamma}{}^{z}& \\
-F_{t\beta\phi}\Gamma^{\phi}{}_{\gamma}{}^{t} - F_{r\beta\phi}\Gamma^{\phi}{}_{\gamma}{}^{r} - F_{\phi\beta\phi}\Gamma^{\phi}{}_{\gamma}{}^{\phi} - F_{z\beta\phi}\Gamma^{\phi}{}_{\gamma}{}^{z}& \\
-F_{t\beta z}\Gamma^{z}{}_{\gamma}{}^{t} - F_{r\beta z}\Gamma^{z}{}_{\gamma}{}^{r} - F_{\phi\beta z}\Gamma^{z}{}_{\gamma}{}^{\phi} - F_{z\beta z}\Gamma^{z}{}_{\gamma}{}^{z}& \\
= 0\ .& \qquad (89.284)
\end{aligned}
$$

Taking various combinations for β and γ gives (no sum on repeated indexes) the following six equations. For $(\beta, \gamma) = (t, t)$, we have

$$
\begin{aligned}
g^{tt}F_{ttt,t} + g^{\phi t}F_{\phi tt,t} + g^{rr}F_{rtt,r} + g^{t\phi}F_{ttt,\phi} + g^{\phi\phi}F_{\phi tt,\phi} + g^{zz}F_{ztt,z}& \\
-F_{ttt}\Gamma^{t}{}_{\mu}{}^{\mu} - F_{rtt}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi tt}\Gamma^{\phi}{}_{\mu}{}^{\mu} - F_{ztt}\Gamma^{z}{}_{\mu}{}^{\mu}& \\
-F_{ttt}\Gamma^{t}{}_{t}{}^{t} - F_{rtt}\Gamma^{t}{}_{t}{}^{r} - F_{\phi tt}\Gamma^{t}{}_{t}{}^{\phi} - F_{ztt}\Gamma^{t}{}_{t}{}^{z}& \\
-F_{trt}\Gamma^{r}{}_{t}{}^{t} - F_{rrt}\Gamma^{r}{}_{t}{}^{r} - F_{\phi rt}\Gamma^{r}{}_{t}{}^{\phi} - F_{zrt}\Gamma^{r}{}_{t}{}^{z}& \\
-F_{t\phi t}\Gamma^{\phi}{}_{t}{}^{t} - F_{r\phi t}\Gamma^{\phi}{}_{t}{}^{r} - F_{\phi\phi t}\Gamma^{\phi}{}_{t}{}^{\phi} - F_{z\phi t}\Gamma^{\phi}{}_{t}{}^{z}& \\
-F_{tzt}\Gamma^{z}{}_{t}{}^{t} - F_{rzt}\Gamma^{z}{}_{t}{}^{r} - F_{\phi zt}\Gamma^{z}{}_{t}{}^{\phi} - F_{zzt}\Gamma^{z}{}_{t}{}^{z}& \\
-F_{ttt}\Gamma^{t}{}_{t}{}^{t} - F_{rtt}\Gamma^{t}{}_{t}{}^{r} - F_{\phi tt}\Gamma^{t}{}_{t}{}^{\phi} - F_{ztt}\Gamma^{t}{}_{t}{}^{z}& \\
-F_{ttr}\Gamma^{r}{}_{t}{}^{t} - F_{rtr}\Gamma^{r}{}_{t}{}^{r} - F_{\phi tr}\Gamma^{r}{}_{t}{}^{\phi} - F_{ztr}\Gamma^{r}{}_{t}{}^{z}& \\
-F_{tt\phi}\Gamma^{\phi}{}_{t}{}^{t} - F_{rt\phi}\Gamma^{\phi}{}_{t}{}^{r} - F_{\phi t\phi}\Gamma^{\phi}{}_{t}{}^{\phi} - F_{zt\phi}\Gamma^{\phi}{}_{t}{}^{z}& \\
-F_{ttz}\Gamma^{z}{}_{t}{}^{t} - F_{rtz}\Gamma^{z}{}_{t}{}^{r} - F_{\phi tz}\Gamma^{z}{}_{t}{}^{\phi} - F_{ztz}\Gamma^{z}{}_{t}{}^{z}& \\
= 0\ .& \qquad (89.285)
\end{aligned}
$$

For $(\beta,\gamma)=(r,r)$, we have

$$
\begin{aligned}
g^{tt}F_{trr,t}+g^{\phi t}F_{\phi rr,t}+g^{rr}F_{rrr,r}+g^{t\phi}F_{trr,\phi}+g^{\phi\phi}F_{\phi rr,\phi}+g^{zz}F_{zrr,z}\\
-F_{trr}\Gamma^{t}{}_{\mu}{}^{\mu}-F_{rrr}\Gamma^{r}{}_{\mu}{}^{\mu}-F_{\phi rr}\Gamma^{\phi}{}_{\mu}{}^{\mu}-F_{zrr}\Gamma^{z}{}_{\mu}{}^{\mu}\\
-F_{ttr}\Gamma^{t}{}_{r}{}^{t}-F_{rtr}\Gamma^{t}{}_{r}{}^{r}-F_{\phi tr}\Gamma^{t}{}_{r}{}^{\phi}-F_{ztr}\Gamma^{t}{}_{r}{}^{z}\\
-F_{trr}\Gamma^{r}{}_{r}{}^{t}-F_{rrr}\Gamma^{r}{}_{r}{}^{r}-F_{\phi rr}\Gamma^{r}{}_{r}{}^{\phi}-F_{zrr}\Gamma^{r}{}_{r}{}^{z}\\
-F_{t\phi r}\Gamma^{\phi}{}_{r}{}^{t}-F_{r\phi r}\Gamma^{\phi}{}_{r}{}^{r}-F_{\phi\phi r}\Gamma^{\phi}{}_{r}{}^{\phi}-F_{z\phi r}\Gamma^{\phi}{}_{r}{}^{z}\\
-F_{tzr}\Gamma^{z}{}_{r}{}^{t}-F_{rzr}\Gamma^{z}{}_{r}{}^{r}-F_{\phi zr}\Gamma^{z}{}_{r}{}^{\phi}-F_{zzr}\Gamma^{z}{}_{r}{}^{z}\\
-F_{trt}\Gamma^{t}{}_{r}{}^{t}-F_{rrt}\Gamma^{t}{}_{r}{}^{r}-F_{\phi rt}\Gamma^{t}{}_{r}{}^{\phi}-F_{zrt}\Gamma^{t}{}_{r}{}^{z}\\
-F_{trr}\Gamma^{r}{}_{r}{}^{t}-F_{rrr}\Gamma^{r}{}_{r}{}^{r}-F_{\phi rr}\Gamma^{r}{}_{r}{}^{\phi}-F_{zrr}\Gamma^{r}{}_{r}{}^{z}\\
-F_{tr\phi}\Gamma^{\phi}{}_{r}{}^{t}-F_{rr\phi}\Gamma^{\phi}{}_{r}{}^{r}-F_{\phi r\phi}\Gamma^{\phi}{}_{r}{}^{\phi}-F_{zr\phi}\Gamma^{\phi}{}_{r}{}^{z}\\
-F_{trz}\Gamma^{z}{}_{r}{}^{t}-F_{rrz}\Gamma^{z}{}_{r}{}^{r}-F_{\phi rz}\Gamma^{z}{}_{r}{}^{\phi}-F_{zrz}\Gamma^{z}{}_{r}{}^{z}\\
=0\ . \qquad (89.286)
\end{aligned}
$$

For $(\beta,\gamma)=(\phi,\phi)$, we have

$$
\begin{aligned}
g^{tt}F_{t\phi\phi,t}+g^{\phi t}F_{\phi\phi\phi,t}+g^{rr}F_{r\phi\phi,r}+g^{t\phi}F_{t\phi\phi,\phi}+g^{\phi\phi}F_{\phi\phi\phi,\phi}+g^{zz}F_{z\phi\phi,z}\\
-F_{t\phi\phi}\Gamma^{t}{}_{\mu}{}^{\mu}-F_{r\phi\phi}\Gamma^{r}{}_{\mu}{}^{\mu}-F_{\phi\phi\phi}\Gamma^{\phi}{}_{\mu}{}^{\mu}-F_{z\phi\phi}\Gamma^{z}{}_{\mu}{}^{\mu}\\
-F_{tt\phi}\Gamma^{t}{}_{\phi}{}^{t}-F_{rt\phi}\Gamma^{t}{}_{\phi}{}^{r}-F_{\phi t\phi}\Gamma^{t}{}_{\phi}{}^{\phi}-F_{zt\phi}\Gamma^{t}{}_{\phi}{}^{z}\\
-F_{tr\phi}\Gamma^{r}{}_{\phi}{}^{t}-F_{rr\phi}\Gamma^{r}{}_{\phi}{}^{r}-F_{\phi r\phi}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{zr\phi}\Gamma^{r}{}_{\phi}{}^{z}\\
-F_{t\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{t}-F_{r\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{r}-F_{\phi\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{\phi}-F_{z\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{z}\\
-F_{tz\phi}\Gamma^{z}{}_{\phi}{}^{t}-F_{rz\phi}\Gamma^{z}{}_{\phi}{}^{r}-F_{\phi z\phi}\Gamma^{z}{}_{\phi}{}^{\phi}-F_{zz\phi}\Gamma^{z}{}_{\phi}{}^{z}\\
-F_{t\phi t}\Gamma^{t}{}_{\phi}{}^{t}-F_{r\phi t}\Gamma^{t}{}_{\phi}{}^{r}-F_{\phi\phi t}\Gamma^{t}{}_{\phi}{}^{\phi}-F_{z\phi t}\Gamma^{t}{}_{\phi}{}^{z}\\
-F_{t\phi r}\Gamma^{r}{}_{\phi}{}^{t}-F_{r\phi r}\Gamma^{r}{}_{\phi}{}^{r}-F_{\phi\phi r}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{z\phi r}\Gamma^{r}{}_{\phi}{}^{z}\\
-F_{t\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{t}-F_{r\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{r}-F_{\phi\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{\phi}-F_{z\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{z}\\
-F_{t\phi z}\Gamma^{z}{}_{\phi}{}^{t}-F_{r\phi z}\Gamma^{z}{}_{\phi}{}^{r}-F_{\phi\phi z}\Gamma^{z}{}_{\phi}{}^{\phi}-F_{z\phi z}\Gamma^{z}{}_{\phi}{}^{z}\\
=0\ . \qquad (89.287)
\end{aligned}
$$

For $(\beta,\gamma)=(t,\phi)$, we have

$$
\begin{aligned}
g^{tt}F_{tt\phi,t}+g^{\phi t}F_{\phi t\phi,t}+g^{rr}F_{rt\phi,r}+g^{t\phi}F_{tt\phi,\phi}+g^{\phi\phi}F_{\phi t\phi,\phi}+g^{zz}F_{zt\phi,z}\\
-F_{tt\phi}\Gamma^{t}{}_{\mu}{}^{\mu}-F_{rt\phi}\Gamma^{r}{}_{\mu}{}^{\mu}-F_{\phi t\phi}\Gamma^{\phi}{}_{\mu}{}^{\mu}-F_{zt\phi}\Gamma^{z}{}_{\mu}{}^{\mu}\\
-F_{tt\phi}\Gamma^{t}{}_{t}{}^{t}-F_{rt\phi}\Gamma^{t}{}_{t}{}^{r}-F_{\phi t\phi}\Gamma^{t}{}_{t}{}^{\phi}-F_{zt\phi}\Gamma^{t}{}_{t}{}^{z}\\
-F_{tr\phi}\Gamma^{r}{}_{t}{}^{t}-F_{rr\phi}\Gamma^{r}{}_{t}{}^{r}-F_{\phi r\phi}\Gamma^{r}{}_{t}{}^{\phi}-F_{zr\phi}\Gamma^{r}{}_{t}{}^{z}\\
-F_{t\phi\phi}\Gamma^{\phi}{}_{t}{}^{t}-F_{r\phi\phi}\Gamma^{\phi}{}_{t}{}^{r}-F_{\phi\phi\phi}\Gamma^{\phi}{}_{t}{}^{\phi}-F_{z\phi\phi}\Gamma^{\phi}{}_{t}{}^{z}\\
-F_{tz\phi}\Gamma^{z}{}_{t}{}^{t}-F_{rz\phi}\Gamma^{z}{}_{t}{}^{r}-F_{\phi z\phi}\Gamma^{z}{}_{t}{}^{\phi}-F_{zz\phi}\Gamma^{z}{}_{t}{}^{z}\\
-F_{ttt}\Gamma^{t}{}_{\phi}{}^{t}-F_{rtt}\Gamma^{t}{}_{\phi}{}^{r}-F_{\phi tt}\Gamma^{t}{}_{\phi}{}^{\phi}-F_{ztt}\Gamma^{t}{}_{\phi}{}^{z}\\
-F_{ttr}\Gamma^{r}{}_{\phi}{}^{t}-F_{rtr}\Gamma^{r}{}_{\phi}{}^{r}-F_{\phi tr}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{ztr}\Gamma^{r}{}_{\phi}{}^{z}\\
-F_{tt\phi}\Gamma^{\phi}{}_{\phi}{}^{t}-F_{rt\phi}\Gamma^{\phi}{}_{\phi}{}^{r}-F_{\phi t\phi}\Gamma^{\phi}{}_{\phi}{}^{\phi}-F_{zt\phi}\Gamma^{\phi}{}_{\phi}{}^{z}\\
-F_{ttz}\Gamma^{z}{}_{\phi}{}^{t}-F_{rtz}\Gamma^{z}{}_{\phi}{}^{r}-F_{\phi tz}\Gamma^{z}{}_{\phi}{}^{\phi}-F_{ztz}\Gamma^{z}{}_{\phi}{}^{z}\\
=0\ . \qquad (89.288)
\end{aligned}
$$

For $(\beta, \gamma) = (r, \phi)$, we have

$$\begin{aligned}
g^{tt}F_{tr\phi,t} + g^{\phi t}F_{\phi r\phi,t} + g^{rr}F_{rr\phi,r} + g^{t\phi}F_{tr\phi,\phi} + g^{\phi\phi}F_{\phi r\phi,\phi} + g^{zz}F_{zr\phi,z} \\
-F_{tr\phi}\Gamma^{t}{}_{\mu}{}^{\mu} - F_{rr\phi}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi r\phi}\Gamma^{\phi}{}_{\mu}{}^{\mu} - F_{zr\phi}\Gamma^{z}{}_{\mu}{}^{\mu} \\
-F_{tt\phi}\Gamma^{t}{}_{r}{}^{t} - F_{rt\phi}\Gamma^{t}{}_{r}{}^{r} - F_{\phi t\phi}\Gamma^{t}{}_{r}{}^{\phi} - F_{zt\phi}\Gamma^{t}{}_{r}{}^{z} \\
-F_{tr\phi}\Gamma^{r}{}_{r}{}^{t} - F_{rr\phi}\Gamma^{r}{}_{r}{}^{r} - F_{\phi r\phi}\Gamma^{r}{}_{r}{}^{\phi} - F_{zr\phi}\Gamma^{r}{}_{r}{}^{z} \\
-F_{t\phi\phi}\Gamma^{\phi}{}_{r}{}^{t} - F_{r\phi\phi}\Gamma^{\phi}{}_{r}{}^{r} - F_{\phi\phi\phi}\Gamma^{\phi}{}_{r}{}^{\phi} - F_{z\phi\phi}\Gamma^{\phi}{}_{r}{}^{z} \\
-F_{tz\phi}\Gamma^{z}{}_{r}{}^{t} - F_{rz\phi}\Gamma^{z}{}_{r}{}^{r} - F_{\phi z\phi}\Gamma^{z}{}_{r}{}^{\phi} - F_{zz\phi}\Gamma^{z}{}_{r}{}^{z} \\
-F_{trt}\Gamma^{t}{}_{\phi}{}^{t} - F_{rrt}\Gamma^{t}{}_{\phi}{}^{r} - F_{\phi rt}\Gamma^{t}{}_{\phi}{}^{\phi} - F_{zrt}\Gamma^{t}{}_{\phi}{}^{z} \\
-F_{trr}\Gamma^{r}{}_{\phi}{}^{t} - F_{rrr}\Gamma^{r}{}_{\phi}{}^{r} - F_{\phi rr}\Gamma^{r}{}_{\phi}{}^{\phi} - F_{zrr}\Gamma^{r}{}_{\phi}{}^{z} \\
-F_{tr\phi}\Gamma^{\phi}{}_{\phi}{}^{t} - F_{rr\phi}\Gamma^{\phi}{}_{\phi}{}^{r} - F_{\phi r\phi}\Gamma^{\phi}{}_{\phi}{}^{\phi} - F_{zr\phi}\Gamma^{\phi}{}_{\phi}{}^{z} \\
-F_{trz}\Gamma^{z}{}_{\phi}{}^{t} - F_{rrz}\Gamma^{z}{}_{\phi}{}^{r} - F_{\phi rz}\Gamma^{z}{}_{\phi}{}^{\phi} - F_{zrz}\Gamma^{z}{}_{\phi}{}^{z} \\
= 0\ .
\end{aligned} \tag{89.289}$$

For $(\beta, \gamma) = (r, t)$, we have

$$\begin{aligned}
g^{tt}F_{trt,t} + g^{\phi t}F_{\phi rt,t} + g^{rr}F_{rrt,r} + g^{t\phi}F_{trt,\phi} + g^{\phi\phi}F_{\phi rt,\phi} + g^{zz}F_{zrt,z} \\
-F_{trt}\Gamma^{t}{}_{\mu}{}^{\mu} - F_{rrt}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi rt}\Gamma^{\phi}{}_{\mu}{}^{\mu} - F_{zrt}\Gamma^{z}{}_{\mu}{}^{\mu} \\
-F_{ttt}\Gamma^{t}{}_{r}{}^{t} - F_{rtt}\Gamma^{t}{}_{r}{}^{r} - F_{\phi tt}\Gamma^{t}{}_{r}{}^{\phi} - F_{ztt}\Gamma^{t}{}_{r}{}^{z} \\
-F_{trt}\Gamma^{r}{}_{r}{}^{t} - F_{rrt}\Gamma^{r}{}_{r}{}^{r} - F_{\phi rt}\Gamma^{r}{}_{r}{}^{\phi} - F_{zrt}\Gamma^{r}{}_{r}{}^{z} \\
-F_{t\phi t}\Gamma^{\phi}{}_{r}{}^{t} - F_{r\phi t}\Gamma^{\phi}{}_{r}{}^{r} - F_{\phi\phi t}\Gamma^{\phi}{}_{r}{}^{\phi} - F_{z\phi t}\Gamma^{\phi}{}_{r}{}^{z} \\
-F_{tzt}\Gamma^{z}{}_{r}{}^{t} - F_{rzt}\Gamma^{z}{}_{r}{}^{r} - F_{\phi zt}\Gamma^{z}{}_{r}{}^{\phi} - F_{zzt}\Gamma^{z}{}_{r}{}^{z} \\
-F_{trt}\Gamma^{t}{}_{t}{}^{t} - F_{rrt}\Gamma^{t}{}_{t}{}^{r} - F_{\phi rt}\Gamma^{t}{}_{t}{}^{\phi} - F_{zrt}\Gamma^{t}{}_{t}{}^{z} \\
-F_{trr}\Gamma^{r}{}_{t}{}^{t} - F_{rrr}\Gamma^{r}{}_{t}{}^{r} - F_{\phi rr}\Gamma^{r}{}_{t}{}^{\phi} - F_{zrr}\Gamma^{r}{}_{t}{}^{z} \\
-F_{tr\phi}\Gamma^{\phi}{}_{t}{}^{t} - F_{rr\phi}\Gamma^{\phi}{}_{t}{}^{r} - F_{\phi r\phi}\Gamma^{\phi}{}_{t}{}^{\phi} - F_{zr\phi}\Gamma^{\phi}{}_{t}{}^{z} \\
-F_{trz}\Gamma^{z}{}_{t}{}^{t} - F_{rrz}\Gamma^{z}{}_{t}{}^{r} - F_{\phi rz}\Gamma^{z}{}_{t}{}^{\phi} - F_{zrz}\Gamma^{z}{}_{t}{}^{z} \\
= 0\ .
\end{aligned} \tag{89.290}$$

To continue calculating, we need to write down the metric used to lower indexes in a frame that is rotating with angular velocity ω about the z-axis of flat Minkowski frame. In cylindrical coordinates with indexes (t, r, ϕ, z), we have[27, Adler, Bazin, Schiffer, eq. (4.83), p. 112] and [257, Klauber, 2001]

$$g_{\mu\nu} = \begin{pmatrix} -1+\omega^2 r^2 & 0 & r^2\omega & 0 \\ 0 & 1 & 0 & 0 \\ r^2\omega & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ . \tag{89.291}$$

The metric used to raise indexes is

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & \omega & 0 \\ 0 & 1 & 0 & 0 \\ \omega & 0 & \frac{1}{r^2}-\omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ . \tag{89.292}$$

We also need the non-zero connections. These are [27, Adler, Bazin, Schiffer, pp. 118-120] and (89.13) and (89.14)

$$\Gamma_{rtt} = \Gamma^{r}{}_{tt} = -\omega^2 r\ . \tag{89.293}$$

$$\Gamma_{ttr} = \Gamma_{trt} = \Gamma_{tt}{}^{r} = \Gamma_{t}{}^{r}{}_{t} = +\omega^2 r \ . \tag{89.294}$$

$$\Gamma_{r\phi\phi} = \Gamma^{r}{}_{\phi\phi} = -r \ . \tag{89.295}$$

$$\Gamma_{\phi\phi r} = \Gamma_{\phi r\phi} = \Gamma_{\phi\phi}{}^{r} = \Gamma_{\phi}{}^{r}{}_{\phi} = +r \ . \tag{89.296}$$

$$\Gamma_{rt\phi} = \Gamma_{r\phi t} = \Gamma^{r}{}_{t\phi} = \Gamma^{r}{}_{\phi t} = -\omega r \ . \tag{89.297}$$

$$\Gamma_{tr\phi} = \Gamma_{t\phi r} = \Gamma_{\phi rt} = \Gamma_{\phi tr} = \Gamma_{t}{}^{r}{}_{\phi} = \Gamma_{t\phi}{}^{r} = \Gamma_{\phi}{}^{r}{}_{t} = \Gamma_{\phi t}{}^{r} = +\omega r \ . \tag{89.298}$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \Gamma^{\phi r}{}_{\phi} = \Gamma^{\phi}{}_{\phi}{}^{r} = \frac{1}{r} \ . \tag{89.299}$$

$$\Gamma^{\phi}{}_{rt} = \Gamma^{\phi}{}_{tr} = \Gamma^{\phi r}{}_{t} = \Gamma^{\phi}{}_{t}{}^{r} = \frac{\omega}{r} \ . \tag{89.300}$$

We can use (89.292) to raise indexes to get the following connections:

$$\Gamma^{r}{}_{t}{}^{t} = \Gamma^{rt}{}_{t} = \Gamma^{tr}{}_{t} = \Gamma^{tt}{}_{r} = \Gamma^{r}{}_{t}{}^{r} = \Gamma^{rr}{}_{t} = 0 \ . \tag{89.301}$$

$$\Gamma^{r}{}_{t}{}^{\phi} = \Gamma^{r\phi}{}_{t} = -\frac{\omega}{r} \ . \tag{89.302}$$

$$\Gamma^{\phi}{}_{t}{}^{r} = \Gamma^{\phi r}{}_{t} = \frac{\omega}{r} \ . \tag{89.303}$$

$$\Gamma^{\phi}{}_{r}{}^{\phi} = \Gamma^{\phi\phi}{}_{r} = \frac{1}{r^3} \ . \tag{89.304}$$

$$\Gamma^{\phi t}{}_{\phi} = \Gamma^{t\phi}{}_{\phi} = \Gamma^{rt}{}_{\phi} = \Gamma^{tr}{}_{\phi} = \Gamma^{r}{}_{\phi}{}^{t} = \Gamma^{\phi\phi}{}_{t} = \Gamma^{\phi t}{}_{t} = \Gamma^{t\phi}{}_{t} = \Gamma^{\phi t}{}_{r} = \Gamma^{t\phi}{}_{r} = 0 \ . \tag{89.305}$$

$$\Gamma^{\phi}{}_{\phi}{}^{r} = \Gamma^{\phi r}{}_{\phi} = \frac{1}{r} \ . \tag{89.306}$$

$$\Gamma^{r}{}_{\phi}{}^{\phi} = \Gamma^{r\phi}{}_{\phi} = -\frac{1}{r} \ . \tag{89.307}$$

In addition, we have the following contractions:

$$\Gamma^{r}{}_{\mu}{}^{\mu} = \Gamma^{r\mu}{}_{\mu} = -\frac{1}{r} \ . \tag{89.308}$$

$$\Gamma^{\alpha r}{}_{\alpha} = \frac{1}{r} \ . \tag{89.309}$$

$$\Gamma^{\phi}{}_{\mu}{}^{\mu} = \Gamma^{\phi\mu}{}_{\mu} = 0 \ . \tag{89.310}$$

$F = -\Gamma$ satisfies (89.250), (89.252), and 89.254 if we take

$$\frac{8\pi}{3}\left(Y_{\beta\gamma} - \frac{1}{2}g_{\beta\gamma}Y\right) = \begin{pmatrix} \omega^2 & 0 & -2\omega & 0 \\ 0 & 0 & 0 & 0 \\ -2\omega & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{89.311}$$

However, any frame for which $F = -\Gamma$ must be a frame defined by a congruence of geodesics. In such a frame, $\Gamma = 0$, so that means that $\omega = 0$. And in that case, the equation is satisfied, although that is not an interesting case. To check further, we must take a case where $F \neq -\Gamma$.

We can use (89.301) through (89.310) to write (89.285) through (89.290) keeping only non-zero terms. For $(\beta,\gamma)=(t,t)$, we have

$$\begin{aligned} g^{tt}F_{ttt,t}+g^{\phi t}F_{\phi tt,t}+g^{rr}F_{rtt,r}+g^{t\phi}F_{ttt,\phi}+g^{\phi\phi}F_{\phi tt,\phi}+g^{zz}F_{ztt,z} \\ -F_{rtt}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi rt}\Gamma^{r}{}_{t}{}^{\phi} \\ -F_{r\phi t}\Gamma^{\phi}{}_{t}{}^{r} \\ -F_{\phi tr}\Gamma^{r}{}_{t}{}^{\phi} \\ -F_{rt\phi}\Gamma^{\phi}{}_{t}{}^{r} \\ =0\ . \end{aligned} \tag{89.312}$$

For $(\beta,\gamma)=(r,r)$, we have

$$\begin{aligned} g^{tt}F_{trr,t}+g^{\phi t}F_{\phi rr,t}+g^{rr}F_{rrr,r}+g^{t\phi}F_{trr,\phi}+g^{\phi\phi}F_{\phi rr,\phi}+g^{zz}F_{zrr,z} \\ -F_{rrr}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi\phi r}\Gamma^{\phi}{}_{r}{}^{\phi} \\ -F_{\phi r\phi}\Gamma^{\phi}{}_{r}{}^{\phi} \\ =0\ . \end{aligned} \tag{89.313}$$

For $(\beta,\gamma)=(\phi,\phi)$, we have

$$\begin{aligned} g^{tt}F_{t\phi\phi,t}+g^{\phi t}F_{\phi\phi\phi,t}+g^{rr}F_{r\phi\phi,r}+g^{t\phi}F_{t\phi\phi,\phi}+g^{\phi\phi}F_{\phi\phi\phi,\phi}+g^{zz}F_{z\phi\phi,z} \\ -F_{r\phi\phi}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi r\phi}\Gamma^{r}{}_{\phi}{}^{\phi} \\ -F_{\phi\phi r}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{r\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{r}-F_{r\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{r} \\ =0\ . \end{aligned} \tag{89.314}$$

For $(\beta,\gamma)=(t,\phi)$, we have

$$\begin{aligned} g^{tt}F_{tt\phi,t}+g^{\phi t}F_{\phi t\phi,t}+g^{rr}F_{rt\phi,r}+g^{t\phi}F_{tt\phi,\phi}+g^{\phi\phi}F_{\phi t\phi,\phi}+g^{zz}F_{zt\phi,z} \\ -F_{rt\phi}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi r\phi}\Gamma^{r}{}_{t}{}^{\phi} \\ -F_{r\phi\phi}\Gamma^{\phi}{}_{t}{}^{r} \\ -F_{\phi tr}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{rt\phi}\Gamma^{\phi}{}_{\phi}{}^{r} \\ =0\ . \end{aligned} \tag{89.315}$$

For $(\beta,\gamma)=(r,\phi)$, we have

$$\begin{aligned} g^{tt}F_{tr\phi,t}+g^{\phi t}F_{\phi r\phi,t}+g^{rr}F_{rr\phi,r}+g^{t\phi}F_{tr\phi,\phi}+g^{\phi\phi}F_{\phi r\phi,\phi}+g^{zz}F_{zr\phi,z} \\ -F_{rr\phi}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi\phi\phi}\Gamma^{\phi}{}_{r}{}^{\phi} \\ -F_{\phi rr}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{rr\phi}\Gamma^{\phi}{}_{\phi}{}^{r} \\ =0\ . \end{aligned} \tag{89.316}$$

For $(\beta,\gamma)=(r,t)$, we have

$$\begin{aligned} g^{tt}F_{trt,t}+g^{\phi t}F_{\phi rt,t}+g^{rr}F_{rrt,r}+g^{t\phi}F_{trt,\phi}+g^{\phi\phi}F_{\phi rt,\phi}+g^{zz}F_{zrt,z} \\ -F_{rrt}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi\phi t}\Gamma^{\phi}{}_{r}{}^{\phi} \\ -F_{\phi rr}\Gamma^{r}{}_{t}{}^{\phi} \\ -F_{rr\phi}\Gamma^{\phi}{}_{t}{}^{r} \\ =0\ . \end{aligned} \tag{89.317}$$

The symmetry (89.39) allows us to combine some terms to give the following 6 equations. For $(\beta,\gamma)=(t,t)$, we have

$$\begin{aligned} g^{tt}F_{ttt,t}+g^{\phi t}F_{\phi tt,t}+g^{rr}F_{rtt,r}+g^{t\phi}F_{ttt,\phi}+g^{\phi\phi}F_{\phi tt,\phi}+g^{zz}F_{ztt,z} \\ -F_{rtt}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi rt}\Gamma^{r}{}_{t}{}^{\phi}-F_{r\phi t}\Gamma^{\phi}{}_{t}{}^{r}-F_{\phi tr}\Gamma^{r}{}_{t}{}^{\phi}-F_{rt\phi}\Gamma^{\phi}{}_{t}{}^{r} \\ =0\ . \end{aligned} \tag{89.318}$$

For $(\beta,\gamma)=(r,r)$, we have

$$\begin{aligned} g^{tt}F_{trr,t}+g^{\phi t}F_{\phi rr,t}+g^{rr}F_{rrr,r}+g^{t\phi}F_{trr,\phi}+g^{\phi\phi}F_{\phi rr,\phi}+g^{zz}F_{zrr,z} \\ -F_{rrr}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi\phi r}\Gamma^{\phi}{}_{r}{}^{\phi}-F_{\phi r\phi}\Gamma^{\phi}{}_{r}{}^{\phi} \\ =0\ . \end{aligned} \tag{89.319}$$

For $(\beta,\gamma)=(\phi,\phi)$, we have

$$\begin{aligned} g^{tt}F_{t\phi\phi,t}+g^{\phi t}F_{\phi\phi\phi,t}+g^{rr}F_{r\phi\phi,r}+g^{t\phi}F_{t\phi\phi,\phi}+g^{\phi\phi}F_{\phi\phi\phi,\phi}+g^{zz}F_{z\phi\phi,z} \\ -F_{r\phi\phi}\Gamma^{r}{}_{\mu}{}^{\mu}-F_{\phi r\phi}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{\phi\phi r}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{r\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{r}-F_{r\phi\phi}\Gamma^{\phi}{}_{\phi}{}^{r} \\ =0\ . \end{aligned} \tag{89.320}$$

For $(\beta,\gamma)=(t,\phi)$, we have

$$\begin{aligned} g^{tt}F_{tt\phi,t}+g^{\phi t}F_{\phi t\phi,t}+g^{rr}F_{rt\phi,r}+g^{t\phi}F_{tt\phi,\phi}+g^{\phi\phi}F_{\phi t\phi,\phi}+g^{zz}F_{zt\phi,z} \\ -F_{rt\phi}\Gamma^{r}{}_{\mu}{}^{\mu}-F_{\phi tr}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{rt\phi}\Gamma^{\phi}{}_{\phi}{}^{r} \\ -F_{\phi r\phi}\Gamma^{r}{}_{t}{}^{\phi}-F_{r\phi\phi}\Gamma^{\phi}{}_{t}{}^{r} \\ =0\ . \end{aligned} \tag{89.321}$$

For $(\beta,\gamma)=(r,\phi)$, we have

$$\begin{aligned} g^{tt}F_{tr\phi,t}+g^{\phi t}F_{\phi r\phi,t}+g^{rr}F_{rr\phi,r}+g^{t\phi}F_{tr\phi,\phi}+g^{\phi\phi}F_{\phi r\phi,\phi}+g^{zz}F_{zr\phi,z} \\ -F_{rr\phi}\Gamma^{r}{}_{\mu}{}^{\mu}-F_{\phi rr}\Gamma^{r}{}_{\phi}{}^{\phi}-F_{rr\phi}\Gamma^{\phi}{}_{\phi}{}^{r} \\ -F_{\phi\phi\phi}\Gamma^{\phi}{}_{r}{}^{\phi} \\ =0\ . \end{aligned} \tag{89.322}$$

For $(\beta,\gamma)=(r,t)$, we have

$$\begin{aligned} g^{tt}F_{trt,t}+g^{\phi t}F_{\phi rt,t}+g^{rr}F_{rrt,r}+g^{t\phi}F_{trt,\phi}+g^{\phi\phi}F_{\phi rt,\phi}+g^{zz}F_{zrt,z} \\ -F_{rrt}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi\phi t}\Gamma^{\phi}{}_{r}{}^{\phi} \\ -F_{\phi rr}\Gamma^{r}{}_{t}{}^{\phi}-F_{rr\phi}\Gamma^{\phi}{}_{t}{}^{r} \\ =0\ . \end{aligned} \tag{89.323}$$

Or, setting derivatives with respect to z to zero gives the following 6 equations. For $(\beta,\gamma) = (t,t)$, we have

$$\begin{aligned} g^{tt}F_{ttt,t} + g^{\phi t}F_{\phi tt,t} + g^{rr}F_{rtt,r} + g^{t\phi}F_{ttt,\phi} + g^{\phi\phi}F_{\phi tt,\phi} \\ -F_{rtt}\Gamma^{r}{}_{\mu}{}^{\mu} \\ -F_{\phi rt}\left(\Gamma^{r}{}_{t}{}^{\phi} + \Gamma^{\phi}{}_{t}{}^{r} + \Gamma^{r}{}_{t}{}^{\phi} + \Gamma^{\phi}{}_{t}{}^{r}\right) = 0 \ . \end{aligned} \tag{89.324}$$

For $(\beta,\gamma) = (r,r)$, we have

$$\begin{aligned} g^{tt}F_{trr,t} + g^{\phi t}F_{\phi rr,t} + g^{rr}F_{rrr,r} + g^{t\phi}F_{trr,\phi} + g^{\phi\phi}F_{\phi rr,\phi} \\ -F_{rrr}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\phi r}\left(\Gamma^{\phi}{}_{r}{}^{\phi} + \Gamma^{\phi}{}_{r}{}^{\phi}\right) = 0 \ . \end{aligned} \tag{89.325}$$

For $(\beta,\gamma) = (\phi,\phi)$, we have

$$\begin{aligned} g^{tt}F_{t\phi\phi,t} + g^{\phi t}F_{\phi\phi\phi,t} + g^{rr}F_{r\phi\phi,r} + g^{t\phi}F_{t\phi\phi,\phi} + g^{\phi\phi}F_{\phi\phi\phi,\phi} \\ -F_{r\phi\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) = 0 \ . \end{aligned} \tag{89.326}$$

For $(\beta,\gamma) = (t,\phi)$, we have

$$\begin{aligned} g^{tt}F_{tt\phi,t} + g^{\phi t}F_{\phi t\phi,t} + g^{rr}F_{rt\phi,r} + g^{t\phi}F_{tt\phi,\phi} + g^{\phi\phi}F_{\phi t\phi,\phi} \\ -F_{rt\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) - F_{\phi r\phi}\left(\Gamma^{r}{}_{t}{}^{\phi} + \Gamma^{\phi}{}_{t}{}^{r}\right) = 0 \ . \end{aligned} \tag{89.327}$$

For $(\beta,\gamma) = (r,\phi)$, we have

$$\begin{aligned} g^{tt}F_{tr\phi,t} + g^{\phi t}F_{\phi r\phi,t} + g^{rr}F_{rr\phi,r} + g^{t\phi}F_{tr\phi,\phi} + g^{\phi\phi}F_{\phi r\phi,\phi} \\ -F_{rr\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) - F_{\phi\phi\phi}\Gamma^{\phi}{}_{r}{}^{\phi} = 0 \ . \end{aligned} \tag{89.328}$$

For $(\beta,\gamma) = (r,t)$, we have

$$\begin{aligned} g^{tt}F_{trt,t} + g^{\phi t}F_{\phi rt,t} + g^{rr}F_{rrt,r} + g^{t\phi}F_{trt,\phi} + g^{\phi\phi}F_{\phi rt,\phi} \\ -F_{rrt}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\phi t}\Gamma^{\phi}{}_{r}{}^{\phi} - F_{\phi rr}\left(\Gamma^{r}{}_{t}{}^{\phi} + \Gamma^{\phi}{}_{t}{}^{r}\right) = 0 \ . \end{aligned} \tag{89.329}$$

Canceling some terms gives the following 6 equations. For $(\beta,\gamma) = (t,t)$, we have

$$\begin{aligned} g^{tt}F_{ttt,t} + g^{\phi t}F_{\phi tt,t} + g^{rr}F_{rtt,r} + g^{t\phi}F_{ttt,\phi} + g^{\phi\phi}F_{\phi tt,\phi} \\ -F_{rtt}\Gamma^{r}{}_{\mu}{}^{\mu} = 0 \ . \end{aligned} \tag{89.330}$$

For $(\beta,\gamma) = (r,r)$, we have

$$\begin{aligned} g^{tt}F_{trr,t} + g^{\phi t}F_{\phi rr,t} + g^{rr}F_{rrr,r} + g^{t\phi}F_{trr,\phi} + g^{\phi\phi}F_{\phi rr,\phi} \\ -F_{rrr}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\phi r}\left(\Gamma^{\phi}{}_{r}{}^{\phi} + \Gamma^{\phi}{}_{r}{}^{\phi}\right) = 0 \ . \end{aligned} \tag{89.331}$$

For $(\beta,\gamma) = (\phi,\phi)$, we have

$$\begin{aligned} g^{tt}F_{t\phi\phi,t} + g^{\phi t}F_{\phi\phi\phi,t} + g^{rr}F_{r\phi\phi,r} + g^{t\phi}F_{t\phi\phi,\phi} + g^{\phi\phi}F_{\phi\phi\phi,\phi} \\ -F_{r\phi\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) = 0 \ . \end{aligned} \tag{89.332}$$

For $(\beta,\gamma) = (t,\phi)$, we have

$$\begin{aligned} g^{tt}F_{tt\phi,t} + g^{\phi t}F_{\phi t\phi,t} + g^{rr}F_{rt\phi,r} + g^{t\phi}F_{tt\phi,\phi} + g^{\phi\phi}F_{\phi t\phi,\phi} \\ -F_{rt\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) = 0 \ . \end{aligned} \tag{89.333}$$

For $(\beta, \gamma) = (r, \phi)$, we have

$$g^{tt}F_{tr\phi,t} + g^{\phi t}F_{\phi r\phi,t} + g^{rr}F_{rr\phi,r} + g^{t\phi}F_{tr\phi,\phi} + g^{\phi\phi}F_{\phi r\phi,\phi} \\ -F_{rr\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) - F_{\phi\phi\phi}\Gamma^{\phi}{}_{r}{}^{\phi} = 0 \; . \tag{89.334}$$

For $(\beta, \gamma) = (r, t)$, we have

$$g^{tt}F_{trt,t} + g^{\phi t}F_{\phi rt,t} + g^{rr}F_{rrt,r} + g^{t\phi}F_{trt,\phi} + g^{\phi\phi}F_{\phi rt,\phi} \\ -F_{rrt}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\phi t}\Gamma^{\phi}{}_{r}{}^{\phi} = 0 \; . \tag{89.335}$$

Using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) allows us to combine some equal terms to give the following 9 equations. For $(\beta, \gamma) = (t, t)$, we have

$$g^{tt}F_{ttt,t} + g^{rr}F_{rtr,t} + 2g^{t\phi}F_{tt\phi,t} + g^{\phi\phi}F_{\phi t\phi,t} - F_{rtt}\Gamma^{r}{}_{\mu}{}^{\mu} = 0 \; . \tag{89.336}$$

For $(\beta, \gamma) = (r, r)$, we have

$$g^{tt}F_{trt,r} + g^{rr}F_{rrr,r} + 2g^{t\phi}F_{tr\phi,r} + g^{\phi\phi}F_{\phi r\phi,r} - F_{rrr}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\phi r}\left(\Gamma^{\phi}{}_{r}{}^{\phi} + \Gamma^{\phi}{}_{r}{}^{\phi}\right) = 0 \; . \tag{89.337}$$

For $(\beta, \gamma) = (\phi, \phi)$, we have

$$g^{tt}F_{t\phi t,\phi} + g^{rr}F_{r\phi r,\phi} + 2g^{t\phi}F_{t\phi\phi,\phi} + g^{\phi\phi}F_{\phi\phi\phi,\phi} - F_{r\phi\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) = 0 \; . \tag{89.338}$$

For $(\beta, \gamma) = (t, \phi)$, we have

$$g^{tt}F_{tt t,\phi} + g^{rr}F_{rrt,\phi} + 2g^{t\phi}F_{t\phi t,\phi} + g^{\phi\phi}F_{\phi\phi t,\phi} - F_{rt\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) = 0 \tag{89.339}$$

and

$$g^{tt}F_{tt\phi,t} + g^{rr}F_{rr\phi,t} + 2g^{t\phi}F_{t\phi\phi,t} + g^{\phi\phi}F_{\phi\phi\phi,t} - F_{rt\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) = 0 \; . \tag{89.340}$$

For $(\beta, \gamma) = (r, \phi)$, we have

$$g^{tt}F_{ttr,\phi} + g^{rr}F_{rrr,\phi} + 2g^{t\phi}F_{t\phi r,\phi} + g^{\phi\phi}F_{\phi\phi r,\phi} - F_{rr\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) - F_{\phi\phi\phi}\Gamma^{\phi}{}_{r}{}^{\phi} = 0 \tag{89.341}$$

and

$$g^{tt}F_{tt\phi,r} + g^{rr}F_{rr\phi,r} + 2g^{t\phi}F_{t\phi\phi,r} + g^{\phi\phi}F_{\phi\phi\phi,r} - F_{rr\phi}\left(\Gamma^{r}{}_{\mu}{}^{\mu} + \Gamma^{r}{}_{\phi}{}^{\phi} + \Gamma^{\phi}{}_{\phi}{}^{r}\right) - F_{\phi\phi\phi}\Gamma^{\phi}{}_{r}{}^{\phi} = 0 \; . \tag{89.342}$$

For $(\beta, \gamma) = (r, t)$, we have

$$g^{tt}F_{ttr,t} + g^{rr}F_{rrr,t} + 2g^{t\phi}F_{t\phi r,t} + g^{\phi\phi}F_{\phi\phi r,t} - F_{rrt}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\phi t}\Gamma^{\phi}{}_{r}{}^{\phi} = 0 \; . \tag{89.343}$$

and

$$g^{tt}F_{ttt,r} + g^{rr}F_{rrt,r} + 2g^{t\phi}F_{t\phi t,r} + g^{\phi\phi}F_{\phi\phi t,r} - F_{rrt}\Gamma^{r}{}_{\mu}{}^{\mu} - F_{\phi\phi t}\Gamma^{\phi}{}_{r}{}^{\phi} = 0 \; . \tag{89.344}$$

We can use (89.292) and (89.301) through (89.310) to write (89.336) through (89.344) as the following 9 equations. For $(\beta, \gamma) = (t, t)$, we have

$$-F_{ttt,t} + F_{rtr,t} + 2\omega F_{tt\phi,t} + \left(\frac{1}{r^2} - \omega^2\right) F_{\phi t\phi,t} + \frac{1}{r}F_{rtt} = 0 \; . \tag{89.345}$$

For $(\beta,\gamma)=(r,r)$, we have

$$-F_{trt,r}+F_{rrr,r}+2\omega F_{tr\phi,r}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi r\phi,r}+\frac{1}{r}F_{rrr}-\frac{2}{r^3}F_{\phi\phi r}=0\ . \tag{89.346}$$

For $(\beta,\gamma)=(\phi,\phi)$, we have

$$-F_{t\phi t,\phi}+F_{r\phi r,\phi}+2\omega F_{t\phi\phi,\phi}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi\phi,\phi}+\frac{1}{r}F_{r\phi\phi}=0\ . \tag{89.347}$$

For $(\beta,\gamma)=(t,\phi)$, we have

$$-F_{ttt,\phi}+F_{rrt,\phi}+2\omega F_{t\phi t,\phi}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi t,\phi}+\frac{1}{r}F_{rt\phi}=0 \tag{89.348}$$

and

$$-F_{tt\phi,t}+F_{rr\phi,t}+2\omega F_{t\phi\phi,t}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi\phi,t}+\frac{1}{r}F_{rt\phi}=0\ . \tag{89.349}$$

For $(\beta,\gamma)=(r,\phi)$, we have

$$-F_{ttr,\phi}+F_{rrr,\phi}+2\omega F_{t\phi r,\phi}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi r,\phi}+\frac{1}{r}F_{rr\phi}-\frac{1}{r^3}F_{\phi\phi\phi}=0 \tag{89.350}$$

and

$$-F_{tt\phi,r}+F_{rr\phi,r}+2\omega F_{t\phi\phi,r}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi\phi,r}+\frac{1}{r}F_{rr\phi}-\frac{1}{r^3}F_{\phi\phi\phi}=0\ . \tag{89.351}$$

For $(\beta,\gamma)=(r,t)$, we have

$$-F_{ttr,t}+F_{rrr,t}+2\omega F_{t\phi r,t}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi r,t}+\frac{1}{r}F_{rrt}-\frac{1}{r^3}F_{\phi\phi t}=0\ . \tag{89.352}$$

and

$$-F_{ttt,r}+F_{rrt,r}+2\omega F_{t\phi t,r}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi t,r}+\frac{1}{r}F_{rrt}-\frac{1}{r^3}F_{\phi\phi t}=0\ . \tag{89.353}$$

We can write (89.345) through (89.353) as the following 9 equations. For $(\beta,\gamma)=(t,t)$, we have

$$\frac{\partial}{\partial t}\left[-F_{ttt}+F_{rtr}+2\omega F_{tt\phi}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi t\phi}\right]+\frac{1}{r}F_{rtt}=0\ . \tag{89.354}$$

For $(\beta,\gamma)=(r,r)$, we have

$$\frac{\partial}{\partial r}\left[-F_{trt}+F_{rrr}+2\omega F_{tr\phi}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi r\phi}\right]+\frac{1}{r}F_{rrr}=0\ . \tag{89.355}$$

For $(\beta,\gamma)=(\phi,\phi)$, we have

$$\frac{\partial}{\partial \phi}\left[-F_{t\phi t}+F_{r\phi r}+2\omega F_{t\phi\phi}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi\phi}\right]+\frac{1}{r}F_{r\phi\phi}=0\ . \tag{89.356}$$

For $(\beta,\gamma)=(t,\phi)$, we have

$$\frac{\partial}{\partial \phi}\left[-F_{ttt}+F_{rrt}+2\omega F_{t\phi t}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi t}\right]+\frac{1}{r}F_{rt\phi}=0 \tag{89.357}$$

and

$$\frac{\partial}{\partial t}\left[-F_{tt\phi}+F_{rr\phi}+2\omega F_{t\phi\phi}+\left(\frac{1}{r^2}-\omega^2\right)F_{\phi\phi\phi}\right]+\frac{1}{r}F_{rt\phi}=0\ . \tag{89.358}$$

For $(\beta, \gamma) = (r, \phi)$, we have

$$\frac{\partial}{\partial \phi}\left[-F_{ttr} + F_{rrr} + 2\omega F_{t\phi r} + \left(\frac{1}{r^2} - \omega^2\right) F_{\phi\phi r}\right] + \frac{1}{r}F_{rr\phi} - \frac{1}{r^3}F_{\phi\phi\phi} = 0 \tag{89.359}$$

and

$$\frac{\partial}{\partial r}\left[-F_{tt\phi} + F_{rr\phi} + 2\omega F_{t\phi\phi} + \left(\frac{1}{r^2} - \omega^2\right) F_{\phi\phi\phi}\right] + \frac{1}{r}F_{rr\phi} + \frac{1}{r^3}F_{\phi\phi\phi} = 0\ . \tag{89.360}$$

For $(\beta, \gamma) = (r, t)$, we have

$$\frac{\partial}{\partial t}\left[-F_{ttr} + F_{rrr} + 2\omega F_{t\phi r} + \left(\frac{1}{r^2} - \omega^2\right) F_{\phi\phi r}\right] + \frac{1}{r}F_{rrt} - \frac{1}{r^3}F_{\phi\phi t} = 0\ . \tag{89.361}$$

and

$$\frac{\partial}{\partial r}\left[-F_{ttt} + F_{rrt} + 2\omega F_{t\phi t} + \left(\frac{1}{r^2} - \omega^2\right) F_{\phi\phi t}\right] + \frac{1}{r}F_{rrt} + \frac{1}{r^3}F_{\phi\phi t} = 0\ . \tag{89.362}$$

If we take the partial derivative of (89.354) with respect to ϕ, and from that, subtract the partial derivative of (89.357) with respect to t, we get [using the symmetries (89.34) and (89.39)]

$$F_{rtt,\phi} - F_{tr\phi,t} = 0\ , \tag{89.363}$$

which we already knew from the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39).

In the same way, using various combinations (89.354) through (89.362), we get the following 8 equations [using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39].

If we take the partial derivative of (89.358) with respect to ϕ, and from that, subtract the partial derivative of (89.356) with respect to t, we get [using the symmetries (89.34) and (89.39)]

$$F_{r\phi\phi,t} - F_{tr\phi,\phi} = 0\ , \tag{89.364}$$

which we already knew from the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39).

If we take the partial derivative of (89.361) with respect to ϕ, and from that, subtract the partial derivative of (89.359) with respect to t, we get [using the symmetries (89.34) and (89.39)]

$$F_{trr,\phi} - F_{\phi rr,t} = 0\ , \tag{89.365}$$

which we already knew from the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39).

If we take the partial derivative of (89.362) with respect to t, and from that, subtract the partial derivative of (89.354) with respect to r, we get [using the symmetries (89.34) and (89.39)]

$$\frac{1}{r^2}F_{rtt} + \frac{1}{r^3}F_{t\phi\phi,t} = 0\ . \tag{89.366}$$

If we take the partial derivative of (89.361) with respect to r, and from that, subtract the partial derivative of (89.355) with respect to t, we get [using the symmetries (89.34) and (89.39)]

$$-\frac{1}{r^2}F_{trr} - \frac{1}{r^3}F_{t\phi\phi,r} + \frac{3}{r^4}F_{t\phi\phi} = 0\ . \tag{89.367}$$

If we take the partial derivative of (89.360) with respect to t, and from that, subtract the partial derivative of (89.358) with respect to r, we get [using the symmetries (89.34) and (89.39)]

$$\frac{1}{r^3}F_{\phi\phi\phi,t} + \frac{1}{r^2}F_{tr\phi} = 0\ . \tag{89.368}$$

If we take the partial derivative of (89.359) with respect to r, and from that, subtract the partial derivative of (89.355) with respect to ϕ, we get [using the symmetries (89.34) and (89.39)]

$$-\frac{1}{r^2}F_{\phi rr} - \frac{1}{r^3}F_{\phi\phi\phi,r} + \frac{3}{r^4}F_{\phi\phi\phi} = 0 \ . \tag{89.369}$$

If we take the partial derivative of (89.360) with respect to ϕ, and from that, subtract the partial derivative of (89.356) with respect to r, we get [using the symmetries (89.34) and (89.39)]

$$\frac{1}{r^3}F_{\phi\phi\phi,\phi} + \frac{1}{r^2}F_{r\phi\phi} = 0 \ . \tag{89.370}$$

If we take the partial derivative of (89.362) with respect to ϕ, and from that, subtract the partial derivative of (89.357) with respect to r, we get [using the symmetries (89.34) and (89.39)]

$$\frac{1}{r^3}F_{t\phi\phi,\phi} + \frac{1}{r^2}F_{tr\phi} = 0 \ . \tag{89.371}$$

Substituting (89.366) into (89.354) and using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) gives

$$\frac{\partial}{\partial t}\left[-F_{ttt} + F_{trr} + 2\omega F_{\phi tt} - \omega^2 F_{t\phi\phi}\right] = 0 \ . \tag{89.372}$$

Substituting (89.370) into (89.356) and using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) gives

$$\frac{\partial}{\partial \phi}\left[-F_{\phi tt} + F_{\phi rr} + 2\omega F_{t\phi\phi} - \omega^2 F_{\phi\phi\phi}\right] = 0 \ . \tag{89.373}$$

Substituting (89.371) into (89.357) and using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) gives

$$\frac{\partial}{\partial \phi}\left[-F_{ttt} + F_{trr} + 2\omega F_{\phi tt} - \omega^2 F_{t\phi\phi}\right] = 0 \tag{89.374}$$

Substituting (89.368) into (89.358) and using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) gives

$$\frac{\partial}{\partial t}\left[-F_{\phi tt} + F_{\phi rr} + 2\omega F_{t\phi\phi} - \omega^2 F_{\phi\phi\phi}\right] = 0 \ . \tag{89.375}$$

Equation (89.355) can be written

$$\frac{\partial}{\partial r}\left[-F_{rtt} + F_{rrr} + 2\omega F_{tr\phi} - \omega^2 F_{r\phi\phi}\right] + \frac{1}{r^2}F_{r\phi\phi,r} - \frac{2}{r^3}F_{r\phi\phi} + \frac{1}{r}F_{rrr} = 0 \ . \tag{89.376}$$

Substituting (89.369) into (89.359) and using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) gives

$$\frac{\partial}{\partial \phi}\left[-F_{rtt} + F_{rrr} + 2\omega F_{t\phi r} - \omega^2 F_{r\phi\phi}\right] + \frac{2}{r^3}F_{\phi\phi\phi} = 0 \ . \tag{89.377}$$

Substituting (89.369) into (89.360) gives

$$\frac{\partial}{\partial r}\left[-F_{\phi tt} + F_{\phi rr} + 2\omega F_{t\phi\phi} - \omega^2 F_{\phi\phi\phi}\right] + \frac{2}{r^3}F_{\phi\phi\phi} = 0 \ . \tag{89.378}$$

Substituting (89.367) into (89.361) and using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) gives

$$\frac{\partial}{\partial t}\left[-F_{rtt}+F_{rrr}+2\omega F_{t\phi r}-\omega^2 F_{r\phi\phi}\right]+\frac{2}{r^3}F_{t\phi\phi}=0\ . \tag{89.379}$$

Substituting (89.367) into (89.362) gives

$$\frac{\partial}{\partial r}\left[-F_{ttt}+F_{trr}+2\omega F_{\phi tt}-\omega^2 F_{t\phi\phi}\right]+\frac{2}{r^3}F_{t\phi\phi}=0\ . \tag{89.380}$$

The easiest way to get an equation with only one dependent variable is to start with (89.379). That gives

$$-F_{rtt,t}+F_{rrr,t}+2\omega F_{t\phi r,t}-\omega^2 F_{r\phi\phi,t}+\frac{2}{r^3}F_{t\phi\phi}=0\ . \tag{89.381}$$

Using the homogeneous field equation (89.40) and the symmetries (89.34) and (89.39) gives

$$-F_{rtt,t}+F_{trr,r}+2\omega F_{tr\phi,t}-\omega^2 F_{t\phi\phi,r}+\frac{2}{r^3}F_{t\phi\phi}=0\ . \tag{89.382}$$

Using (89.366), (89.367), and (89.371) in (89.382) gives

$$-\left(-\frac{1}{r}F_{t\phi\phi,t}\right)_{,t}+\left(-\frac{1}{r}F_{t\phi\phi,r}+\frac{3}{r^2}F_{t\phi\phi}\right)_{,r}+2\omega\left(-\frac{1}{r}F_{t\phi\phi,\phi}\right)_{,t}-\omega^2F_{t\phi\phi,r}+\frac{2}{r^3}F_{t\phi\phi}=0\ . \tag{89.383}$$

Or,

$$\frac{1}{r}F_{t\phi\phi,t,t}-\frac{1}{r}F_{t\phi\phi,r,r}+\frac{3}{r^2}F_{t\phi\phi,r}+\frac{1}{r^2}F_{t\phi\phi,r}-\frac{6}{r^3}F_{t\phi\phi}-\frac{2\omega}{r}F_{t\phi\phi,\phi,t}-\omega^2F_{t\phi\phi,r}+\frac{2}{r^3}F_{t\phi\phi}=0\ . \tag{89.384}$$

Or,

$$\frac{1}{r}F_{t\phi\phi,t,t}-\frac{1}{r}F_{t\phi\phi,r,r}-\frac{2\omega}{r}F_{t\phi\phi,\phi,t}+\left(\frac{4}{r^2}-\omega^2\right)F_{t\phi\phi,r}-\frac{4}{r^3}F_{t\phi\phi}=0\ . \tag{89.385}$$

It is possible to find a solution of (89.385) as a product of a function of t times a function of r times a function of ϕ, but it does not lead to the expected solution for F_{rtt}. Either I have made an algebraic error, or a bad assumption somewhere.

The appropriate non-zero terms of the gravitational field tensor in cylindrical coordinates that lead to centrifugal force should be:

$$F_{rtt}=F_{trt}=F_{ttr}=\omega^2 r\ . \tag{89.386}$$

There may be more non-zero terms. I have to check this. Also, this is the Newtonian value. I need to get this right.

$$R_{\alpha\beta\gamma\delta}=-F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}+F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}=0\ . \tag{89.387}$$

$$\Lambda_{DE}\equiv F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta}=3F^{r00}F_{r00}\ . \tag{89.388}$$

$$\Lambda_{DM}\equiv -F^{\theta}{}_{\epsilon\theta}F^{\epsilon\mu}{}_{\mu}=-F^{0}{}_{r0}F^{r0}{}_{0}\ . \tag{89.389}$$

89.12.3 Schwarzschild metric

Without showing details, in Schwarzschild coordinates, we have

$$\frac{8\pi}{3}\left(Y_{00}-\frac{1}{2}g_{00}Y\right)=\frac{r_s}{2r^3}\left[1-\frac{r_s}{r}-\frac{1}{2}\left(\frac{r_s}{r}\right)^3\right]=\frac{GM}{r^3}\left[1-\frac{r_s}{r}-\frac{1}{2}\left(\frac{r_s}{r}\right)^3\right]=-\frac{g}{r}-2g^2-4g^4r^2\ . \tag{89.390}$$

However, any frame for which $F=-\Gamma$ must be a frame defined by a congruence of geodesics. In such a frame, $\Gamma=0$, so that means that $g=M=0$. And in that case, the equation is satisfied, although that is not an interesting case. To check further, we must take a case where $F\neq-\Gamma$.

I have to multiply the coupling term by G to get the units right.

Spherical polar coordinates.

$$F_{r00}=F_{0r0}=F_{00r}=\ . \tag{89.391}$$

There may be more non-zero terms. I have to check this. See [27, Adler-Bazin-Schiffer, p. 164]. The connection includes also coordinate effects. I have to consider whether $F_{\alpha\beta\gamma}$ also includes coordinate effects.

$$R_{\alpha\beta\gamma\delta}=-F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}+F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}=0\ . \tag{89.392}$$

$$\Lambda_{DE}\equiv F^{\epsilon\theta\alpha}F_{\alpha\epsilon\theta}=3F^{r00}F_{r00}\ . \tag{89.393}$$

$$\Lambda_{DM}\equiv -F^{\theta}{}_{\epsilon\theta}F^{\epsilon\mu}{}_{\mu}=-F^{0}{}_{r0}F^{r0}{}_{0}\ . \tag{89.394}$$

89.12.4 Robertson-Walker metric

89.12.5 Gravitational waves

89.13 Dark energy, dark matter, and inflation

89.14 Discussion

The symmetries of the gravitational field tensor and the homogeneous field equation arise from the symmetries of the Riemann tensor. The inhomogeneous field equation arises from the variation of $R=-F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon\gamma}{}_{\gamma}+F^{\alpha}{}_{\epsilon\gamma}F^{\epsilon\gamma}{}_{\alpha}$, combined with the minimal coupling term $+\frac{1}{6}A_{\mu\nu}T^{\mu\nu}$.

We can represent gravity waves easier in terms of a gravitational field representation rather than as a geometric representation.

However, solutions of the field equations for the gravitational field tensor depend on boundary conditions in addition to sources. The various solutions for different boundary conditions for the same sources are all realized in the sense that different observers or bodies will see different gravitational fields. Thus, there will not be a consensus on what is the actual gravitational field. This is actually a different kind of relativity. The reality seen by each observer is valid. It makes deciding which reality is the correct one to use in a given application challenging.

89.15 Acknowledgments

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89.16 Appendix

89.16.1 Ansatz ca 2012

We can write the Ricci scalar as

$$R = g^{\theta\phi}(-F^{\alpha}{}_{\theta\phi;\alpha} + F^{\alpha}{}_{\theta\alpha;\phi} - F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\theta\phi} + F^{\alpha}{}_{\epsilon\phi}F^{\epsilon}{}_{\theta\alpha}) \ . \tag{89.395}$$

Or,

$$R = -g^{\theta\phi}F^{\alpha}{}_{\theta\phi;\alpha} + g^{\theta\phi}F^{\alpha}{}_{\theta\alpha;\phi} - g^{\theta\phi}F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\theta\phi} + g^{\theta\phi}F^{\alpha}{}_{\epsilon\phi}F^{\epsilon}{}_{\theta\alpha} \ . \tag{89.396}$$

Using (89.412), (89.413), (89.414), and (89.415) in (89.396) gives

$$R = -c^{\alpha}{}_{\alpha} + d^{\alpha}{}_{\alpha} - g^{\theta\phi}F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\theta\phi} + g^{\theta\phi}F^{\alpha}{}_{\epsilon\phi}F^{\epsilon}{}_{\theta\alpha} \ . \tag{89.397}$$

Direct calculation from the definitions in (89.412), (89.413), (89.414), and (89.415) shows that $c^{\alpha}{}_{\alpha} = a^{\alpha}{}_{\alpha}$ and $d^{\alpha}{}_{\alpha} = b^{\alpha}{}_{\alpha}$. Therefore (89.397) becomes

$$R = -a^{\alpha}{}_{\alpha} + b^{\alpha}{}_{\alpha} - g^{\theta\phi}F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\theta\phi} + g^{\theta\phi}F^{\alpha}{}_{\epsilon\phi}F^{\epsilon}{}_{\theta\alpha} \ . \tag{89.398}$$

Imposing the symmetry (89.422) gives $b^{\alpha}{}_{\alpha} = a^{\alpha}{}_{\alpha}$. Therefore, (89.398) beecomes

$$R = g^{\theta\phi}\left(-F^{\alpha}{}_{\epsilon\alpha}F^{\epsilon}{}_{\theta\phi} + F^{\alpha}{}_{\epsilon\phi}F^{\epsilon}{}_{\theta\alpha}\right) \ . \tag{89.399}$$

89.16.2 Symmetry conditions on $F_{\alpha\beta\gamma}$ from symmetries of the connection and the Riemann tensor using the ansatz ca 2012

$F_{\mu\alpha\beta}$ has a symmetry because of a symmetry of $\Gamma_{\mu\alpha\beta}$. Specifically,

$$\Gamma_{\mu\alpha\beta} = \Gamma_{\mu\beta\alpha} \tag{89.400}$$

leads to the symmetry

$$F_{\mu\alpha\beta} = F_{\mu\beta\alpha} \ , \tag{89.401}$$

at least in the frame where $F_{\mu\alpha\beta}$ is defined to be the negative of the connection. In other frames, we can require that symmetry because the force equation depends only on the symmetric part of $F_{\mu\alpha\beta}$ on the last two indexes.

Equation (89.32) can be rewritten with lowered indexes as

$$R_{\alpha\beta\gamma\delta} = -F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta;} - F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} \ . \tag{89.402}$$

The symmetry in (89.24) is identically satisfied by (89.402). The symmetry in (89.25) is a consequence of the symmetry in (89.23) and (89.22). The symmetry (89.26) is identically satisfied by (89.402) if $F_{\mu\alpha\beta}$ also satisfies the symmetry (89.401). Equation (89.22) and the symmetry in (89.23) give two independent conditions for $F_{\beta\mu\epsilon}$ to satisfy.

Substituting (89.402) into (89.22) gives

$$-F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} = -F_{\gamma\delta\beta;\alpha} + F_{\gamma\delta\alpha;\beta} - F_{\gamma\epsilon\alpha}F^{\epsilon}{}_{\delta\beta} + F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta\alpha} \ . \tag{89.403}$$

Substituting (89.402) into the symmetry in (89.23) gives

$$-F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} - F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} + F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} = F_{\beta\alpha\delta;\gamma} - F_{\beta\alpha\gamma;\delta} + F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha\delta} - F_{\beta\epsilon\delta}F^{\epsilon}{}_{\alpha\gamma} \ . \tag{89.404}$$

Equations (89.403) and (89.404) can be rewritten as

$$-F_{\alpha\beta\delta;\gamma} + F_{\alpha\beta\gamma;\delta} + F_{\gamma\delta\beta;\alpha} - F_{\gamma\delta\alpha;\beta} = F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta} - F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma} - F_{\gamma\epsilon\alpha}F^{\epsilon}{}_{\delta\beta} + F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta\alpha} \tag{89.405}$$

and

$$-F_{\alpha\beta\delta;\gamma}+F_{\alpha\beta\gamma;\delta}-F_{\beta\alpha\delta;\gamma}+F_{\beta\alpha\gamma;\delta}=F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}-F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}+F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha\delta}-F_{\beta\epsilon\delta}F^{\epsilon}{}_{\alpha\gamma}\ . \tag{89.406}$$

Equations (89.405) and (89.406) can be rewritten as

$$F_{\gamma\delta\beta;\alpha}-F_{\gamma\delta\alpha;\beta}-F_{\alpha\beta\delta;\gamma}+F_{\alpha\beta\gamma;\delta}=e_{\alpha\beta\gamma\delta} \tag{89.407}$$

and

$$-F_{\alpha\beta\delta;\gamma}+F_{\alpha\beta\gamma;\delta}-F_{\beta\alpha\delta;\gamma}+F_{\beta\alpha\gamma;\delta}=f_{\alpha\beta\gamma\delta}\ . \tag{89.408}$$

where

$$e_{\alpha\beta\gamma\delta}\equiv F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}-F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F_{\gamma\epsilon\alpha}F^{\epsilon}{}_{\delta\beta}+F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta\alpha} \tag{89.409}$$

and

$$f_{\alpha\beta\gamma\delta}\equiv F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}-F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}+F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha\delta}-F_{\beta\epsilon\delta}F^{\epsilon}{}_{\alpha\gamma}\ . \tag{89.410}$$

Equations (89.407) and (89.408) might be analogous to the electromagnetic equations

$$F_{\alpha\beta;\gamma}+F_{\beta\gamma;\alpha}+F_{\gamma\alpha;\beta}=0\ , \tag{89.411}$$

since they involve no source terms.

We shall need some contractions of $F_{\alpha\beta\gamma;\delta}$. Because of the symmetry (89.401), there are only four independent contractions of $F_{\alpha\beta\gamma;\delta}$. These are

$$a_{\alpha\beta}\equiv g^{\gamma\delta}F_{\gamma\alpha\beta;\delta}\ , \tag{89.412}$$

$$b_{\alpha\beta}\equiv g^{\gamma\delta}F_{\gamma\alpha\delta;\beta}\ , \tag{89.413}$$

$$c_{\alpha\beta}\equiv g^{\gamma\delta}F_{\alpha\gamma\delta;\beta}\ , \tag{89.414}$$

and

$$d_{\alpha\beta}\equiv g^{\gamma\delta}F_{\alpha\gamma\beta;\delta}\ . \tag{89.415}$$

Also because of the symmetry (89.401), we have

$$a_{\alpha\beta}=a_{\beta\alpha}\ . \tag{89.416}$$

We can take various contractions of (89.407) and (89.408) to give various relations among $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, and $d_{\alpha\beta}$. Contracting (89.407) with $g^{\alpha\gamma}$ gives

$$b_{\delta\beta}=b_{\beta\delta}\ . \tag{89.417}$$

Contracting (89.407) with $g^{\gamma\delta}$ gives an equivalent result. Contracting (89.407) with $g^{\beta\delta}$ gives

$$c_{\alpha\gamma}-d_{\alpha\gamma}=c_{\gamma\alpha}-d_{\gamma\alpha}\ . \tag{89.418}$$

Contracting (89.408) with $g^{\alpha\delta}$ gives

$$\begin{aligned}-a_{\beta\gamma}+b_{\beta\gamma}+c_{\beta\gamma}-d_{\beta\gamma}&=g^{\alpha\delta}f_{\alpha\beta\delta\gamma}\\&=g^{\alpha\delta}\left(F_{\alpha\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F_{\alpha\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}+F_{\beta\epsilon\delta}F^{\epsilon}{}_{\alpha\gamma}-F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha\delta}\right)\\&=F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F^{\delta}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}+F_{\beta\epsilon}{}^{\alpha}F^{\epsilon}{}_{\alpha\gamma}-F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\alpha}{}^{\alpha}\\&=F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\beta\gamma}-F^{\delta}{}_{\epsilon\gamma}F^{\epsilon}{}_{\beta\delta}+F_{\beta\epsilon}{}^{\delta}F^{\epsilon}{}_{\delta\gamma}-F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\delta}{}^{\delta}\ .\end{aligned} \tag{89.419}$$

We can reverse the indexes in (89.419) to give

$$-a_{\gamma\beta}+b_{\gamma\beta}+c_{\gamma\beta}-d_{\gamma\beta}=F^{\delta}{}_{\epsilon\delta}F^{\epsilon}{}_{\gamma\beta}-F^{\delta}{}_{\epsilon\beta}F^{\epsilon}{}_{\gamma\delta}+F_{\gamma\epsilon}{}^{\delta}F^{\epsilon}{}_{\delta\beta}-F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta}{}^{\delta}\ . \tag{89.420}$$

Because of the symmetries (89.416), (89.417), and (89.418), (89.419) and (89.420) must be equal. Therfore, making use of the symmetry (89.401) gives

$$F_{\beta\epsilon}{}^{\delta}F^{\epsilon}{}_{\delta\gamma}-F_{\beta\epsilon\gamma}F^{\epsilon}{}_{\delta}{}^{\delta}=F_{\gamma\epsilon}{}^{\delta}F^{\epsilon}{}_{\delta\beta}-F_{\gamma\epsilon\beta}F^{\epsilon}{}_{\delta}{}^{\delta}\ . \tag{89.421}$$

It would be sufficient for $F_{\gamma\epsilon\beta}$ to be symmetric on the first and third indexes to satisfy (89.421).

89.16.3 The second symmetry

There are two possible conditions to satisfy (89.421). The most reasonable of these is (89.440), which gives the symmetry

$$F_{\alpha\beta\gamma} = F_{\beta\alpha\gamma} = F_{\gamma\beta\alpha} \ . \tag{89.422}$$

In that case, there are only two independent contractions of $F_{\alpha\beta\gamma;\delta}$ since

$$a_{\alpha\beta} = a_{\beta\alpha} = g^{\gamma\delta} F_{\gamma\beta\alpha;\delta} = g^{\gamma\delta} F_{\alpha\beta\gamma;\delta} = d_{\alpha\beta} \ , \tag{89.423}$$

and

$$b_{\alpha\beta} = g^{\gamma\delta} F_{\gamma\alpha\delta;\beta} = g^{\gamma\delta} F_{\gamma\delta\alpha;\beta} = g^{\gamma\delta} F_{\alpha\delta\gamma;\beta} = c_{\alpha\beta} \ . \tag{89.424}$$

Taking the symmetries (89.401) and (89.422) means that $F_{\alpha\beta\gamma}$ is symmetric for all permutations of the indexes. Such a tensor has 20 independent parameters in four dimensions, exactly the number of independent parameters in the Riemann tensor, which suggests that we might on the right track.

Substituting (89.401) and (89.422) into (89.408) gives

$$-F_{\alpha\beta\gamma;\delta} + F_{\alpha\beta\delta;\gamma} = \frac{1}{2} f_{\alpha\beta\delta\gamma} \ . \tag{89.425}$$

Substituting (89.423) and (89.424) into (89.419) gives

$$-2a_{\beta\gamma} + 2b_{\beta\gamma} = F^{\delta}{}_{\epsilon\delta} F^{\epsilon}{}_{\beta\gamma} - F^{\delta}{}_{\epsilon\gamma} F^{\epsilon}{}_{\beta\delta} + F_{\beta\epsilon}{}^{\delta} F^{\epsilon}{}_{\delta\gamma} - F_{\beta\epsilon\gamma} F^{\epsilon}{}_{\delta}{}^{\delta} \ . \tag{89.426}$$

Substituting (89.423) and (89.424) into (89.420) gives

$$-2a_{\gamma\beta} + 2b_{\gamma\beta} = F^{\delta}{}_{\epsilon\delta} F^{\epsilon}{}_{\gamma\beta} - F^{\delta}{}_{\epsilon\beta} F^{\epsilon}{}_{\gamma\delta} + F_{\gamma\epsilon}{}^{\delta} F^{\epsilon}{}_{\delta\beta} - F_{\gamma\epsilon\beta} F^{\epsilon}{}_{\delta}{}^{\delta} \ . \tag{89.427}$$

Contracting (89.407) with $g^{\alpha\delta}$ and $g^{\beta\gamma}$ and contracting (89.408) with $g^{\alpha\gamma}$, $g^{\beta\gamma}$, $g^{\gamma\delta}$, and $g^{\beta\delta}$ gives consistent results with the above, but adds nothing new.

Equations (89.426) and (89.427) can be written

$$a_{\beta\gamma} - b_{\beta\gamma} = -\frac{1}{2} F^{\delta}{}_{\epsilon\delta} F^{\epsilon}{}_{\beta\gamma} + \frac{1}{2} F^{\delta}{}_{\epsilon\gamma} F^{\epsilon}{}_{\beta\delta} - \frac{1}{2} F_{\beta\epsilon}{}^{\delta} F^{\epsilon}{}_{\delta\gamma} + \frac{1}{2} F_{\beta\epsilon\gamma} F^{\epsilon}{}_{\delta}{}^{\delta} \tag{89.428}$$

and

$$a_{\gamma\beta} - b_{\gamma\beta} = -\frac{1}{2} F^{\delta}{}_{\epsilon\delta} F^{\epsilon}{}_{\gamma\beta} + \frac{1}{2} F^{\delta}{}_{\epsilon\beta} F^{\epsilon}{}_{\gamma\delta} - \frac{1}{2} F_{\gamma\epsilon}{}^{\delta} F^{\epsilon}{}_{\delta\beta} + \frac{1}{2} F_{\gamma\epsilon\beta} F^{\epsilon}{}_{\delta}{}^{\delta} \ . \tag{89.429}$$

Further, (89.428) and (89.429) gives

$$a_{\beta\gamma} = b_{\beta\gamma} \ . \tag{89.430}$$

I now know that (89.440) is too strong a condition, so we look for other possibilities.

89.16.4 Other possibilities

To test for other possibilities for the second symmetry, we rearrange terms in (89.421) to get

$$F_{\beta\epsilon}{}^{\delta} F^{\epsilon}{}_{\delta\gamma} - F_{\gamma\epsilon}{}^{\delta} F^{\epsilon}{}_{\delta\beta} - F_{\beta\epsilon\gamma} F^{\epsilon}{}_{\delta}{}^{\delta} + F_{\gamma\epsilon\beta} F^{\epsilon}{}_{\delta}{}^{\delta} = 0 \ . \tag{89.431}$$

Equation (89.431) can be rewritten as

$$g_{\delta\phi} g_{\beta\mu} g_{\alpha\gamma} g_{\epsilon\theta} F^{\mu\delta\theta} F^{\epsilon\alpha\phi} - g_{\delta\phi} g_{\beta\mu} g_{\alpha\gamma} g_{\epsilon\theta} F^{\alpha\epsilon\phi} F^{\delta\mu\theta} - g_{\beta\phi} g_{\alpha\gamma} F^{\phi\alpha\epsilon} F_{\epsilon\delta}{}^{\delta} + g_{\beta\phi} g_{\alpha\gamma} F^{\alpha\phi\epsilon} F_{\epsilon\delta}{}^{\delta} = 0 \ , \tag{89.432}$$

where I have used (89.401). Or,

$$g_{\delta\phi}g_{\beta\mu}g_{\alpha\gamma}g_{\epsilon\theta}\left(F^{\mu\delta\theta}F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}F^{\delta\mu\theta}\right)+g_{\beta\phi}g_{\alpha\gamma}\left(F^{\alpha\phi\epsilon}-F^{\phi\alpha\epsilon}\right)F_{\epsilon\delta}{}^{\delta}=0\ . \tag{89.433}$$

$F^{\alpha\beta\gamma}$ is symmetric on indexes 2 and 3. We are wondering about the symmetry on indexes 1 and 2 or 1 and 3. We can write (89.433) as

$$\frac{1}{2}g_{\delta\phi}g_{\beta\mu}g_{\alpha\gamma}g_{\epsilon\theta}\left[\left(F^{\mu\delta\theta}-F^{\delta\mu\theta}\right)\left(F^{\epsilon\alpha\phi}+F^{\alpha\epsilon\phi}\right)+\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)\left(F^{\mu\delta\theta}+F^{\delta\mu\theta}\right)\right]$$
$$+g_{\beta\phi}g_{\alpha\gamma}\left(F^{\alpha\phi\epsilon}-F^{\phi\alpha\epsilon}\right)F_{\epsilon\delta}{}^{\delta}=0\ . \tag{89.434}$$

Multiplying through and changing some dummy indexes in (89.434) gives

$$\frac{1}{2}\left[g_{\delta\phi}g_{\beta\mu}g_{\alpha\gamma}g_{\epsilon\theta}\left(F^{\mu\delta\theta}-F^{\delta\mu\theta}\right)\left(F^{\epsilon\alpha\phi}+F^{\alpha\epsilon\phi}\right)+g_{\delta\phi}g_{\beta\mu}g_{\alpha\gamma}g_{\epsilon\theta}\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)\left(F^{\mu\delta\theta}+F^{\delta\mu\theta}\right)\right]$$
$$+g_{\beta\epsilon}g_{\alpha\gamma}\left(F^{\alpha\epsilon\phi}-F^{\epsilon\alpha\phi}\right)F_{\phi\delta}{}^{\delta}=0 \tag{89.435}$$

Changing some dummy indexes on (89.435) gives

$$\frac{1}{2}\left[g_{\alpha\theta}g_{\beta\epsilon}g_{\delta\gamma}g_{\mu\phi}\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)\left(F^{\mu\delta\theta}+F^{\delta\mu\theta}\right)+g_{\delta\phi}g_{\beta\mu}g_{\alpha\gamma}g_{\epsilon\theta}\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)\left(F^{\mu\delta\theta}+F^{\delta\mu\theta}\right)\right]$$
$$-g_{\beta\epsilon}g_{\alpha\gamma}\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)F_{\phi\delta}{}^{\delta}=0 \tag{89.436}$$

Factoring (89.436) gives

$$\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)\left\{\frac{1}{2}\left[g_{\alpha\theta}g_{\beta\epsilon}g_{\delta\gamma}g_{\mu\phi}\left(F^{\mu\delta\theta}+F^{\delta\mu\theta}\right)+g_{\delta\phi}g_{\beta\mu}g_{\alpha\gamma}g_{\epsilon\theta}\left(F^{\mu\delta\theta}+F^{\delta\mu\theta}\right)\right]\right.$$
$$\left.-g_{\beta\epsilon}g_{\alpha\gamma}F_{\phi\delta}{}^{\delta}\right\}=0\ . \tag{89.437}$$

We can write (89.437) as

$$\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)\left[\frac{1}{2}g_{\beta\epsilon}\left(F_{\phi\gamma\alpha}+F_{\gamma\phi\alpha}\right)+\frac{1}{2}g_{\alpha\gamma}\left(F_{\beta\phi\epsilon}+F_{\phi\beta\epsilon}\right)-g_{\beta\epsilon}g_{\alpha\gamma}F_{\phi\delta}{}^{\delta}\right]=0\ . \tag{89.438}$$

We can rewrite (89.438) as

$$\left(F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}\right)\left[g_{\beta\epsilon}F_{(\phi\gamma)\alpha}+g_{\alpha\gamma}F_{(\phi\beta)\epsilon}-g_{\beta\epsilon}g_{\alpha\gamma}F_{\phi\delta}{}^{\delta}\right]=0\ , \tag{89.439}$$

where parentheses denote symmetrization.

I now know that setting either factor in (89.439) is too restrictive. We must use the whole equation (89.439).

89.16.5 The simple solution

One solution of (89.439) is

$$F^{\epsilon\alpha\phi}-F^{\alpha\epsilon\phi}=0\ . \tag{89.440}$$

If (89.440) is correct, then combined with (89.401) implies that $F^{\epsilon\phi\alpha}$ is unchanged for any permutation of its indexes.

However, I now know that (89.440) is too restrictive.

89.16.6 The second solution

Another solution of (89.439) can be found from

$$g_{\beta\epsilon}F_{(\phi\gamma)\alpha} + g_{\alpha\gamma}F_{(\phi\beta)\epsilon} - g_{\beta\epsilon}g_{\alpha\gamma}F_{\phi\delta}{}^{\delta} = 0 \ . \tag{89.441}$$

Raising indexes allows us to write (89.441) as

$$\delta^{\epsilon}_{\beta}F_{(\phi\gamma)}{}^{\alpha} + \delta^{\alpha}_{\gamma}F_{(\phi\beta)}{}^{\epsilon} - \delta^{\epsilon}_{\beta}\delta^{\alpha}_{\gamma}F_{\phi\delta}{}^{\delta} = 0 \ . \tag{89.442}$$

If we consider (89.442) to be an equation to determine $F_{\phi\delta}{}^{\delta}$, then it is overdetermined, since it is at least 16 equations for only four quantities. However, contracting indexes gives

$$\delta^{\beta}_{\beta}F_{(\phi\gamma)}{}^{\gamma} + \delta^{\gamma}_{\gamma}F_{(\phi\beta)}{}^{\beta} - \delta^{\beta}_{\beta}\delta^{\gamma}_{\gamma}F_{\phi\delta}{}^{\delta} = 0 \ . \tag{89.443}$$

Or,

$$F_{(\phi\gamma)}{}^{\gamma} + F_{(\phi\beta)}{}^{\beta} - 4F_{\phi\delta}{}^{\delta} = 0 \ . \tag{89.444}$$

Changing dummy indexes gives

$$F_{(\phi\delta)}{}^{\delta} + F_{(\phi\delta)}{}^{\delta} - 4F_{\phi\delta}{}^{\delta} = 0 \ . \tag{89.445}$$

Combining equal terms gives

$$F_{\phi\delta}{}^{\delta} = \frac{1}{2}F_{(\phi\delta)}{}^{\delta} \ . \tag{89.446}$$

If it were not for that factor of $\frac{1}{2}$, (89.446) would be consistent with (89.440). Equation (89.446) is a necessary condition to satisfy (89.442), not a sufficient condition. To find the rest of the conditions, we substitute (89.446) into (89.442) to give

$$\delta^{\epsilon}_{\beta}F_{(\phi\gamma)}{}^{\alpha} + \delta^{\alpha}_{\gamma}F_{(\phi\beta)}{}^{\epsilon} - \frac{1}{2}\delta^{\epsilon}_{\beta}\delta^{\alpha}_{\gamma}F_{(\phi\delta)}{}^{\delta} = 0 \ . \tag{89.447}$$

We can rewrite (89.447) as

$$\delta^{\epsilon}_{\beta}\left[F_{(\phi\gamma)}{}^{\alpha} - \frac{1}{4}\delta^{\alpha}_{\gamma}F_{(\phi\delta)}{}^{\delta}\right] + \delta^{\alpha}_{\gamma}\left[F_{(\phi\beta)}{}^{\epsilon} - \frac{1}{4}\delta^{\epsilon}_{\beta}F_{(\phi\delta)}{}^{\delta}\right] = 0 \ . \tag{89.448}$$

Both quantities inside of the brackets have the same form. Setting one of them to zero gives

$$F_{(\phi\gamma)}{}^{\alpha} = \frac{1}{4}\delta^{\alpha}_{\gamma}F_{(\phi\delta)}{}^{\delta} \ . \tag{89.449}$$

Equations (89.449) taken together with (89.446) give necessary and sufficient conditions to satisfy (89.442). We can rewrite (89.449) as

$$F_{(\phi\gamma)}{}^{\alpha} = 0 \text{ for } \alpha \neq \gamma, \text{ or} \tag{89.450}$$

$$F_{(\phi\gamma)}{}^{\alpha} = \frac{1}{4}F_{(\phi\delta)}{}^{\delta} \text{ for } \alpha = \gamma. \tag{89.451}$$

Taken together, these imply that

$$F_{(\phi0)}{}^{0} = F_{(\phi1)}{}^{1} = F_{(\phi2)}{}^{2} = F_{(\phi3)}{}^{3} \ . \tag{89.452}$$

Frankly, these conditions seem somewhat bizarre. Therefore, I am inclined to take (89.440) as the correct condition. However, if it turns out that a different decision is appropriate, we can return to this one.

I now know that setting either factor in (89.439) is too restrictive. We must use the whole equation (89.439).

Chapter 90

A realistic interpretation of quantum field theory[1]

abstract[2]

Conceptual difficulties such as wave function collapse, Schrödinger's cat, or the EPR paradox cannot be explained by standard quantum theory. At least quantum field theory is required. Standard quantum theory (solutions of Schrödinger's equation or the Dirac equation) can be used only to calculate solutions for comparison with ensemble averages.

Considering the fluctuating fields associated with wave functions as real physical fields (even though they cannot be calculated) gives a realistic interpretation of quantum field theory that avoids the need for wave function collapse, many worlds, or a transaction with the future.

90.1 Introduction

Among the many difficulties in interpreting quantum wave functions, the most basic is whether a wave function represents reality (an actual physical wave) or whether it represents only knowledge or information. Neither interpretation is satisfactory, in that the first disagrees with experiment and the second does not answer how the world functions or survives without observers. Various combinations of these two alternatives have been suggested, including:

1. A wave function represents an ensemble of identically prepared systems, not a member of the ensemble.
2. Because the wave function represents only knowledge or information, its collapse presents no problem.
3. The world splits.
4. A transaction with the future occurs.

That none of these interpretations, nor any of many others, has received universal acceptance explains why, even after 84 years, there is no general agreement about the interpretation of quantum theory.

We know from quantum field theory that there are fluctuations superimposed on wave functions. Quantum field theory can estimate the magnitude of these fluctuations, but cannot predict their exact spatial or temporal values.

[1] February 2010, additional comment 18 March 2011

[2] 16 February 2010

I propose that these fluctuations are real in the sense that they represent actual fields that fluctuate in space and time (even though we cannot calculate them in detail).[3]

I propose further that wave functions calculated as solutions to Schrödinger's or Dirac's equation represent not real fields, nor knowledge, but that $\int_V \psi^* \psi dV$ represents the probability that one quantum of energy of the field associated with the wave function ψ is actually present in the volume V.

If there happens to be something in that volume that could interact with that field with 100% efficiency, then that field (which is actually there) will be absorbed and detected.[4] Such interaction and detection does not cause wave function collapse. The wave function is not real. It does not cause any real fields to collapse either. Those fields were continually collapsing anyway through fluctuations.

I will begin showing how this works with a quantized electromagnetic field (photons), and then show why the same explanation applies to other wave functions as well.

The following analogy should help shed some light on the nature of quantum theory.[5] The Navier-Stokes equations approximate the lumpiness of molecular motions by a continuous fluid. Including turbulence gives equations that apply only to an ensemble average. In the same way, I suggest that the Dirac equation, Schrödinger equation, and Klein-Gordon equation apply only to an ensemble, and are approximations to quantum field theory rather than being fundamental equations.

90.2 A quantum electromagnetic field

When a photon is produced by an interaction, it is often represented by a spherical wave. As that wave expands, the wave is confined mostly to a spherical shell, whose thickness is roughly the pulse length of that wave function. It is well known, however, that this simple picture is not the whole story. We know from quantum field theory, that the electric and magnetic fields associated with this photon fluctuate in a way that is not described by the picture of a simple spherical wave. For example, if the number of photons is fixed, then the field strength is completely uncertain[239, e.g., Sakurai, p. 33]. However, averages of field strength over some volume have finite fluctuations. Thus, for such a single-photon spherical wave, the actual situation is that the associated fields fluctuate wildly, even though they are mostly confined to that spherical shell. Within the spherical shell, they vary wildly, sometimes being here, sometimes being on the opposite side of the shell. The probability of finding the photon somewhere is given by the wave function for that photon.

For any normalized wave function ψ,

$$\int \psi^* \psi dV = 1 \ . \tag{90.1}$$

If the wave function corresponds to an electromagnetic wave, then

$$\int (E^* E + H^* H) dV = \hbar\omega \ . \tag{90.2}$$

Since $|E| = |H|$ for an electromagnetic wave in free space in the appropriate system of units, $2|E|^2 = |\psi|^2 \hbar\omega$, so $E = \sqrt{\hbar\omega/2}\psi$. The probability that there is one photon's energy in volume V is

$$\int_V \psi^* \psi dV = \frac{2}{\hbar\omega} \int_V E^* E dV = \int_V \frac{1}{\hbar\omega} (|E|^2 + |H|^2) dV \ .^6 \tag{90.3}$$

[3]It is beyond the scope of this paper to consider whether a theory that could calculate fluctuating fields in detail is even possible.

[4]Of course, if the efficiency is less than 100%, then that must be included in the calculation.

[5]added 18 March 2011

Equation (90.3) is the probability that the particle (photon, in this case) is inside the volume V. It is also the probability of detecting a particle with an instrument that is 100% effective in the volume V. It is also the probability that there is within the volume V an energy $\hbar\omega$.

So far, we have been considering only monochromatic electromagnetic waves. In reality, we have electric and magnetic fields that vary arbitrarily in time and space. Because photons are Bosons, individual photons have no separate identity. Electric and magnetic fields simply add. To consider this generalization, we need to consider something like the following

$$\int_V \psi^* \psi dV \Delta t \Delta\omega \tag{90.4}$$

to be proportional to the probability that within the volume V in the interval Δt and the frequency band $\Delta\omega$ there is an amount of energy in the field equal to $\hbar\omega$.

To finish the calculations, I need to use creation and annihilation operators and take the calculation literally.

90.3 Einstein-Podolsky-Rosen[8]

In the Einstein-Podolsky-Rosen experiment, a spin zero system decays into two particles. To conserve linear momentum, they must go off in opposite directions. To conserve angular momentum, they must have opposite spin. However, the directions they go and the spins they have is not determined.

For the directions they go, we consider that each propagates as a spherical wave, but the two waves are correlated. The situations for the spins is similar.

The situation is usually put forward that we have two observers, sometimes referred to as Alice and Bob, situated in opposite directions from the original spin-zero state. Because of conservation of angular momentum, if Alice observes a particle of spin up, then Bob will observe a particle of spin down if he has oriented his apparatus parallel to that of Alice. If they do not have their apparatuses parallel, then various correlations can occur, which have been calculated.

The usual difficulty is that, although the state of the whole system is a pure state, the state for each of those two particles is a mixed state, that must be described by a density matrix rather than a wave function. However, if Alice makes a measurement, then the state of the other particle becomes pure rather than mixed. It is simply a complicated wave-function collapse.

The other problem is that we cannot describe the system in terms of actual fields that have actual values.

How does my interpretation of quantum field theory deal with this situation? First, we treat each of the two particles as spherical waves in the same way as I did before[9]. So, each particle is expanding as a spherical wave, and these two spherical waves are correlated. In addition, the fields associated with each particle are fluctuating wildly within the spherical shell of a spherical wave, and those fluctuations are correlated.

Notice, we do not have a superposition of states; we have correlated fluctuations of actual fields. If the fields corresponding to one of these particles happens to appear within Alice's detection apparatus, then with some probability that depends on the suitability of that apparatus for detecting

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[6] I need to use quantum field theory at this point to calculate how much fluctuation there is in E and H.

[8] 16 February 2010

[9] One interesting point in this regard, that is usually ignored in the EPR situation, is that since each particle is expanding in a spherical wave, the detection of one of these particles by either Alice or Bob will be very small. The probability can be calculated, however, and if Alice and Bob are patient, they will eventually detect a particle. If Alice does detect a particle, however, the probability for Bob detecting a particle increases enormously.

such a particle, she may detect it. Because the fluctuations between the two particles are correlated, Bob may also detect the other particle.

The spins of the particles also fluctuate, and those flutuations of the spins the two particles are correlated. Because in my interpretation, I am proposing actual fields, that have actual values, then when Alice detected a particle, that particle had an actual spin state that Alice could detect, depending on the orientation of her apparatus.

One possible way to do the correlations is to have the fluctuations exactly mirror the superpositions predicted by quantum field theory. For example, if quantum field theory predicts the state

$$(|\uparrow> |\downarrow> - |\downarrow> |\uparrow>)/\sqrt{2}\ , \tag{90.5}$$

then we simply have the spins of the two particles fluctuate between $|\uparrow> |\downarrow>$ and $|\downarrow> |\uparrow>$. Any measurements on such a system will give exactly the same results as quantum field theory. Notice that this is a hidden-variable theory, but not a local hidden-variable theory.

90.4 Schrödinger's cat

In the Schrödinger's cat paradox, we have a superposition of a decayed and an undecayed state, and correspondingly, a superposition of a dead and an alive cat, which, of course, makes no sense. In the inperpretation I propose here, we do not have a superposition of a decayed and an undecayed state. Instead, we have a system that fluctuates between a decayed and an undecayed state[11].

As the state fluctuates between decayed and undecayed, there is some probability at any instant, that the state has decayed. In addition, we suppose we have an apparatus that can detect a decayed state with some efficiency or probability. We can then calculate the probability of such detection in the usual way. Eventually, detection may occur. Once that has happened, there is no longer any fluctuation between a decayed and an undecayed state. The point is, that in the process of fluctuating between a decayed and an undecayed state, there are also fluctuations that localize the fields corresponding to a decay particle at the apparatus with some probability. Once the decay particle has been localized through fluctuations, there is a probability of detecting it.

90.5 Further considerations

I want to make it perfectly clear that I am not proposing a new theory. It is not standard quantum theory. It is quantum field theory with a new interpretation of the significance of quantum fluctuations. Standard quantum mechanics provides only an ensemble average of $\psi^*\psi$. Call solutions of Schrödinger's equation, Dirac's equation, or Maxwell's equations ψ_e, where e means ensemble. Distinguish that from ψ_a (actual). This is the fluctuating field that consists of the ensemble average plus the fluctuating part. We have no way to calculate ψ_a. Quantum field theory tells us that ψ fluctuates. Quantum field theory does not give ψ_a[12].

[11] This requires that nuclear decays, even though they involve tunneling through a potential barrier, must have fluctuations between decayed and undecayed states, not superpositions. The latter is the ordinary quantum mechanical view. In quantum field theory, we have fluctuations.

[12] I make no speculations about whether there might be the possibility for a set of equations that would give ψ_a. I do not consider whether it is possible to have a theory that gives a detailed description of the quantum fluctuations. Such questions are beyond the scope of this paper.

90.6 Analogy with turbulence[13]

First, consider the unification we have so far between electron wave functions and photon wave functions. A photon wave function does not have fluctuations, but the actual electric and magnetic fields associated with an electromagnetic wave do fluctuate according to quantum field theory. An electron wave function does not fluctuate, but that wave function applies only to an ensemble of identically prepared systems. The interpretation here is that the actual electron wave is a fluctuating field. In both cases, we have a theory that gives the non-fluctuating wave function, but the actual wave fluctuates, and we have no theory to tell us what that fluctuating wave is. Is there anything in some field of physics where we have a theory that gives us ensemble averages, but not the actual physical thing?

Yes, in turbulence theory. The Navier-Stokes equations tell us the behavior of a continuous fluid, which we take as an approximation for an actual fluid that is made up of molecules. We apply such a continuous-fluid approximation to the atmosphere, the ocean, and to cosmology, the latter as Einstein's field equations in General Relativity, which treats a system composed of stars, planets, and other objects as a continuous fluid. We know this is an approximation that does not represent the real physical situation, but we know that the approximation is usually valid, and most of the time we are not interested in the details of the molecular behavior.

Now we add turbulence. We add Reynold's stress and some other quantities to represent an averaged effect of turbulence. We know this is an approximation, but we really are not interested in all of the details of the fluid flow on all scales, so we approximate the smaller-scale flow by ensemble averages. When we get solutions to our equations, they give us only these ensemble averages. It is possible to consider waves propagating in a turbulent medium in which each turbulence quantity (which is an ensemble average) has a background part and a wave-associated part[258, 259, Jones & Hooke, 1986]. This gives a wave, some of whose components are ensemble averages. Thus, such a wave solution applies only to an ensemble, not to a specific instance.

That is the analogy with quantum theory. Quantum theory and quantum field theory give only ensemble averages, even though we know the actual fields fluctuate. Does that help us better interpret wave functions?

90.7 Comments by Dave Peterson, and my responses

1. Interesting and stimulating and makes a lot of sense.

2. I would just like to see it go further (i.e., more mechanisms, more details/examples - I'm hard to satisfy).

3. You say (section 90.3), 'we have correlated fluctuations of actual fields.' Of course, I agree that ψ for EM has to have an EM part - after all it negotiates pathways through apparatus that often involve reflection, refraction, thin metallic film beam splitting, polarizers, phase shifters, attenuators, non-linear crystals, vector addition interferences...and there could be fluctuations. I think of fluctuations as Random - so why would fluctuations correlate for two particles?

   My response: Since I have a realistic interpretation of quantum theory, I am assuming that the fields are real (including the fluctuations), so if a detector is there, it has the possibility of detecting the particle. If there are two detectors on opposite sides, then both have the possibility to detect particles. The fluctuations have to be correlated to conserve momentum. Otherwise, we have the possibility of detecting particles that do not conserve momentum. The

[13] 14 October 2010

spins have to be correlated also, to conserve angular momentum. I do not have a mechanism for the correlations. Maybe Cramer's theory is needed for that.

4. Cramer would say that they always have back and forth communication in time via the source for non-local agreements - without that mechanism - I can't think why/how correlation should occur. If a measuring apparatus is a perfect absorber - then materialization somewhere on it is guaranteed - it shouldn't depend on also having a fortuitous fluctuation of the ψ field.

   My response: First, I don't have fluctuations of the ψ field. ψ is the wave function (that does not fluctuate) and whose absolute square gives the probability for detection of the particle in an ensemble. It is the actual field (which we cannot predict) that fluctuates. A reference for fluctuating E and B fields is in Sakurai, p. 32. Second, since my theory (or interpretation) is a realistic theory, I require that the actual field (E or B in the case of photons or whatever field electrons are in the case of electrons) must actually concentrate a quantum of stuff in the neighborhood of an apparatus for it to be detected. Since we know that fluctuations occur, then it seems reasonable to assume that the fluctuations are sufficient to concentrate a quantum as needed. Actually, it should be possible to calculate that.

5. QM has contextuality so that ψ and absorber are tied together. The fluctuations (of some certain threshold amplitude?) cause (momentary materialization) 'collapse' (to particleness? or quanta? or variable degree of solidity). It sounds to me like you have a 'Selection principle' here - a way to get candidate final decisions. And that is important! I think an 'interpretation' needs a little more. Suppose Bob and Alice are at different distances - Alice's ψ - EM field momentary materialization may not be at the right place or time as that needed for Bob.

   My response: Suppose Alice is closer than Bob. Suppose also, that the fluctuations are such that Alice detects a particle. In my interpretation, that means a quanta of the field corresponding to that particle was actually concentrated in or near enough to her apparatus. Once her apparatus has absorbed that quanta (which was already concentrated there), then that field will no longer fluctuate. Notice that her measurements did not cause a wave function to collapse. The wave was already concentrated there. What does happen is that the wave will not jump around any more.

   Now, what happens to the particle/quanta/wave field corresponding to the other particle, the one that may be near Bob? Because the fluctuations are correlated, that wave is now a concentrated wave packet moving in the opposite direction from Alice, with a spin opposite to that detected by Alice (if she measured spin). As that wave packet gets close to Bob's apparatus, he has the possibility of detecting it. Again, I have no mechanism for the correlations. Maybe Cramer's theory can do that.

6. Cramer would arrange a handshaking regardless of the distance to the two absorbers. I really like Cramer (even though he is horribly under-developed).

A few more thoughts:

1. Does QFT really aid QM Interpretation? Usefulness of QFT for QM Interpretation:

   - 'quantum-statistics' explanation, variable superposition of indefinite number of particles, particles vs fields, zitterbewegung, localization with Compton wavelength, quantum fluctuations, – but limited usefulness. ψ(QFT) is incompatable with ψ(NR) cannot even deduce $\psi^*\psi$ probability from QFT.

     My response: Dave, are you sure about that?
   - A successful interpretation may not require QFT or strings or TOE.

2. Requirements of a Quantum Interpretation:

   - Explain what a wavefunction or state-vector is.
     My response: It applies to an ensemble.
   - Explain "Collapse of the wave-function."
     My response: The wave (the actual wave, not the wave function) is collapsing all of the time because of fluctuations. Because the wave function applies only to an ensemble, it does not collapse because it is not the real wave.
   - A selection principle versus randomness.
     My response: Dave, I'm not sure what you mean here.
   - Discuss reality below the level of classical observers.
     My response: The reality is fluctuating waves.
   - Explain contextuality dependence on a final 'absorber.'
     My response: The field is fluctuating. If during its fluctuations, it is concentrated enough as something that could interact with it, it could be 'detected'.
   - Satisfy generalized Einstein-locality but also explain the non-locality of entanglement.
     My response: My interpretation does not explain entanglement. It just assumes it. Maybe Cramer is needed to explain it.

3. The ensemble interpretation, unlike other interpretations to the Copenhagen Interpretation, does not attempt to justify, or otherwise derive, or explain quantum mechanics from any deterministic process, or make any other statement about the real nature of quantum phenomena; it is simply a statement as to the manner of wave function interpretation. It is a minimal interpretation. [WIK]. Mermin says that individual systems have to be addressed.

   My response: I think I agree. wave functions apply only to ensembles. Individual systems see the fluctuating waves.

4. New knowledge from this year:

   - De Broglie/Bohm is consistent and works for particles and photons and can be made relativistic and even 'symmetric QM' retrocausality. [5/10].
   - Quantum interference can occur with Buckyballs and neutrons (using magnetic moments) appears that 'particle' actually exists in multiple places at the same time.
     My response: I am still having trouble understanding that one.
   - ψ probabilities are associated with physical fields but detectors see their localized derivatives (d/dx, d/dt, or $d/d\phi$) to obtain information on momentum, energy or angular momentum.
     My response: I'm not sure I have anything to say about that.
   - Collapse scenarios then have to nullify not only the unused probabilities but also the old physical fields as well. How?
     My response: I think I have covered this. Nothing actually collapses, except the fluctuating fields that could be considered to be continually 'collapsing.'
   - Photons can carry orbital angular momentum as well as spin.
     My response: I think that is not a problem.
   - Entangled photons can act as a group: $\lambda(N) = \lambda(1)/N$ e.g., 'biphotons' with non-local wavelengths.
     My response: Again, I am still having trouble understanding this one.

- Valence quarks only occupy the inner quarter radius of the nucleon.
  My response: Interesting.
- Having an FT pair $f(x)$ to $F(s)$ satisfies the derivative operator $DF = sF$.
  My response: Can you elaborate?

Later comments:

1. Conservation laws have to be obeyed and the idea of exchange of quanta has to be presently presumed. My present ideas about mechanisms behind the scene are only a first guess - The idea is to select the best of many such sets of guesses.

2. You know, Bohm with its deterministic trajectories is also a statistical theory: An 'ensemble' as a 'conceptual set of replicas of one particle in its experimental surroundings' is analogous in Bohm theory to the set of all initial conditions (the set of all Bohmian trajectories of a Bohmian particle through a system). There is a fuzzy $\psi^*\psi$ with many particular concrete instances behind it, an ensemble wavefunction with a hidden detailed reality too.

   My response: I have been skeptical of Bohmian mechanics ever since I discovered that his trajectories were the same as those we calculate in ray tracing for WKB approximations to the wave equation. It is the same for any kind of wave, including radio waves, acoustic waves, and electron waves. In the case of classical waves, we know the trajectories are just a mathematical artifice that is useful for calculating fields.

3. Also Cramer calls the selection of one possibility for a transaction a 'stochastic choice' based on the strength of $\psi^*\psi$ but determined at the SOURCE rather than at the detector!

4. On QFT not yielding NR $\psi^*\psi$ probability [It was mentioned in my monthly notes November '09 and September '10 one referring to the "Brooks Blog" and the other to Nicolic]. QFT does not explain field collapse, and In QFT ψ is not probability it is actual intensity of real fields. However, Zee [p173] says QFT began as relativistic Klein-Gordon to Schrödinger in Nr limit, $\phi = exp(-imt)/\sqrt(2m)\phi(x, t \text{ NR})$. Noether current reduces to $J_o = \phi^\dagger\phi$ (which sounds good). But for NR Schrödinger $(d/dt)(\int \psi^*\psi) = 0$ not true for KG so $\psi^*\psi$ not Prob. See Nicolic:

   Ref: http://arxiv.org/pdf/quant-ph/0609163v2

   My response: Quantum Field Theory does allow one to calculate the amplitude for a process (such as the outcome for a particular measurement). Taking the absolute square gives the probability for observing that outcome.

5. On a selection principle versus randomness it could really be that Nature does random selections I just don't like it. My hidden mechanism of matching pointers would reduce the randomness.

6. The apparent appearance of a 'particle' at different places at the same time is the main problem of QM , we hope to resolve it.

   My response: If the field associated with a particle is fluctuating, it will not be possible for a full quantum of that field to exist at different places at the same time. Therefore, it is not possible to detect the same particle at different places at the same time.

7. My new belief is that Nature has networks between particles and that quanta transfer occurs by information transfer along the network paths. Just as for electronic channels, there has to be sequences of requests and acknowledgments (REQ/ACK's) for the reliable and limited

transfer of quanta information. The physics we discuss comes from here , but there is a sub-physics below that transmits and IS the information. Taking derivatives of ψ is a way of extracting information by itself.

8. I still feel that nullifying the combination of physical field and probability field is a problem , It is hard to picture it all being done with the word 'fluctuation.'

   My response: The probability field does not fluctuate. It applies only to an ensemble. The physical field is the real field. It fluctuates, but we cannot predict the details of that fluctuation.

9. You're not shocked by the existence of single photon orbital angular momentum? Perhaps the lesson is that anything that EM fields can do can be considered as a set of photons. A laser beam is coherent EM field - quantize it.

   My response: I think the orbital angular momentum of a single photon must depend on coordinate system. Like impact parameter in scattering theory.

10. Biphotons is new and strange (from 2002), but experimentally verified so we somehow have to get used to it. It is consistent with the entanglement concept of total group energy/momentum.

11. The Fourier Transform pair $f(x)$ to $F(s)$ or between position and momentum wavefunctions is equivalent to having p represented by the operator d/dx (i's and $\hbar$s to be added). And differentiation of the wavefunction $\psi(x)$ gives $p\psi(x)$. So, as a postulate, there is a free choice between stating $p = d/dx$ or requiring Fourier Transforms between $\psi(x)$ and $\phi(p)$. Also, $dF/F = -isdx$ implies $F = a\exp(-isx)$.

    My response: OK, now I understand.

12. There are different kinds of fluctuations: Nelson stochastic diffusion via Brownian motion fluctuations, and ZPE. Nelson derives Schrodinger, but that doesn't explain collapse.

    My response: Brownian motion is something else. I would guess that my fluctuations are probably consistent with zero-point energy.

13. Penrose talks about zig-zag electron motion (more on zitterbewegung– and he believes in it)– So the "frame of the electron" is a hard concept. The expectation value is always +/- c (both ways alternating), but net motion is v.

    My response: My 'frame of the electron' calculations were non-quantum general relativity calculations. I don't know how to reconcile them with quantum theory.

Later comments:

1. Wow! So many multiple usage naming problems: without fluctuations, I use ψ and wavefunction for both probability amplitude and field amplitude , need separate names! 'ϕ' is also overused, and E could be energy or electric field. We could use 'f' for 'field' energy amplitude (e.g., both E and H combined and with appropriate factors μ, ϵ, fractions), 'force' isn't relevant here. ψ is normalized, f^2 is limited by quanta energy. In many cases, f is proportional to ψ. For EM, H energy matches E energy, so E suffices. Clauser talks about E^2 having to match quanta $h\nu$.

   My response: Yes, there is a notation problem that we have to work out. The wave function is normalized, but the fields are not. In the case of photons, we already have E and H for the fields, but we have to come up with something for the electron wave fields. It would help if we know what those fields were. They might be also E and H.

2. It seems to me that if $f(r)$ starts with integrated energy $h\nu$, after long distance spherical amplitude falloff , it is unlikely that a fluctuation would ever attain $h\nu$ energy anywhere again.

   My response: Yes, the field is getting weaker, so the probability of detection is getting smaller. However, we can calculate that probability, and it is not zero. In my realistic interpretation, it is equal to the probability that one quanta of the field is actually concentrated in that region.

3. I'm missing something basic: If we remove fluctuations from discussion for the moment, what are some examples of the real f fields differing from the 'ensemble' ψ fields?

   My response: The only difference is the fluctuations.

4. Are you thinking of things like spherical ψ fields composed of ray f fields? Wouldn't a ray still diverge? Laser beams diverge. Or cylindrical slit falloff to single ray absorption points on a screen?

   My response: Yes, the rays diverge, and the fields get weaker.

5. Sounds like Bohmian rays in the ensemble ψ again. But photons can't be localized.

   My response: That is right. A photon is not localized. It does not even have an individual identity. Electric fields add. Magnetic fields add. A photon exists only as a quantum during creation or absorption.

6. You are picturing something in your head that I'm not yet able to match in my head. Bohm trajectories being an 'artifice' would go with them only having position and velocity and no other physical properties. But there are sure a lot of people now getting on the Bohm bandwagon. But not Bohm himself. He hated the name 'Bohmian Mechanics' and only intended it as a counterexample with more future physics needed. "I think the orbital angular momentum of a single photon must depend on coordinate system. Like impact parameter in scattering theory." Yes, the case in point is the cylindrical coordinate system of the laser beam with a well-defined center ray. The orbit goes around that axis. The problem for me is that the tight laser beam depends on having lots of neighbor bosons all doing the same thing. For single photons, there are no longer any neighbors. There must be a past memory of being in a group and that defines a ghost coordinate system.

   My response: I don't think there is any ghost memory for photons. Photons merge and lose their identity. The electric fields add.

7. What did you think of Nikolic's argument about QFT not giving probability ψ's (QM Myths and facts – pg 23-24)? Apparently, you don't agree. Ref: http://arxiv.org/pdf/quant-ph/0609163v2

   My response: I read only the first two pages (the table of contents). The paper looks interesting, but long. Maybe I should read all of it. I am guessing that it is simply a matter of notation. Although ψ may not give probability, Quantum Field Theory does give the amplitude for processes, and the absolute square of those amplitudes is the probability for the process to occur.

Chapter 91

Dirac equation in the frame of the electron[1]

abstract

Writing the Dirac equation in the frame of the electron partly solves the problem of separating gravitation from geometry.

91.1 Introduction

We write the Dirac equation in an arbitrary frame. Then we look for how to specialize that to the frame fixed with the electron.

First, we consider a free particle. In that case, we get for a solution, time harmonic in both space and time. In the frame of the particle, the spatial harmonic disappears. The corresponding Dirac equation would have no spatial derivatives.

For the general case, we can try doing the same thing. Simply take out the spatial derivatives to get the Dirac equation in the frame of the electron. This may not be correct, but I can think of no other way to generalize from the free-particle case.

To check on the validity, I can consider some simple cases. First, a constant acceleration in a local region. I don't want to consider the global case, because that gets into Rindler spaces. I would set up the situation in an inertial frame, transform to the frame of the electron, solve the Dirac equation in the frame of the electron, and transform back.

Second, a harmonic oscillator. Again, I would set up the situation in an inertial frame, transform to the frame of the electron, solve the Dirac equation in the frame of the electron, and transform back. I would then compare with the known solution.

It would be nice to solve the one-electron atom this way, but that might be too difficult.

91.2 Additions, 21-23 June 2015

In Newtonian theory, motion in a straight line at constant speed in a single direction is normal. It a takes a force to change that. In General Relativity, it is similar. Without matter, we would have a flat Minkowski spacetime as normal. In quantum theory, it is similar. Without interaction, the solution is plane waves in an inertial frame.

[1]May 2010, updated January 18, 2016

Although General Relativity can determine the inertial frames, it doesn't determine the strength of the inertia[2]. Only Sciama's 1953 paper, "On the origin of inertia," [12] was able to calculate the strength of inertia, but in a non-rigorous way. We learned that G was not a gravitational constant, but an inertial constant.

91.2.1 Integral form of Einstein's field equations

The SWG 1969 paper [16] gave a rigorous way to calculate inertia from General Relativity. Using the SWG 69 equations, I can get the intensity of inertia in the frame of the particle.

To see the inertial effect from induction by the rest of matter in the universe, we use a result from the 1969 paper by Dennis Sciama, P. C. Waylen, and Robert Gilman on the integral formulation of Einstein's field equations[16, equation (14)] and a follow-up paper by Robert Gilman [156, equation (1)]. The metric tensor at one point is equal to a volume integral over spacetime plus a surface integral: [3]

$$g^{\alpha'\beta'}(x') = 2\kappa \int_\Omega G^{-\alpha'\beta'\nu}{}_\mu \left(T^\mu{}_\nu - \frac{1}{2}T^\lambda{}_\lambda \delta^\mu{}_\nu - \frac{\Lambda}{\kappa}\delta^\mu{}_\nu\right)[-g(x)]^{1/2} d^4x + \int_{\partial\Omega} G^{-\alpha'\beta'\nu}{}_\nu{}^{;\sigma}[-g(x)]^{1/2} dS_\sigma \ , \tag{91.1}$$

where $G^-(x', x)$ is the retarded Green's functional that gives the contribution to the metric at x' from the stress-energy tensor at x, and $\kappa \equiv 8\pi G/c^2$. Equation (91.1) shows how $g^{\alpha\beta}$ here depends on the matter distribution in the universe through the volume integral.[4] The surface integral accounts for sources outside of the volume plus initial values and boundary values. That the relative rotation of local inertial frames and distant matter is so far unmeasurable (in the absence of local Lense-Thirring effects) suggests that for our universe, the volume integral dominates over the surface integral in (91.1).

Equation (91.1) is actually an integral equation that must be solved for the whole cosmology in the same way that Einstein's field equations must be solved. The significance of (91.1) is that the Green's functional G is not changed for first-order variation in the sources.

Once all 16 of the $g^{\alpha\beta}$ have been determined from (91.1), the inverse of $g^{\alpha\beta}$ (as a matrix) gives the components of $g_{\alpha\beta}$.

Equation (91.1) can also be written

$$g^{\alpha'\beta'}(x') = 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{\nu\mu} \left(T^{\mu\nu} - \frac{1}{2}T^\lambda{}_\lambda g^{\mu\nu} - \frac{\Lambda}{\kappa}g^{\mu\nu}\right)[-g(x)]^{1/2} d^4x + \int_{\partial\Omega} G^{-\alpha'\beta'\nu}{}_\nu{}^{;\sigma}[-g(x)]^{1/2} dS_\sigma \ . \tag{91.2}$$

We can take for the stress-energy tensor[20, p. 132]

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu} \ . \tag{91.3}$$

Thus, (91.2) becomes

$$g^{\alpha'\beta'}(x') = 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{\nu\mu}(x', x)(\rho + p)U^\mu U^\nu[-g(x)]^{1/2} d^4x + g_\Lambda^{\alpha'\beta'}(x') + g_{\text{surface}}^{\alpha'\beta'}(x') \ , \tag{91.4}$$

where

$$g_\Lambda^{\alpha'\beta'}(x') \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{\nu\mu}(x', x)\left(\frac{1}{2}(\rho - p) - \frac{\Lambda}{\kappa}\right) g^{\mu\nu}[-g(x)]^{1/2} d^4x \tag{91.5}$$

and

$$g_{\text{surface}}^{\alpha'\beta'}(x') \equiv \int_{\partial\Omega} G^{-\alpha'\beta'\nu}{}_\nu{}^{;\sigma}(x', x)[-g(x)]^{1/2} dS_\sigma \tag{91.6}$$

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[2]Not quite right, see further on.

[3]This equation is also repeated in (92.24).

[4]Note added 11 July 2014: Altshuler [260] showed that the integral formulation has ghosts unless the dimension of spacetime equals 4. This would cause the background to be unstable.

is the contribution from the surface integral, which we do not consider here.

The differential equation for the retarded Green's function $G^{-\alpha'\beta'}{}_{\nu\mu}(x',x)$ is given by Sciama, Waylen, and Gilman in 1969, [16, equation (9)] and is [5]

$$\Box G^{-\alpha'\beta'}{}_{\mu\nu}(x',x) - 2R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu} G^{-\alpha'\beta'}{}_{\rho\sigma}(x',x) = g^{\alpha'}{}_{(\mu} g^{\beta'}{}_{\nu)} [g(x')g(x)]^{-1/4} \delta(x',x) \ , \tag{91.7}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian, ∇_μ is a covariant derivative, parentheses on indexes denote symmetrization, $R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}$ is the Riemann tensor, $\delta(x',x)$ denotes the four-dimensional δ distribution of the two points x' and x, $g^{\alpha'}{}_{\mu}(x',x)$ denotes the two-point vector of geodesic parallel transport which is covariant at x and contravariant at x', and satisfies

$$\lim_{x'\to x} g^{\alpha'}{}_{\mu}(x',x) = \delta^{\alpha'}{}_{\mu}(x',x) \ . \tag{91.8}$$

For the Robertson-Walker metric, many of the components of $R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}$ are zero. Robertson and Noonan's book[261, page 343] gives the values of $R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}$. These are:

$$R_{0i0j} = -S\ddot{S}h_{ij} \ , \tag{91.9}$$

where S is proportional to the radius of curvature[6], h_{ij} is the spatial metric, and

$$R_{ijkl} = +S^2(\dot{S}^2 + k)(h_{ik}h_{jl} - h_{il}h_{jk}) \ , \tag{91.10}$$

where a dot denotes differentiation with respect to cosmic time, and I have converted from Robertson and Noonan's convention to Sciama, Waylen, and Gilman's convention because they define the Riemann Tensor $R_{\alpha\beta\gamma\delta}$ with the opposite sign.

We know from measurement that the universe is spatially flat within measurement error. Therefore, we take $k = 0$ and take $h_{ij} = \delta_{ik}$ for Cartesian coordinates. This gives

$$R^{0}{}_{1}{}^{0}{}_{1} = R^{0}{}_{2}{}^{0}{}_{2} = R^{0}{}_{3}{}^{0}{}_{3} = R^{1}{}_{0}{}^{1}{}_{0} = R^{2}{}_{0}{}^{2}{}_{0} = R^{3}{}_{0}{}^{3}{}_{0} = R^{0}{}_{1}{}^{1}{}_{0} = R^{0}{}_{2}{}^{2}{}_{0} = R^{0}{}_{3}{}^{3}{}_{0} = -S\ddot{S} \tag{91.11}$$

and

$$R^{1}{}_{2}{}^{1}{}_{2} = R^{1}{}_{3}{}^{1}{}_{3} = R^{2}{}_{3}{}^{2}{}_{3} = R^{2}{}_{1}{}^{2}{}_{1} = R^{3}{}_{1}{}^{3}{}_{1} = R^{3}{}_{2}{}^{3}{}_{2} = +S^2\dot{S}^2 \ . \tag{91.12}$$

Because of the particular elements of the Riemann tensor that are non-zero, (91.7) can be separated into several uncoupled equations. These are

$$\left(\Box + m_A^2\right) G^{-\alpha'\beta'}{}_{A}(x',x) = -\left[\frac{m_A^2}{3m_E^2}\left(g^{\alpha'}{}_{1}g^{\beta'}{}_{1} + g^{\alpha'}{}_{2}g^{\beta'}{}_{2} + g^{\alpha'}{}_{3}g^{\beta'}{}_{3}\right) + g^{\alpha'}{}_{0}g^{\beta'}{}_{0}\right][g(x')g(x)]^{-1/4}\delta(x',x) \ , \tag{91.13}$$

$$\left(\Box + m_B^2\right) G^{-\alpha'\beta'}{}_{B}(x',x) = \left[\left(g^{\alpha'}{}_{1}g^{\beta'}{}_{1} + g^{\alpha'}{}_{2}g^{\beta'}{}_{2} + g^{\alpha'}{}_{3}g^{\beta'}{}_{3}\right) - \frac{m_A^2}{m_E^2} g^{\alpha'}{}_{0}g^{\beta'}{}_{0}\right][g(x')g(x)]^{-1/4}\delta(x',x) \ , \tag{91.14}$$

$$\left(\Box + m_C^2\right) G^{-\alpha'\beta'}{}_{C}(x',x) = \left(g^{\alpha'}{}_{2}g^{\beta'}{}_{2} - g^{\alpha'}{}_{1}g^{\beta'}{}_{1}\right)[g(x')g(x)]^{-1/4}\delta(x',x) \ , \tag{91.15}$$

$$\left(\Box + m_D^2\right) G^{-\alpha'\beta'}{}_{D}(x',x) = \left(g^{\alpha'}{}_{2}g^{\beta'}{}_{2} + g^{\alpha'}{}_{3}g^{\beta'}{}_{3} - 2g^{\alpha'}{}_{1}g^{\beta'}{}_{1}\right)[g(x')g(x)]^{-1/4}\delta(x',x) \ , \tag{91.16}$$

$$\left(\Box + m_E^2\right) G^{-\alpha'\beta'}{}_{iE}(x',x) = \left(g^{\alpha'}{}_{0}g^{\beta'}{}_{i} + g^{\alpha'}{}_{i}g^{\beta'}{}_{0}\right)[g(x')g(x)]^{-1/4}\delta(x',x) \ , \tag{91.17}$$

and

$$\left(\Box + m_F^2\right) G^{-\alpha'\beta'}{}_{iF}(x',x) = 0 \ , \tag{91.18}$$

[5] This equation is also repeated in (92.77).

[6] For the case of the spatial metric being flat, S is an arbitrary constant.

where

$$G^{-\alpha'\beta'}{}_A(x',x) \equiv -\frac{m_A^2}{3m_E^2}\left(G^{-\alpha'\beta'}{}_{11}(x',x)+G^{-\alpha'\beta'}{}_{22}(x',x)+G^{-\alpha'\beta'}{}_{33}(x',x)\right)-G^{-\alpha'\beta'}{}_{00}(x',x)\ , \tag{91.19}$$

$$G^{-\alpha'\beta'}{}_B(x',x) \equiv \left(G^{-\alpha'\beta'}{}_{11}(x',x)+G^{-\alpha'\beta'}{}_{22}(x',x)+G^{-\alpha'\beta'}{}_{33}(x',x)\right)-\frac{m_A^2}{m_E^2}G^{-\alpha'\beta'}{}_{00}(x',x)\ , \tag{91.20}$$

$$G^{-\alpha'\beta'}{}_C(x',x) \equiv G^{-\alpha'\beta'}{}_{22}(x',x)-G^{-\alpha'\beta'}{}_{11}(x',x)\ , \tag{91.21}$$

$$G^{-\alpha'\beta'}{}_D(x',x) \equiv G^{-\alpha'\beta'}{}_{22}(x',x)+G^{-\alpha'\beta'}{}_{33}(x',x)-2G^{-\alpha'\beta'}{}_{11}(x',x)\ , \tag{91.22}$$

$$G^{-\alpha'\beta'}{}_{iE}(x',x) \equiv G^{-\alpha'\beta'}{}_{0i}(x',x)+G^{-\alpha'\beta'}{}_{i0}(x',x)\ , \tag{91.23}$$

$$G^{-\alpha'\beta'}{}_{iF}(x',x) \equiv G^{-\alpha'\beta'}{}_{0i}(x',x)-G^{-\alpha'\beta'}{}_{i0}(x',x)\ , \tag{91.24}$$

$$m_A^2 \equiv -2R^1{}_2{}^1{}_2+2\sqrt{\left(R^1{}_2{}^1{}_2\right)^2+3\left(R^0{}_1{}^0{}_1\right)^2} = -2S^2\dot S^2+2\sqrt{S^4\dot S^4+3S^2\ddot S^2}>0\ , \tag{91.25}$$

$$m_B^2 \equiv -2R^1{}_2{}^1{}_2-2\sqrt{\left(R^1{}_2{}^1{}_2\right)^2+3\left(R^0{}_1{}^0{}_1\right)^2} = -2S^2\dot S^2-2\sqrt{S^4\dot S^4+3S^2\ddot S^2}<0\ , \tag{91.26}$$

$$m_C^2 = m_D^2 \equiv +2R^1{}_2{}^1{}_2 = +2S^2\dot S^2>0\ , \tag{91.27}$$

$$m_E^2 \equiv -2R^0{}_1{}^1{}_0 = +2S\ddot S>0\ , \tag{91.28}$$

and

$$m_F^2 \equiv +2R^0{}_1{}^1{}_0 = -2S\ddot S<0\ . \tag{91.29}$$

Notice that the above equations for the Green's functions also describe the propagation of fluctuations in the metric or in the gravitational field. Therefore, these are also the equations for gravitational waves or gravitons. Thus, the above effective masses are the effective masses of gravitons. To evaluate the above (time-dependent) effective masses, we take the present values for $S=1$, $\dot S=22$km/s/Mega light year, and $\ddot S=\dot S^2$ to give

$$m_A \approx m_C = m_D \approx m_E \approx \sqrt{2}\dot S \approx \frac{1}{10^{10}\ \text{light years}}\ , \tag{91.30}$$

and for the two tachyon masses

$$m_B \approx \pm i\sqrt{6}\dot S \approx \frac{\pm i}{6\times 10^9\ \text{light years}} \tag{91.31}$$

and

$$m_F \approx \pm i\sqrt{2}\dot S \approx \frac{\pm i}{10^{10}\ \text{light years}}\ . \tag{91.32}$$

Of course, the values of the effective masses would be different in different eras. In addition, these effective masses are global averages for the whole universe. Locally (in a galaxy, say), the effective masses could be significantly larger.

From (91.19) and (91.20), we have

$$G^{-\alpha'\beta'}{}_{00}(x',x) = \frac{m_B^2G^{-\alpha'\beta'}{}_A(x',x)-m_E^2G^{-\alpha'\beta'}{}_B(x',x)}{(m_A^2-m_B^2)} \tag{91.33}$$

and

$$G^{-\alpha'\beta'}{}_{11}(x',x)+G^{-\alpha'\beta'}{}_{22}(x',x)+G^{-\alpha'\beta'}{}_{33}(x',x) = \frac{-3m_E^2G^{-\alpha'\beta'}{}_A(x',x)-m_B^2G^{-\alpha'\beta'}{}_B(x',x)}{(m_A^2-m_B^2)}\ . \tag{91.34}$$

From (91.21), (91.22), and (91.34), we have

$$G^{-\alpha'\beta'}{}_{11}(x',x) = -\frac{m_E^2 G^{-\alpha'\beta'}{}_A(x',x)}{(m_A^2 - m_B^2)} - \frac{1}{3}\frac{m_B^2 G^{-\alpha'\beta'}{}_B(x',x)}{(m_A^2 - m_B^2)} - \frac{1}{3}G^{-\alpha'\beta'}{}_D(x',x)\ , \tag{91.35}$$

$$G^{-\alpha'\beta'}{}_{22}(x',x) = -\frac{m_E^2 G^{-\alpha'\beta'}{}_A(x',x)}{(m_A^2 - m_B^2)} - \frac{1}{3}\frac{m_B^2 G^{-\alpha'\beta'}{}_B(x',x)}{(m_A^2 - m_B^2)} + G^{-\alpha'\beta'}{}_C(x',x) - \frac{1}{3}G^{-\alpha'\beta'}{}_D(x',x)\ , \tag{91.36}$$

and

$$G^{-\alpha'\beta'}{}_{33}(x',x) = -\frac{m_E^2 G^{-\alpha'\beta'}{}_A(x',x)}{(m_A^2 - m_B^2)} - \frac{1}{3}\frac{m_B^2 G^{-\alpha'\beta'}{}_B(x',x)}{(m_A^2 - m_B^2)} - G^{-\alpha'\beta'}{}_C(x',x) + \frac{2}{3}G^{-\alpha'\beta'}{}_D(x',x)\ . \tag{91.37}$$

Equations (91.13) through (91.18) represent Yukawa potentials. The Robertson-Walker metric is spatially homogeneous. In addition, based on observations, we are taking the metric here to be spatially flat. Therefore, we can take the solutions of (91.13) through (91.18) to depend only on $|\mathbf{r}-\mathbf{r}'|$. This gives the following Yukawa solutions.

$$G^{-\alpha'\beta'}{}_A(x',x) = -\frac{\frac{m_A^2}{3m_E^2}\left(\delta^{\alpha'}{}_1\delta^{\beta'}{}_1 + \delta^{\alpha'}{}_2\delta^{\beta'}{}_2 + \delta^{\alpha'}{}_3\delta^{\beta'}{}_3\right) + \delta^{\alpha'}{}_0\delta^{\beta'}{}_0}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{-m_A|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.38}$$

$$G^{-\alpha'\beta'}{}_B(x',x) = \frac{\left(\delta^{\alpha'}{}_1\delta^{\beta'}{}_1 + \delta^{\alpha'}{}_2\delta^{\beta'}{}_2 + \delta^{\alpha'}{}_3\delta^{\beta'}{}_3\right) - \frac{m_A^2}{m_E^2}\delta^{\alpha'}{}_0\delta^{\beta'}{}_0}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{\pm i|m_B||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.39}$$

$$G^{-\alpha'\beta'}{}_C(x',x) = \frac{\delta^{\alpha'}{}_2\delta^{\beta'}{}_2 - \delta^{\alpha'}{}_1\delta^{\beta'}{}_1}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{-m_C|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.40}$$

$$G^{-\alpha'\beta'}{}_D(x',x) = \frac{\delta^{\alpha'}{}_2\delta^{\beta'}{}_2 + \delta^{\alpha'}{}_3\delta^{\beta'}{}_3 - 2\delta^{\alpha'}{}_1\delta^{\beta'}{}_1}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{-m_D|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.41}$$

$$G^{-\alpha'\beta'}{}_{iE}(x',x) = \frac{\delta^{\alpha'}{}_0\delta^{\beta'}{}_i + \delta^{\alpha'}{}_i\delta^{\beta'}{}_0}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{-m_E|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.42}$$

and

$$G^{-\alpha'\beta'}{}_{iF}(x',x) = 0\ . \tag{91.43}$$

The non-zero solutions of (91.38) through (91.43) are:

$$G^{-0'0'}{}_A(x',x) = -\frac{1}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{-m_A|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.44}$$

$$G^{-1'1'}{}_A(x',x) = G^{-2'2'}{}_A(x',x) = G^{-3'3'}{}_A(x',x) = -\frac{m_A^2/m_E^2}{12\pi[g(x')g(x)]^{1/4}}\frac{e^{-m_A|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.45}$$

$$G^{-0'0'}{}_B(x',x) = \frac{-m_A^2/m_E^2}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{\pm i|m_B||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.46}$$

$$G^{-1'1'}{}_B(x',x) = G^{-2'2'}{}_B(x',x) = G^{-3'3'}{}_B(x',x) = \frac{1}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{\pm i|m_B||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.47}$$

$$G^{-2'2'}{}_C(x',x) = -G^{-1'1'}{}_C(x',x) = \frac{1}{4\pi[g(x')g(x)]^{1/4}}\frac{e^{-m_C|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{91.48}$$

$$G^{-1'1'}{}_D(x',x) = \frac{-2}{4\pi[g(x')g(x)]^{1/4}} \frac{e^{-m_D|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \ , \tag{91.49}$$

$$G^{-2'2'}{}_D(x',x) = G^{-3'3'}{}_D(x',x) = \frac{1}{4\pi[g(x')g(x)]^{1/4}} \frac{e^{-m_D|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \ , \tag{91.50}$$

and

$$G^{-0'i'}{}_{iE}(x',x) = G^{-i'0'}{}_{iE}(x',x) = \frac{1}{4\pi[g(x')g(x)]^{1/4}} \frac{e^{-m_E|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{91.51}$$

If we neglect the Λ term and the boundary term in (91.4), and we use the isotropy of the Robertson-Walker metric to take $U^1U^1 = U^2U^2 = U^3U^3$, then from (91.4), (91.33), (91.34), (91.44), and (91.46), we have

$$\begin{aligned} g^{0'0'}(x') = &-2\kappa \int_\Omega \frac{m_B^2 U^0U^0 - 3m_E^2U^1U^1}{m_A^2 - m_B^2} (\rho + p) \frac{e^{-m_A|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \left[\frac{g(x)}{g(x')}\right]^{1/4} |\mathbf{r}-\mathbf{r}'|^2 \, \mathrm{d}r \, \mathrm{d}t \\ &+2\kappa \int_\Omega \frac{m_E^2 U^0U^0 + m_B^2U^1U^1}{m_A^2 - m_B^2} \frac{m_A^2}{m_E^2} (\rho + p) \frac{e^{\pm i|m_B||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \left[\frac{g(x)}{g(x')}\right]^{1/4} |\mathbf{r}-\mathbf{r}'|^2 \, \mathrm{d}r \, \mathrm{d}t \ . \end{aligned} \tag{91.52}$$

Similarly, from (91.4), (91.33), (91.34), (91.45), and (91.47), we have

$$\begin{aligned} &g^{1'1'}(x') = g^{2'2'}(x') = g^{3'3'}(x') = \\ &-2\kappa \int_\Omega \tfrac{m_B^2 U^0U^0 - 3m_E^2U^1U^1}{3(m_A^2 - m_B^2)} \tfrac{m_A^2}{m_E^2} (\rho + p) \tfrac{e^{-m_A|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \left[\tfrac{g(x)}{g(x')}\right]^{1/4} |\mathbf{r}-\mathbf{r}'|^2 \, \mathrm{d}r \, \mathrm{d}t \\ &-2\kappa \int_\Omega \tfrac{m_E^2 U^0U^0 + m_B^2U^1U^1}{m_A^2 - m_B^2} (\rho + p) \tfrac{e^{\pm i|m_B||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \left[\tfrac{g(x)}{g(x')}\right]^{1/4} |\mathbf{r}-\mathbf{r}'|^2 \, \mathrm{d}r \, \mathrm{d}t \ . \end{aligned} \tag{91.53}$$

From (91.17) and (91.18), we find that (in a coordinate system in which the metric tensor is diagonal) the only non-zero components of $G^{-0'i'}{}_{\nu\mu}(x',x)$ in (91.4) are $G^{-0'i'}{}_{0i}(x',x)$ and $G^{-0'i'}{}_{i0}(x',x)$. Thus, (91.4) gives

$$g^{i'0'}(x') = g^{0'i'}(x') = 2\kappa \int_\Omega G^{-0'i'}{}_{iE}(x',x)(\rho+p)U^0U^i[-g(x)]^{1/2} d^4x \ . \tag{91.54}$$

Or, using (91.51) gives

$$g^{i'0'}(x') = g^{0'i'}(x') = 2\kappa \int_\Omega (\rho+p)U^0U^i \frac{e^{-m_E|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \left[\frac{g(x)}{g(x')}\right]^{1/4} |\mathbf{r}-\mathbf{r}'|^2 \, \mathrm{d}r \, \mathrm{d}t \ . \tag{91.55}$$

We can calculate the integrals in (91.52), (91.53), and (91.55) since we know $p(t)$ and $\rho(t)$ from the standard model for the radiation era, the matter era, and the dark energy era.

91.2.2 Dirac equation in any frame

Using the Brill-Wheeler 1957 [198] equations, I can write the Dirac equation in any frame, including the frame of the particle (electron or proton or positron). (See Chapters 72 and 92.) In doing that, I use the metric determined by the SWG 69 equations with the correct intensity of inertia. (See Chapters 92 and 105.)

In this way, I have a form of the Dirac equation in terms that explicitly has both EM effects and gravitation effects, including inertial effects. To get more insight, I can make the same calculation in the frame of either particle or in the center-of-mass frame, and compare the results.

However, the gravitational effects are still such that gravitation forms the background upon which everything else happens.

This section is based on the 1957 paper of Brill and Wheeler[198], which uses a $(+++-)$ signature, so some of the formulas may have to be changed for a different choice of signature.

We start with the Dirac equation in curved space[198, 262].

$$\gamma^\alpha \nabla_\alpha \psi + \mu\psi = 0 \ , \tag{91.56}$$

where $\mu \equiv mc/\hbar$, there is no confusion between this use of μ and μ used as a subscript or superscript,

$$\nabla_\alpha \psi = \frac{\partial \psi}{\partial x^\alpha} - \Gamma_\alpha \psi \ , \tag{91.57}$$

$$\Gamma_k = \frac{1}{4} g_{\mu\alpha} \left[\frac{\partial {b_\nu}^\beta}{\partial x^k} {a^\alpha}_\beta - {\Gamma_{\nu k}}^\alpha \right] S^{\mu\nu} + a_k \mathbf{1} \ , \tag{91.58}$$

${a^\alpha}_\beta$ and ${b_\nu}^\beta$ give transformations between the x frame to a local Lorentzian frame $\tilde{x}$ such that

$$dx^\alpha = {a^\alpha}_\beta d\tilde{x}^\beta \ , \tag{91.59}$$

$$d\tilde{x}^\beta = {b_\nu}^\beta dx^\nu \ , \tag{91.60}$$

$${\Gamma_{\nu k}}^\alpha = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{k\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^k} - \frac{\partial g_{\nu k}}{\partial x^\beta} \right) \ , \tag{91.61}$$

$$S^{ij} = \frac{1}{2}(\gamma^i\gamma^j - \gamma^j\gamma^i) \ , \tag{91.62}$$

a_k is arbitrary,

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ik}\mathbf{1} \ , \tag{91.63}$$

$\mathbf{1}$ is the unit matrix, and all subscripts and superscripts run from 1 to 4.

Taking

$$a_k \equiv \frac{ieA_k}{\hbar} \tag{91.64}$$

allows us to include A_k as an electromagnetic interaction so that we can apply the Dirac equation to electrons, positrons, or protons rather than neutrinos, [7]

Substituting (91.57), (91.58), and (91.64) into (91.56) gives

$$\gamma^\alpha \left[\frac{\partial}{\partial x^\alpha} - \frac{1}{4} g_{\mu\gamma} \left(\frac{\partial {b_\nu}^\beta}{\partial x^\alpha} {a^\gamma}_\beta - {\Gamma_{\nu\alpha}}^\gamma \right) S^{\mu\nu} - \frac{ieA_\alpha}{\hbar} \right] \psi + \mu\psi = 0 \ . \tag{91.65}$$

as the Dirac equation. Equation (91.65) may finally give us the chance to separate gravitation from geometry. All but the last term inside the square brackets are gravitational terms. The factor γ^α is a geometric term [through (91.63)]. The mass term is also a gravitational term.

Consider the four terms in the square brackets. For a plane wave, the first term becomes $ip_\alpha/\hbar$. The second term gives the contribution to the gravitational potential for a non-inertial frame. The third term combines with the derivative of the first term to make a covariant derivative for a curved spacetime. The fourth term gives the electromagnetic potential.

[7] That (91.56) contains no weak force suggests that (91.56) is not really a complete equation for a neutrino (or for an electron, for that matter).

91.3 Trying to reconcile General Relativity and Quantum Theory

It is well known that General Relativity and Quantum Theory are inconsistent with each other. The most widely known example is the information-loss paradox for black holes. In General Relativity, a black hole is characterized by its mass, electric charge, and angular momentum. However, in Quantum Theory, baryon number and lepton number are also conserved quantities. In General Relativity, it doesn't matter if the mass, electric charge, and angular momentum come from baryons or leptons. Therefore, from the viewpoint of Quantum Theory, that information is lost when baryons and leptons fall into a black hole.

Most attempts to reconcile the information-loss paradox implicitly assume that either General Relativity is correct or that Quantum Theory is correct, with obvious results. A more likely solution is to look at possible flaws in both theories.

One possible flaw in both theories is that both theories are based on coordinate systems in which to represent the two theories. By requiring the theories to be covariant (that is, valid in any arbitrary coordinate system), we hope that our theories may be correct. Covariance may be a necessary condition for a theory to be valid, but it may not be sufficient. When physical law is expressed in some arbitrary coordinate system, it is expressed as something looked at from the outside, rather than something seen from within the system itself. Thus, such formulations may be useful to us to try to understand the situation, but such a formulation probably does not represent how the physical system itself works.

It is not clear how we should formulate physical laws in such a way that they are close to how the physical system itself works, but one possibility is to write equations for each body or particle in a frame of reference attached to that body or particle. If we do this for each body or particle, then all of those equations taken together should define the system.

In the case of classical physics (or, in the classical approximation to the quantum system), we would write the geodesic equation for each body or particle in the frame of the body or particle. In the frame of the body, the 4-velocity has only a time component, and all acceleration terms are zero. To write the geodesic equation for each body or particle in the frame of the body or particle, we would use the gravitational field (connection) and the electromagnetic field in that frame from all of the other bodies and particles in the universe. Each body or particle would move in such a way that the total force on that body or particle is zero. If each body or particle satisfies the geodesic equation in its own frame we would have a complete system to determine the solution for the whole cosmology. To understand how that works from our point of view, we could transform each of those geodesic equations to a common coordinate system of our own choosing. When we do that, we would end up with standard General Relativity (or whatever gravitational theory turns out to be correct) combined with Maxwell theory. Of course, we would need to include the strong and weak interactions also.

In the case of quantum mechanics (as opposed to quantum field theory), we would write the Dirac equation (for Fermions) or the Klein-Gordon equation (for Bosons) for each particle in the frame of that particle. To do that, we would use the gravitational potential and the electromagnetic potential in that frame due to all of the other particles. Each particle would move in such a way that its own wave equation would be satisfied in its own coordinate system. For us to look at how this works in a unified way, we would transform each of the wave equations from the frame of the particle to some common coordinate system. Presumably, this would give a formulation of the quantum many-body problem.

I am trying to decide if the appropriate wave equation for a particle in its own frame would involve only time derivatives, since the particle has no information outside of itself. Maybe to answer that question, we need to see which way gives the usual Dirac equation when we transform to an arbitrary frame. The answer is that the particle in question is at the origin of the coordinate system, but the coordinate system includes the whole universe with all of the other particles, which may be moving with arbitrary velocities. Therefore, all of the derivative terms are still there. If all of the derivative terms are still there in the frame of the electron, what does distinguish the Dirac equation in the frame of the electron? Maybe it is the values of all of the gamma matrices, since they are determined by the metric, and that might be different in the frame of the electron.

For quantum field theory, we would still write the Dirac equation or Klein-Gordon equation for each particle in the frame of that particle. The difference now, is that in calculating the gravitational and electromagnetic potentials due to all of the other particles, we have to take into account that all of those other particles are actually waves, described by a wave function. I am not sure quite how to do that in general, but there are some special cases to consider. If we neglect the curvature of space-time, then we can expand gravitational and electromagnetic potentials in a Fourier series, and denote the coefficients in that expansion as creation and annihilation operators. These would be the creation and annihilation operators for gravitons and photons.

Actually, developing creation and annihilation operators this way may not work for gravitons because there is a gravitational field even without curvature.

Also, for quantum field theory, we have to consider higher-order Feynman diagrams, and we have to do renormalization.

One possibility, is that Feynman and Hibbs [21, Chapter 9] express quantum electrodynamics in terms of path integrals using a Lagrangian for the non-relativistic treatment of the electron. Maybe something similar can be done here for the relativistic case.

91.4 Understanding the Dirac equation

If we write the Schrödinger equation for two particles (say an electron and a proton), we can separate the equation into one equation for the center of mass of the hydrogen atom, and one equation for the hydrogen-atom wave functions in terms of the reduced mass of the electron. The latter equation gives a wave function in terms of the relative coordinates between the two particles. The symmetry of the situation shows that it is a wave function for the internal state of the hydrogen atom, not for the electron.

This separation does not work for the Dirac equation. I now realize the reason is that a hydrogen atom is a Boson, not a Fermion. Therefore, we must use the Klein-Gordon equation for the wave function of the center of mass of the hydrogen atom. For the hydrogen-atom wave function, we use the Dirac equation with the reduced mass of the electron. Again, this gives a wave function for the internal state of the hydrogen atom, not for the electron.

Because the electron and proton must be treated symmetrically in the Dirac equation, some slight change in notation is necessary. The charge of the proton should be factored out of the 4-vector potential, so that the electron and proton are treated symmetrically. This changes nothing in the solution of the equation. The independent variables are the relative coordinates of the electron and proton. The first and fourth terms inside the brackets in (91.65) and the mass term satisfy this symmetry condition, but it is not clear that the two middle terms inside the brackets satisfy this condition.

We have to realize that we have not only the relative coordinates of the electron and proton to consider, but also the coordinates of the two particles relative to an inertial frame. Usually, we write the Dirac equation in an inertial frame, and that takes care of it. But, if we want to explicitly include the gravitational interaction (including inertia), we have to do it differently. In one sense, the properties of the inertial frame give part of the gravitational field.

Here is one way in which the electromagnetic interaction in the hydrogen-atom problem differs from the gravitational interaction. The electromagnetic interaction is a function of the relative coordinates of the electron and proton, whereas the gravitational (inertial) interaction depend on the coordinates of the electron and proton relative to an inertial frame.

I have been trying to find out how Dirac theory is applied to the many-body problem, for example, the many-electron atom, and I can find only really simple formulations, such as Schrödinger theory plus adding spin, or really complicated quantum electrodynamics (QED). So, I am going to take a guess at what the exact formulation should be. I am guessing that we should use standard one-particle Dirac equation for each electron and for the nucleus. In each of these Dirac equations, we need to put the electromagnetic 4-potential A_μ acting on that particle due to all of the other particles. For that, we simply add up the contributions assuming we know all of the relative positions. One slight modification. We assume a single wave function that is a function of the positions of all of the particles. We get a Dirac equation for each particle that involves derivatives of the total wave function, but derivatives only by the coordinates of that particle.

In equations (neglecting at first, curvature of spacetime), this is

$$\left[\gamma^\alpha \left(\frac{\partial}{\partial x_j^\alpha} - \frac{ie_j A_{j\alpha}}{\hbar}\right) + \mu_j\right]\psi = 0 \ , \tag{91.66}$$

where ψ is a spinor that depends on the coordinates of all of the particles, there is a separate equation for each particle, j labels the particle, and $A_{j\alpha}$ gives the value of the 4-potential at the jth particle due to all of the other particles.

I do not know if (91.66) is a correct formulation of the problem, but if it is, there are some questions about its application. The Dirac equation applies only to Fermions, so for nuclei that have even spin, we would need to somehow use the Klein-Gordon for those particles. Even then, it is not clear if the Dirac equation applies to particles other than Leptons.

To change notation slightly, multiply (91.66) by $\hbar$ to give

$$\left[\gamma^\alpha \left(\hbar\frac{\partial}{\partial x_j^\alpha} - ie_j A_{j\alpha}\right) + m_j c\right]\psi = 0 \ . \tag{91.67}$$

To change notation again, we choose a slightly different definition of γ^α to give

$$\left[\gamma^\alpha \left(-i\hbar\frac{\partial}{\partial x_j^\alpha} - e_j A_{j\alpha}\right) + m_j c\right]\psi = 0 \ . \tag{91.68}$$

Or,

$$\left[\gamma^\alpha \left(p_j - e_j A_{j\alpha}\right) + m_j c\right]\psi = 0 \ . \tag{91.69}$$

Equations similar to (91.69) have been used for the two-body Dirac equations [263, Crater:Wong:Van-Alstine:2006], with more complicated formulas for the 4-vector potential and the mass. These are:

$$\gamma_{5j}\left[\gamma_j \cdot \left(p_j - \tilde{A}_j\right) + m_j + \tilde{S}_j\right]\psi = 0 \ , \tag{91.70}$$

where $j = 1, 2$ for the two-body case,

$$\psi = \left[\psi_1, \psi_2, \psi_3, \psi_4\right] \ , \tag{91.71}$$

in which each ψ_i is a four-component Pauli spinor for two spin-one-half particles,

$$\tilde{A}_j^\mu = \tilde{A}_j^\mu\left(A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2\right) \ , \tag{91.72}$$

and

$$\tilde{S}_j = \tilde{S}_j\left(S(r), A(r), p_\perp, \hat{P}, w, \gamma_1, \gamma_2\right) \ . \tag{91.73}$$

However, these equations do not seem to be easily generalized beyond two bodies.

There does not seem to be a closed-form formalism for the Dirac many-body problem [264, Lindgren:2011:book]. It is interesting that Lindgren's book [264, Lindgren:2011:book] does not cite any papers by Crater, Wong, or Van-Alstin, nor does the paper by Crater, Wong, and Van-Alstin [263, Crater:Wong:Van-Alstine:2006] cite the earlier second edition of Lindgren and Morrison's book [265, Lindgren:Morrison:1986:book].

Another method is to use a Lagrangian and path integrals, following, for example, the book by Feynman and Hibbs [21]. The problem is, that the book by Feynman and Hibbs [21] is mostly non-relativistic and does not include spin.

The paper by Kulczycki and Maled [266] has made some calculations of gravitational waves. I cannot tell how they compare with the calculation of gravitational waves using the SWG formulas [16]. They do not cite the SWG paper [16].

91.5 Quantizing the gravitational field

The SWG 69 [16] paper gives equations for the 2-point Green function. These are gravitational waves. To quantize them, we can use the same procedure as for photons. That is, we expand the solutions in a Fourier series, and designate appropriate terms as creation and annihilation operators[8].

[8] As I have explained in other chapters, however, I do not consider that method to be the correct method, but it would be interesting to see what happens.

Chapter 92

Inertia of light[1]

abstract

We know that light travels in straight lines in inertial frames, and that light has inertia. We also know that inertial frames should not accelerate with respect to the average matter in the universe. Here, I show a mechanism using Maxwell's equations in spacetime with an integral formulation of Einstein's field equations for the bending of light in frames that accelerate relative to matter.

92.1 Introduction

Before getting to gravitation, I want to make a slight digression to try to explain why electromagnetic waves travel in straight lines in inertial frames. Although I said that putting Maxwell's equations in integral form would help to separate gravitation from geometry, that has not yet worked. Somehow the integral form of Maxwell's equations still have gravitation in terms of inertial frames and geometry, but the gravitation is not explicit. To begin, we use Cartesian coordinates.

92.2 Generalization of Maxwell's equations to account for a gravitational interaction with matter

Notice that F' [defined in (86.10)], which is related to the sources (ρ and J), is in terms of D and H, whereas F [defined in (86.6)] is in terms of E and B. Because B and E are defined in terms of accelerations produced on charged test particles, the values of those fields will depend on the inertia of those test particles.

In 1953, Sciama [12] made a simple calculation of the inertia that would exist on a body if gravitation behaved like electromagnetism. In doing that, he was able to show that inertia would arise from an induction force. To do that calculation, he used coordinates fixed on the test body (which was in an accelerating frame). In 1969, Sciama, Waylen, and Gilman [16] showed that such a calculation is valid in General Relativity. Although a rigorous calculation would differ in detail from Sciama's 1953 result, the principle of getting inertia from an induction force in General Relativity is valid.

Because E and B are defined in terms of the accelerations that they would cause to charged test bodies, they are not purely electromagnetic quantities, since their magnitudes depend on the amount and distribution of matter in the universe.

Here, I would like to define new fields, E' and B' that would not depend on the amount or distribution of matter in the universe. To do that, we use Sciama's 1953 [12] approximate result

[1] January - June 2011 - 20 April 2012 - 11 July 2014 - 30 June 2015

that inertia due to induction would be proportional to $\rho\tau^2$, where ρ is the mean matter density in the universe, and τ is the Hubble time. In equating his derived formula for motion of a body in the presence of a local gravitational force and inertia from induction with the usual formula, he came up with the approximate formula

$$G\rho\tau^2 = 1 \; , \tag{92.1}$$

where G is the gravitational constant. Thus, G is not a fundamental constant of gravitation, but an inertial constant that depends on the amount and distribution of matter in the universe. However, to achieve our goal of finding E' and B' that do not depend on inertia, we can define

$$E' = E\rho\tau^2 \tag{92.2}$$

and

$$B' = B\rho\tau^2 \; . \tag{92.3}$$

The understanding of (92.2) and (92.3) is that as there is more matter in the universe, E and B will get smaller, while E' and B' will remain unchanged.

For practical definitions, however, since we do not know the density of matter in the universe very well, it is more useful to make the following definitions.

$$E' = E/G \tag{92.4}$$

and

$$B' = B/G \; . \tag{92.5}$$

Again, the understanding of (92.4) and (92.5) is that as there is more matter in the universe, G, E, and B will get smaller, while E' and B' will remain unchanged.

Notice that we are here treating G as an inertial constant rather than a gravitational constant. In equations involving gravitation and electromagnetic forces involving E' and B', G will not appear.

Actually, my attempt to say that D and H are purely electromagnetic quantities because they depend only on the sources (charges and currents) is artificial. Both E and D and both B and H have inertial effects because of the way that all four of these couple to make an electromagnetic wave. I need to think of something else.

Our final goal, however, is not just to get field definitions that are independent of the amount of matter in the universe, but to find out how it is that electromagnetic waves seem to propagate as straight lines only in inertial frames and how inertia affects electromagnetic waves. To do that, we try to do Sciama's 1953 calculation for electromagnetic waves. We cannot go to the rest frame of a photon, as is well known. However, we can come close, in the following respect. If we transform to a frame moving at a velocity v in the direction of the wave at a very high speed, the frequency of the wave can be Doppler shifted to an arbitrarily small frequency. Thus, we could Doppler shift the wave down to a period of say 1000 years. For all practical purposes, we would have a static electric and magnetic field.

We know the energy density of that field. It is [244, equation (10.24), p. 197] $\frac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2)$. We now consider a small volume V that contains that electromagnetic energy ($U = \frac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2)V$). If we now consider the induction force on that energy by the rest of matter in the universe that is now moving at a velocity $-v$ in the same way that Sciama did for a mass in 1953, we would get a force equal to $(U/c^2)2\pi\rho\tau^2\frac{\partial v}{\partial t}$.

This means that if that electromagnetic energy is accelerating relative to a frame that has the average velocity of matter in the universe, there will be a force on it. That is, the electromagnetic field will move in straight lines only in inertial frames.

It should be possible to use this calculation to calculate bending of light as it passes by a massive body, such as the Sun. To make that calculation in detail, it would be necessary to include Lorentz contraction effects because we transformed to a frame that is moving close to the speed of light. Actually, this calculation would probably give only half of the correct deflection of light by the sun because that is what simple calculations do. General Relativity would give the correct answer for light deflection by a mass, and would also have light moving along straight lines in inertial frames because it was designed to do that.

OK, so we have a way of seeing how light would move in straight lines in inertial frames based on the energy of the wave. However, what I really want is a generalization of Maxwell's equations that account for a gravitational interaction with matter.

So, how do we modify Maxwell's equations to put in correctly a gravitational interaction with matter? We know that if we combine all of Maxwell's equations, we will get a wave equation. If we include General Relativity, along with Maxwell's equations, we will get light moving in straight lines only in inertial frames. However, such a calculation does not directly give a gravitational interaction of electric and magnetic fields with all of the matter in the universe. We have to solve Einstein's field equations first to get the inertial frames. This digression is not working out very well.

92.3 Generalization of Maxwell's equations to account for a gravitational interaction with matter - II

Let us consider another tack. We start with the differential form of Maxwell's equations [20, p. 88]

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 \tag{92.6}$$

and

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^{\alpha} \ , \tag{92.7}$$

and we take ϵ_0 and μ_0 to be unity. Looking at components, each of (92.6) and (92.7) has four components, giving a total of eight equations. However, one of each of those four equations involves no time derivative, so they are initial-value constraints rather than dynamical equations. Thus, we have six independent dynamical equations. $F_{\alpha\beta}$ has six independent components, as does $F^{\alpha\beta}$. However, they depend on each other through

$$F^{\mu}{}_{\beta} = g_{\nu\beta} F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} \ . \tag{92.8}$$

If we take as our dynamical variables the 12 quantities F^{23}, F^{31}, F^{12}, F^{01}, F^{02}, F^{03}, F_{01}, F_{02}, F_{03}, F_{23}, F_{31}, and F_{12}, then we can take as our independent dynamical equations the three of (92.7) for α equal 1, 2, and then 3; the six of (92.8) for $(\alpha\nu)$ equal (01), (02), (03), (23), (31), (12); and the three of (92.6) that do not involve 1, then 2, and then 3. This gives 12 variables and 12 equations, which, for free space when there are no charges or currents, can be expressed as

$$\begin{pmatrix} -i\hat{\Omega} & i\hat{K} & 0 & 0 \\ -\hat{g}_{00} & \hat{g}_{i0} & \hat{g}^{ij} & 0 \\ -\hat{g}_{0j} & -\hat{g}_{ii} & -\hat{g}^{i0} & \hat{g}^{jj} \\ 0 & 0 & i\hat{K} & -i\hat{\Omega} \end{pmatrix} \begin{pmatrix} F^{0j} \\ F^{ij} \\ F_{0j} \\ F_{ij} \end{pmatrix} = \begin{pmatrix} -4\pi J \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \ , \tag{92.9}$$

where each of the elements in the matrix in (92.9) is a 3×3 matrix, and each of the elements in the column vector in (92.9) is a 3-element column vector. Thus, (92.9) is a system of 12 equations with 12 variables.

We have the following definitions.

$$F^{ij} \equiv \begin{pmatrix} F^{23} \\ F^{31} \\ F^{12} \end{pmatrix} \tag{92.10}$$

$$F^{0j} \equiv \begin{pmatrix} F^{01} \\ F^{02} \\ F^{03} \end{pmatrix} \tag{92.11}$$

$$F_{0j} \equiv \begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \end{pmatrix} \tag{92.12}$$

$$F_{ij} \equiv \begin{pmatrix} F_{23} \\ F_{31} \\ F_{12} \end{pmatrix} \tag{92.13}$$

The 3×3 matrices are defined as follows:

$$-i\hat{\Omega} \equiv \begin{pmatrix} -i\hat{\omega} & 0 & 0 \\ 0 & -i\hat{\omega} & 0 \\ 0 & 0 & -i\hat{\omega} \end{pmatrix} \tag{92.14}$$

$$i\hat{K} \equiv \begin{pmatrix} 0 & i\hat{k}_3 & -i\hat{k}_2 \\ -i\hat{k}_3 & 0 & i\hat{k}_1 \\ i\hat{k}_2 & -i\hat{k}_1 & 0 \end{pmatrix} \tag{92.15}$$

$$-\hat{g}_{00} \equiv \begin{pmatrix} -g_{00} & 0 & 0 \\ 0 & -g_{00} & 0 \\ 0 & 0 & -g_{00} \end{pmatrix} \tag{92.16}$$

$$-\hat{g}_{ii} \equiv \begin{pmatrix} -g_{22} & g_{12} & 0 \\ 0 & -g_{33} & g_{23} \\ g_{31} & 0 & -g_{11} \end{pmatrix} \tag{92.17}$$

$$\hat{g}^{ij} \equiv \begin{pmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{pmatrix} \tag{92.18}$$

$$\hat{g}^{jj} \equiv \begin{pmatrix} g^{33} & 0 & -g^{31} \\ -g^{12} & g^{11} & 0 \\ 0 & -g^{23} & g^{22} \end{pmatrix} \tag{92.19}$$

$$\hat{g}_{i0} \equiv \begin{pmatrix} 0 & -g_{30} & g_{20} \\ g_{30} & 0 & -g_{10} \\ -g_{20} & g_{10} & 0 \end{pmatrix} \tag{92.20}$$

$$-\hat{g}_{0j} \equiv \begin{pmatrix} 0 & 0 & -g_{02} \\ -g_{03} & 0 & 0 \\ 0 & -g_{01} & 0 \end{pmatrix} \tag{92.21}$$

$$-\hat{g}^{i0} \equiv \begin{pmatrix} 0 & -g^{30} & 0 \\ 0 & 0 & -g^{10} \\ -g^{20} & 0 & 0 \end{pmatrix} \tag{92.22}$$

The differential operators are defined such that $-i\hat{\omega} \equiv \partial/\partial t$, $i\hat{k_1} \equiv \partial/\partial x^1$, $i\hat{k_2} \equiv \partial/\partial x^2$, and $i\hat{k_3} \equiv \partial/\partial x^3$.

Writing out (92.9) in full gives

$$\begin{pmatrix}
-i\hat{\omega} & 0 & 0 & 0 & i\hat{k}_3 & -i\hat{k}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\hat{\omega} & 0 & -i\hat{k}_3 & 0 & i\hat{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i\hat{\omega} & i\hat{k}_2 & -i\hat{k}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g_{00} & 0 & 0 & 0 & -g_{30} & g_{20} & g^{11} & g^{12} & g^{13} & 0 & 0 & 0 \\
0 & -g_{00} & 0 & g_{30} & 0 & -g_{10} & g^{21} & g^{22} & g^{23} & 0 & 0 & 0 \\
0 & 0 & -g_{00} & -g_{20} & g_{10} & 0 & g^{31} & g^{32} & g^{33} & 0 & 0 & 0 \\
0 & 0 & -g_{02} & -g_{22} & g_{12} & 0 & 0 & -g^{30} & 0 & g^{33} & 0 & -g^{31} \\
-g_{03} & 0 & 0 & 0 & -g_{33} & g_{23} & 0 & 0 & -g^{10} & -g^{12} & g^{11} & 0 \\
0 & -g_{01} & 0 & g_{31} & 0 & -g_{11} & -g^{20} & 0 & 0 & 0 & -g^{23} & g^{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i\hat{k}_3 & -i\hat{k}_2 & -i\hat{\omega} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i\hat{k}_3 & 0 & i\hat{k}_1 & 0 & -i\hat{\omega} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i\hat{k}_2 & -i\hat{k}_1 & 0 & 0 & 0 & -i\hat{\omega}
\end{pmatrix}
\begin{pmatrix} F^{01} \\ F^{02} \\ F^{03} \\ F^{23} \\ F^{31} \\ F^{12} \\ F_{01} \\ F_{02} \\ F_{03} \\ F_{23} \\ F_{31} \\ F_{12} \end{pmatrix} = 0 \ . \tag{92.23}$$

Equation(92.23) is an exact representation of Maxwell's equation in a curved spacetime. Solving it would give the correct bending of light in non-inertial frames.

To see how electromagnetic wave propagation is affected by the rest of matter in the universe, especially to see the inertial effect from induction by the rest of matter in the universe, we use a result from the 1969 paper by Dennis Sciama, P. C. Waylen, and Robert Gilman on the integral formulation of Einstein's field equations[16, equation (14)] and a follow-up paper by Robert Gilman [156, equation (1)]. The metric tensor at one point is equal to a volume integral over spacetime plus a surface integral:

$$g^{\alpha'\beta'}(x') = 2\kappa \int_{\Omega} G^{-\alpha'\beta'\nu}{}_{\mu} \left(T^{\mu}{}_{\nu} - \frac{1}{2} T^{\lambda}{}_{\lambda} \delta^{\mu}{}_{\nu} - \frac{\Lambda}{\kappa} \delta^{\mu}{}_{\nu} \right) [-g(x)]^{1/2} d^4x + \int_{\partial\Omega} G^{-\alpha'\beta'\nu}{}_{\nu}{}^{;\sigma} [-g(x)]^{1/2} dS_{\sigma} \ , \tag{92.24}$$

where $G^{-}(x', x)$ is the retarded Green's functional that gives the contribution to the metric at x' from the stress-energy tensor at x, and $\kappa \equiv 8\pi G/c^2$.[2] Equation (92.24) shows how $g^{\alpha\beta}$ here depends on the matter distribution in the universe through the volume integral. The surface integral accounts for sources outside of the volume plus initial values and boundary values. That the relative rotation of local inertial frames and distant matter is so far unmeasurable (in the absence of local Lense-Thirring effects) suggests that for our universe, the volume integral dominates over the surface integral in (92.24).

Equation (92.24) is actually an integral equation that must be solved for the whole cosmology in the same way that Einstein's field equations must be solved. The significance of (92.24) is that the Green's functional G is not changed for first-order variation in the sources.

Once all 16 of the $g^{\alpha\beta}$ have been determined from (92.24), the inverse of $g^{\alpha\beta}$ (as a matrix) gives the components of $g_{\alpha\beta}$. Then, both can be substituted into (92.23) to see how the propagation of electromagnetic waves depends on the distribution of matter in the universe. At this point, we can forget that the $g^{\alpha\beta}$ and $g_{\alpha\beta}$ in (92.23) came from raising and lowering indices, and simply consider them as fields given by (92.24).

A dispersion relation for an electromagnetic wave in free space is found by replacing the differential operators by a frequency and wavenumbers: $-i\hat{\omega} \to -i\omega$, $i\hat{k_1} \to ik_1$, $i\hat{k_2} \to ik_2$, $i\hat{k_3} \to ik_3$,

[2]Note added 11 July 2014: Altshuler [260] showed that the integral formulation has ghosts unless the dimension of spacetime equals 4. This would cause the background to be unstable.

and setting the determinant of the matrix in (92.9) to zero.

$$\begin{vmatrix} -i\Omega & iK & 0 & 0 \\ -\hat{g}_{00} & \hat{g}_{i0} & \hat{g}^{ij} & 0 \\ -\hat{g}_{0j} & -\hat{g}_{ii} & -\hat{g}^{i0} & \hat{g}^{jj} \\ 0 & 0 & iK & -i\Omega \end{vmatrix} = 0 \ , \tag{92.25}$$

where

$$-i\Omega \equiv \begin{pmatrix} -i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix} \tag{92.26}$$

$$iK \equiv \begin{pmatrix} 0 & ik_3 & -ik_2 \\ -ik_3 & 0 & ik_1 \\ ik_2 & -ik_1 & 0 \end{pmatrix} \ . \tag{92.27}$$

The determinant (92.25) is actually a 12×12 determinant. Writing it out in full gives

$$\begin{vmatrix}
-i\omega & 0 & 0 & 0 & ik_3 & -ik_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\omega & 0 & -ik_3 & 0 & ik_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i\omega & ik_2 & -ik_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g_{00} & 0 & 0 & 0 & -g_{30} & g_{20} & g^{11} & g^{12} & g^{13} & 0 & 0 & 0 \\
0 & -g_{00} & 0 & g_{30} & 0 & -g_{10} & g^{21} & g^{22} & g^{23} & 0 & 0 & 0 \\
0 & 0 & -g_{00} & -g_{20} & g_{10} & 0 & g^{31} & g^{32} & g^{33} & 0 & 0 & 0 \\
0 & 0 & -g_{02} & -g_{22} & g_{12} & 0 & 0 & -g^{30} & 0 & g^{33} & 0 & -g^{31} \\
-g_{03} & 0 & 0 & 0 & -g_{33} & g_{23} & 0 & 0 & -g^{10} & -g^{12} & g^{11} & 0 \\
0 & -g_{01} & 0 & g_{31} & 0 & -g_{11} & -g^{20} & 0 & 0 & 0 & -g^{23} & g^{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & ik_3 & -ik_2 & -i\omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -ik_3 & 0 & ik_1 & 0 & -i\omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ik_2 & -ik_1 & 0 & 0 & 0 & -i\omega
\end{vmatrix} = 0 \ . \tag{92.28}$$

As a check, let us calculate the determinant in (92.28) for the case where the metric tensor $g_{\mu\nu}$ can be made diagonal. Then we have $g^{00}g_{00} = g^{11}g_{11} = g^{22}g_{22} = g^{33}g_{33} = 1$, and the determinant in (92.28) becomes

$$-(\omega/g^{00})^2(g^{00}\omega^2 + g^{11}k_1^2 + g^{22}k_2^2 + g^{33}k_3^2)^2 = 0 \ , \tag{92.29}$$

which has the dispersion relation:

$$g^{00}\omega^2 + g^{11}k_1^2 + g^{22}k_2^2 + g^{33}k_3^2 = 0 \ , \tag{92.30}$$

which includes both a retarded and an advanced solution. (Remember, g^{00} has the opposite sign from g^{11}, g^{22}, and g^{33}.)

In general, there are additional terms in the dispersion relation. Next, we consider the case where $g^{\mu\nu}$ is nonzero for off-diagonal terms. Then, the value of the determinant in (92.28) is

$$-\left(\frac{(g^{11}g^{22}g^{33} - g^{12}g^{23}g^{31})\omega^2 \left|g^{(3)}\right|}{|g|^2}\right)(g^{\mu\nu}k_\mu k_\nu)^2 \ , \tag{92.31}$$

where $k_0 \equiv -\omega$. The dispersion relation is

$$g^{\mu\nu}k_\mu k_\nu = 0 \ . \tag{92.32}$$

Once all 16 of the $g^{\alpha\beta}$ have been determined from (92.24), they can be substituted into (92.32) to see how the dispersion relation for electromagnetic waves depends on the distribution of matter in the universe.

Equation (92.24) can also be written[3]

$$g^{\alpha'\beta'}(x') = 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{\nu\mu}\left(T^{\mu\nu} - \frac{1}{2}T^\lambda{}_\lambda g^{\mu\nu} - \frac{\Lambda}{\kappa}g^{\mu\nu}\right)[-g(x)]^{1/2}d^4x + \int_{\partial\Omega} G^{-\alpha'\beta'\nu}{}_\nu{}^{;\sigma}[-g(x)]^{1/2}dS_\sigma \ . \tag{92.33}$$

We can take for the stress-energy tensor[20, p. 132]

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu} \ . \tag{92.34}$$

Thus, (92.33) becomes

$$g^{\alpha'\beta'}(x') = 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{\nu\mu}(x', x)(\rho + p)U^\mu U^\nu[-g(x)]^{1/2}d^4x + g_\Lambda^{\alpha'\beta'}(x') + g_{\text{surface}}^{\alpha'\beta'}(x') \ , \tag{92.35}$$

where

$$g_\Lambda^{\alpha'\beta'}(x') \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{\nu\mu}(x', x)\left(\frac{1}{2}(\rho - p) - \frac{\Lambda}{\kappa}\right)g^{\mu\nu}[-g(x)]^{1/2}d^4x \tag{92.36}$$

and

$$g_{\text{surface}}^{\alpha'\beta'}(x') \equiv \int_{\partial\Omega} G^{-\alpha'\beta'\nu}{}_\nu{}^{;\sigma}(x', x)[-g(x)]^{1/2}dS_\sigma \tag{92.37}$$

is the contribution from the surface integral, which we do not consider here. We can also write

$$g^{\alpha'\beta'}(x') = \phi_{\nu\mu}^{\alpha'\beta'\mu\nu}(x') + g_\Lambda^{\alpha'\beta'}(x') + g_{\text{surface}}^{\alpha'\beta'}(x') \ , \tag{92.38}$$

where

$$\phi_{\nu\mu}^{\alpha'\beta'\tau\sigma}(x') \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{\nu\mu}(x', x)(\rho + p)U^\tau U^\sigma[-g(x)]^{1/2}d^4x \ . \tag{92.39}$$

In a frame moving with the center of mass of the universe, we have

$$\phi_{\text{cm}}^{\alpha'\beta'\tau\sigma}{}_{\nu\mu}(x') \equiv 2\kappa \int_\Omega G_{\text{cm}}^{-\alpha'\beta'}{}_{\nu\mu}(x', x)(\rho + p)\delta U^\tau \delta U^\sigma[-g(x)]^{1/2}d^4x \ . \tag{92.40}$$

To calculate the integral in (92.35), we can use $p(t)$ and $\rho(t)$ from the standard model for the radiation era, the matter era, and the dark energy era. We also need to know $G^{-\alpha'\beta'}{}_{\nu\mu}$, which we discuss in section 92.3.3. The U^i factor (for $i = 1 \to 3$) has two components. First is the random component from stars and galaxies moving with random velocities relative to the center of mass frame for the whole universe. This component averages to zero. Then there is the component $\overline{U^i}$ from the motion of the center-of-mass frame relative to the primed frame. If velocities simply added, then we could take that component outside of the integral. However, with Lorentz transformations, it is more complicated.

The calculation can be done in two ways. For a direct calculation, we simply use the general addition of 4-velocities in the primed frame. That is considered in section 92.3.1. The other method is to first calculate the metric in the center-of-mass frame of the universe, and then make a Lorentz transformation to get the metric in a different frame. This is considered in section 92.3.2. In principle, both methods should give the same result, but it is not easy to show that.

[3]Equation (92.24) has the property that a first-order change in the sources has only a second-order change in the Green's function. Unfortunately, equation (92.33) does not have this property.

92.3.1 Direct calculation of metric

The calculation below is incorrect. A correct calculation is through the Green's function, but it seems difficult.

For the purposes of the integral in (92.35), we use cartesian coordinates. Strictly speaking, that limits the calculations to flat space, but measurements now show that space is nearly flat. In cartesian coordinates, four-velocities U and V add to form a sum S as follows[20, p. 69][267]

$$S^0 = V^0 U^0 + V \cdot U \tag{92.41}$$

and

$$S^i = U^i + V^i \left[U^0 + \frac{V \cdot U}{V \cdot V}(V^0 - 1) \right] = U^i + V^i \left[U^0 + \frac{V \cdot U}{V^0 + 1} \right] , \tag{92.42}$$

where I have used $V^0 V^0 - V \cdot V = 1$ (which is the same as $V^\mu V_\mu = -1$).

In our case, the U^μ in (92.35) are composed of four-vectors δU that are relative to the center-of-mass frame of the universe (whose spatial components average to zero) and four velocity $\overline{U}$ of the center-of-mass frame relative to the primed frame. Using the above formulas, we have

$$U^0 = \overline{U}^0 \delta U^0 + \overline{U} \cdot \delta U \tag{92.43}$$

and

$$U^i = \delta U^i + \overline{U}^i \left[\delta U^0 + \frac{\overline{U} \cdot \delta U}{\overline{U} \cdot \overline{U}}(\overline{U}^0 - 1) \right] = \delta U^i + \overline{U}^i \left[\delta U^0 + \frac{\overline{U} \cdot \delta U}{\overline{U}^0 + 1} \right] . \tag{92.44}$$

Substituting into (92.35) gives

$$g^{0'0'}(x') = g_\Lambda^{0'0'} + \overline{U}^0 \overline{U}^0 \phi_{00}^{0'0'} + \overline{U}^k \, \overline{U}^k \phi_{00k}^{0'0'} + g_{\text{surface}}^{0'0'} , \tag{92.45}$$

$$g^{0'i'}(x') = g_\Lambda^{0'i'} + \overline{U}^i \phi_{0ii}^{0'i'} + \overline{U}^0 \overline{U}^i \phi_{0i}^{0'i'} + \overline{U}^i \, \overline{U}^k \overline{U}^k \phi_{0ik}^{0'i'} + g_{\text{surface}}^{0'i'} , \tag{92.46}$$

$$g^{i'j'}(x') = g_\Lambda^{i'j'} + \delta^{i'j'} \left(\phi_i^{i'i'} + \frac{2\overline{U}^i \, \overline{U}^i}{1+\gamma} \phi_i^{i'j'} \right) + \overline{U}^i \, \overline{U}^j \phi_{ij}^{i'j'} + \frac{\overline{U}^i \, \overline{U}^j \overline{U}^k \overline{U}^k}{(1+\gamma)^2} \phi_{ijk}^{i'j'} + g_{\text{surface}}^{i'j'} , \tag{92.47}$$

where

$$\phi_{00}^{\alpha'\beta'} \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{00}(\rho + p)\overline{\delta U^0 \delta U^0}[-g(x)]^{1/2} d^4x , \tag{92.48}$$

$$\phi_{00k}^{\alpha'\beta'} \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{00}(\rho + p)\overline{\delta U^k \delta U^k}[-g(x)]^{1/2} d^4x \text{ , (no sum on } k) \tag{92.49}$$

$$\phi_{0ii}^{\alpha'\beta'} \equiv 2\kappa \int_\Omega (G^{-\alpha'\beta'}{}_{0i} + G^{-\alpha'\beta'}{}_{i0})(\rho + p)\overline{\delta U^i \delta U^i}[-g(x)]^{1/2} d^4x \text{ , (no sum on } i) \tag{92.50}$$

$$\phi_{0i}^{\alpha'\beta'} \equiv 2\kappa \int_\Omega (G^{-\alpha'\beta'}{}_{0i} + G^{-\alpha'\beta'}{}_{i0})(\rho + p)\overline{\delta U^k \delta U^k}[-g(x)]^{1/2} d^4x \text{ , (sum on } k) \tag{92.51}$$

$$\phi_{0ik}^{\alpha'\beta'} \equiv 2\kappa \int_\Omega (G^{-\alpha'\beta'}{}_{0i} + G^{-\alpha'\beta'}{}_{i0})(\rho + p)\overline{\delta U^k \delta U^k}[-g(x)]^{1/2} d^4x \text{ , (no sum on } k) \tag{92.52}$$

$$\phi_i^{\alpha'\beta'} \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{ii}(\rho + p)\overline{\delta U^i \delta U^i}[-g(x)]^{1/2} d^4x \text{ , (no sum on } i) \tag{92.53}$$

$$\phi_{ij}^{\alpha'\beta'} \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{ij}(\rho + p)\overline{\delta U^k \delta U^k}[-g(x)]^{1/2} d^4x \text{ , (sum on } k) \tag{92.54}$$

and

$$\phi_{ijk}^{\alpha'\beta'} \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}{}_{ij}(\rho + p)\overline{\delta U^k \delta U^k}[-g(x)]^{1/2} d^4x \text{ , (no sum on } k) \tag{92.55}$$

which depend only on the cosmological model.

92.3.2 Calculating the metric from Lorentz transformations

In this subsection, the primed frame is a center-of-mass frame. We also take the unprimed frame to be a center-of-mass frame. The double-primed frame is arbitrary. $\overline{U}^\alpha$ is the 4-velocity of the primed frame with respect to the double-primed frame. $\delta U^\alpha(x)$ is the 4-velocity of matter in the center-of-mass frame. $\delta U^i(x)$ is not necessarily small. Some of the results below are based on the discussion of the Green's functional in subsection 92.3.3. In that section, we make the approximation that the cosmology is approximately described by the Robertson-Walker metric. Because of that, only some components of the Green's functional are non-zero. For example, the only nonzero component of $G^{-0'0'}_{\alpha\beta}$ is $G^{-0'0'}_{00}$.

Then, using (92.38) in the rest frame, and that the only nonzero component of $G^{-0'0'}_{\alpha\beta}$ is $G^{-0'0'}_{00}$, we get

$$g^{0'0'}_{\text{cm}}\ (x') = \phi^{0'0'}_{\text{cm } 00}(x') + g^{0'0'}_{\text{cm } \Lambda}(x') + g^{0'0'}_{\text{cm surface}}(x')\ , \tag{92.56}$$

where, from (92.40),

$$\phi^{0'0'}_{\text{cm } 00}(x') \equiv 2\kappa \int_\Omega G^{-0'0'}_{\text{cm } 00}(x', x)(\rho + p)\overline{\delta U^0 \delta U^0}[-g(x)]^{1/2} d^4x\ , \tag{92.57}$$

and from (92.36),

$$g^{\alpha'\beta'}_{\text{cm } \Lambda}(x') \equiv 2\kappa \int_\Omega G^{-\alpha'\beta'}_{\text{cm } \nu\mu}(x', x)\left(\frac{1}{2}(\rho - p) - \frac{\Lambda}{\kappa}\right) g^{\mu\nu}[-g(x)]^{1/2} d^4x\ , \tag{92.58}$$

and from (92.37),

$$g^{\alpha'\beta'}_{\text{cm surface}}(x') \equiv \int_{\partial\Omega} G^{-\alpha'\beta'\nu\ \ ;\sigma}_{\text{cm } \ \ \ \ \nu}(x', x)[-g(x)]^{1/2} dS_\sigma \tag{92.59}$$

is the contribution from the surface integral, which we do not consider here.

Again using (92.38) in the rest frame, and that the only nonzero component of $G^{-i'i'}_{\alpha\beta}$ is $\delta_{ii'} G^{-i'i'}_{ii}$, we get

$$g^{i'i'}_{\text{cm}}\ (x') = \delta_{ii'}\phi^{i'i'}_{\text{cm } i}(x') + g^{i'i'}_{\text{cm } \Lambda}(x') + g^{i'i'}_{\text{cm surface}}(x')\ , \tag{92.60}$$

where, from (92.40),

$$\phi^{i'i'}_{\text{cm } i}(x') \equiv 2\kappa \int_\Omega G^{-i'i'}_{\text{cm } ii}(x', x)(\rho + p)\overline{\delta U^i \delta U^i}[-g(x)]^{1/2} d^4x\ , \text{ (no sum on } i \text{ or } i'). \tag{92.61}$$

Notice that $g^{\alpha'\beta'}_{\text{cm}}\ (x')$ is diagonal.

Now we want to transform from the primed (center-of-mass) frame to an arbitrary (double-primed) frame. In the double-primed frame, the center-of-mass frame has the 4-velocity ($\overline{U}^0$, $\overline{U}^i$). We make a Lorentz transformation by ($\overline{U}^0$, $-\overline{U}^i$). Then we get

$$g^{0'0'}(x'') = g^{\alpha'\beta'}_{\text{cm}}\ (x')\overline{U}^\alpha\ \overline{U}^\beta\ , \text{ sum on } \alpha = \alpha' \text{ and } \beta = \beta' \tag{92.62}$$

$$g^{0'i'}(x'') = -\overline{U}^i\left[g^{i'i'}_{\text{cm}}\ (x') + g^{0'0'}_{\text{cm}}\ (x')\overline{U}^0\delta_{00'} + \frac{g^{i'j'}_{\text{cm}}\ (x')\overline{U}^i\ \overline{U}^j}{\gamma + 1}\right]\ , \text{ sum on } i = i' \text{ and } j = j' \tag{92.63}$$

$$g^{i'j'}(x'') = g^{i'j'}_{\text{cm}}\ (x') + \overline{U}^i\ \overline{U}^j\left[g^{0'\ 0'}_{\text{cm}}(x') + \frac{g^{i'i'}_{\text{cm}}\ (x') + g^{j'j'}_{\text{cm}}\ (x')}{\gamma + 1} + \frac{g^{i'j'}_{\text{cm}}\ (x')\overline{U}^i\ \overline{U}^j}{(\gamma + 1)^2}\right], \text{ sum on } i = i',\ j = j' \tag{92.64}$$

We can rewrite (92.62) as

$$g^{0'0'}(x'') = g^{0'0'}_{\text{cm}}\ (x')\overline{U}^0\ \overline{U}^0\delta_{00'} + g^{i'i'}_{\text{cm}}\ (x')\overline{U}^i\ \overline{U}^i\delta_{ii'}\ , \text{ sum on } i = i' \tag{92.65}$$

Because $g_{\rm cm}^{\alpha'\beta'}(x')$ is diagonal, we can write (92.63) as

$$g^{0'i'}(x'') = -\overline{U}^{i'}\left[g_{\rm cm}^{i'i'}(x') + g_{\rm cm}^{0'0'}(x')\overline{U}^0\delta_{00'} + \frac{g_{\rm cm}^{i'i'}(x')\overline{U}^{i'}\ \overline{U}^{i'}}{\gamma+1}\right] , \tag{92.66}$$

and we can write (92.64) as

$$g^{i'j'}(x'') = g_{\rm cm}^{i'i'}(x')\delta_{i'j'} + \overline{U}^{i'}\ \overline{U}^{j'}\left[g_{\rm cm}^{0'\ 0'}(x') + \frac{g_{\rm cm}^{i'i'}(x') + g_{\rm cm}^{j'j'}(x')}{\gamma+1} + \delta_{i'j'}\frac{g_{\rm cm}^{i'i'}(x')\overline{U}^{i'}\ \overline{U}^{i'}}{(\gamma+1)^2}\right] . \tag{92.67}$$

Substituting (92.56) and (92.60) into (92.65) , (92.66) , and (92.67) gives
(THE NEXT 3 EQUATIONS STILL NEED WORK.)

$$g^{0'0'}(x') = g_\Lambda^{0'0'} + \overline{U}^0\overline{U}^0\phi_{00}^{0'0'} + \overline{U}^k\ \overline{U}^k\phi_{00k}^{0'0'} + g_{\rm surface}^{0'0'} , \tag{92.68}$$

$$g^{0'i'}(x') = g_\Lambda^{0'i'} + \overline{U}^i\phi_{0ii}^{0'i'} + \overline{U}^0\overline{U}^i\phi_{0i}^{0'i'} + \overline{U}^i\ \overline{U}^k\overline{U}^k\phi_{0ik}^{0'i'} + g_{\rm surface}^{0'i'} , \tag{92.69}$$

$$g^{i'j'}(x') = g_\Lambda^{i'j'} + \delta^{i'j'}\left(\phi_i^{i'i'} + \frac{2\overline{U}^i\ \overline{U}^i}{1+\gamma}\phi_i^{i'j'}\right) + \overline{U}^i\ \overline{U}^j\phi_{ij}^{i'j'} + \frac{\overline{U}^i\ \overline{U}^j\overline{U}^k\overline{U}^k}{(1+\gamma)^2}\phi_{ijk}^{i'j'} + g_{\rm surface}^{i'j'} , \tag{92.70}$$

where

$$\phi_{{\rm cm}\ 00k}^{\alpha'\beta'}(x') \equiv 2\kappa\int_\Omega G_{\rm cm}^{-\alpha'\beta'}{}_{00}(x',x)(\rho+p)\overline{\delta U^k\delta U^k}[-g(x)]^{1/2}d^4x\ , \text{(no sum on } k) \tag{92.71}$$

$$\phi_{{\rm cm}\ 0ii}^{\alpha'\beta'}(x') \equiv 2\kappa\int_\Omega (G_{\rm cm}^{-\alpha'\beta'}{}_{0i}(x',x) + G_{\rm cm}^{-\alpha'\beta'}{}_{i0}(x',x))(\rho+p)\overline{\delta U^i\delta U^i}[-g(x)]^{1/2}d^4x\ , \text{(no sum on } i) \tag{92.72}$$

$$\phi_{{\rm cm}\ 0i}^{\alpha'\beta'}(x') \equiv 2\kappa\int_\Omega (G_{\rm cm}^{-\alpha'\beta'}{}_{0i}(x',x) + G_{\rm cm}^{-\alpha'\beta'}{}_{i0}(x',x))(\rho+p)\overline{\delta U^k\delta U^k}[-g(x)]^{1/2}d^4x\ , \text{(sum on } k) \tag{92.73}$$

$$\phi_{{\rm cm}\ 0ik}^{\alpha'\beta'}(x') \equiv 2\kappa\int_\Omega (G_{\rm cm}^{-\alpha'\beta'}{}_{0i}(x',x) + G_{\rm cm}^{-\alpha'\beta'}{}_{i0}(x',x))(\rho+p)\overline{\delta U^k\delta U^k}[-g(x)]^{1/2}d^4x\ , \text{(no sum on } k) \tag{92.74}$$

$$\phi_{{\rm cm}\ ij}^{\alpha'\beta'}(x') \equiv 2\kappa\int_\Omega G_{\rm cm}^{-\alpha'\beta'}{}_{ij}(x',x)(\rho+p)\overline{\delta U^k\delta U^k}[-g(x)]^{1/2}d^4x\ , \text{(sum on } k) \tag{92.75}$$

and

$$\phi_{{\rm cm}\ ijk}^{\alpha'\beta'}(x') \equiv 2\kappa\int_\Omega G_{\rm cm}^{-\alpha'\beta'}{}_{ij}(x',x)(\rho+p)\overline{\delta U^k\delta U^k}[-g(x)]^{1/2}d^4x\ , \text{(no sum on } k) \tag{92.76}$$

92.3.3 Calculating the Green's functions

The differential equation for the retarded Green's function $G^{-\alpha'\beta'}{}_{\nu\mu}(x',x)$ is given by Sciama, Waylen, and Gilman in 1969, [16, equation (9)] and is

$$\Box G^{-\alpha'\beta'}{}_{\mu\nu}(x',x) - 2R^\rho{}_\mu{}^\sigma{}_\nu G^{-\alpha'\beta'}{}_{\rho\sigma}(x',x) = g^{\alpha'}{}_{(\mu}g^{\beta'}{}_{\nu)}[g(x')g(x)]^{-1/4}\delta(x',x) , \tag{92.77}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian, ∇_μ is a covariant derivative, parentheses on indexes denote symmetrization, $R^\rho{}_\mu{}^\sigma{}_\nu$ is the Riemann tensor, $\delta(x',x)$ denotes the four-dimensional δ distribution of the two points x' and x, $g^{\alpha'}{}_\mu(x',x)$ denotes the two-point vector of geodesic parallel transport which is covariant at x and contravariant at x', and satisfies

$$\lim_{x'\to x} g^{\alpha'}{}_\mu(x',x) = \delta^{\alpha'}{}_\mu(x',x) . \tag{92.78}$$

For the Robertson-Walker metric, many of the components of $R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}$ are zero. Robertson and Noonan's book[261, page 343] gives the values of $R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}$. From this, we find that (in a coordinate system in which the metric tensor is diagonal) the only non-zero components of $G^{-0'1'}{}_{\nu\mu}(x',x)$ in (92.54) are $G^{-0'1'}{}_{01}(x',x)$ and $G^{-0'1'}{}_{10}(x',x)$. Thus, (92.54) gives

$$\phi^{0'1'}_{01} + \phi^{0'1'}_{10} = 2\kappa \int_{\Omega} \left(G^{-0'1'}{}_{01} + G^{-0'1'}{}_{10} \right) (\rho + p) \overline{\delta U \cdot \delta U} [-g(x)]^{1/2} d^4x \ . \tag{92.79}$$

(The above equation and the discussion below apply as well to $\phi^{0'2'}_{02} + \phi^{0'2'}_{20}$ and $\phi^{0'3'}_{03} + \phi^{0'3'}_{30}$.)

The differential equations for $G^{-0'1'}{}_{01}(x',x)$ and $G^{-0'1'}{}_{10}(x',x)$ are coupled. We have

$$\Box G^{-0'1'}{}_{01}(x',x) - 2R^{1}{}_{0}{}^{0}{}_{1} G^{-0'1'}{}_{10}(x',x) = \frac{g^{0'}{}_{0} g^{1'}{}_{1} + g^{0'}{}_{1} g^{1'}{}_{0}}{2} [g(x')g(x)]^{-1/4} \delta(x',x) \ , \tag{92.80}$$

and

$$\Box G^{-0'1'}{}_{10}(x',x) - 2R^{0}{}_{1}{}^{1}{}_{0} G^{-0'1'}{}_{01}(x',x) = \frac{g^{0'}{}_{0} g^{1'}{}_{1} + g^{0'}{}_{1} g^{1'}{}_{0}}{2} [g(x')g(x)]^{-1/4} \delta(x',x) \ . \tag{92.81}$$

These two equations can be split into two uncoupled equations involving linear combinations of the two Green's functions. Because $R^{1}{}_{0}{}^{0}{}_{1} = R^{0}{}_{1}{}^{1}{}_{0}$, the two independent combinations are $\psi_1(x',x) \equiv G^{-0'1'}{}_{01}(x',x) + G^{-0'1'}{}_{10}(x',x)$ and $\psi_2(x',x) \equiv G^{-0'1'}{}_{01}(x',x) - G^{-0'1'}{}_{10}(x',x)$. Because $g^{0'}{}_{1} g^{1'}{}_{0} \delta(x',x) = \delta^{0'}{}_{1} \delta^{1'}{}_{0} \delta(x',x) = 0$ and $g^{0'}{}_{0} g^{1'}{}_{1} \delta(x',x) = \delta^{0'}{}_{0} \delta^{1'}{}_{1} \delta(x',x) = \delta(x',x)$, the uncoupled equations are

$$\left(\Box - 2R^{0}{}_{1}{}^{1}{}_{0} \right) \psi_1(x',x) = [g(x')g(x)]^{-1/4} \delta(x',x) \ , \tag{92.82}$$

and

$$\left(\Box + 2R^{0}{}_{1}{}^{1}{}_{0} \right) \psi_2(x',x) = 0 \ . \tag{92.83}$$

Thus, (92.79) becomes

$$\phi^{0'1'}_{01} + \phi^{0'1'}_{10} = 2\kappa \int_{\Omega} \psi_1(x',x)(\rho + p) \overline{\delta U \cdot \delta U} [-g(x)]^{1/2} d^4x \ , \tag{92.84}$$

and we do not need to consider $\psi_2(x',x)$. Because $R^{2}{}_{0}{}^{0}{}_{2} = R^{0}{}_{2}{}^{2}{}_{0} = R^{3}{}_{0}{}^{0}{}_{3} = R^{0}{}_{3}{}^{3}{}_{0} = R^{1}{}_{0}{}^{0}{}_{1} = R^{0}{}_{1}{}^{1}{}_{0}$, we have

$$\phi^{0'2'}_{02} + \phi^{0'2'}_{20} = \phi^{0'3'}_{03} + \phi^{0'3'}_{30} = \phi^{0'1'}_{01} + \phi^{0'1'}_{10} \ . \tag{92.85}$$

If we can calculate $\psi_1(x',x)$, then we can calculate the integral in (92.84) since we know $p(t)$ and $\rho(t)$ from the standard model for the radiation era, the matter era, and the dark energy era.

92.4 Hamilton's equations

To calculate propagation of the electromagnetic wave, we use the dispersion relation as a Hamiltonian for the system. In that system, t and x^i are the generalized coordinates, and ω and k_i are the conjugate momenta. We write the dispersion relation as $H(t, x^i, \omega, k_i) = 0$. In a general coordinate frame, the $g^{\alpha\beta}$ and $g_{\alpha\beta}$ that are elements of the matrix in (92.28) will depend on t and x^i from motion of the sources with respect to the local coordinate frame, and the time derivatives and gradients of those terms will give refraction of the electromagnetic wave as seen in a non-inertial frame.

The easiest way to see how bending of light occurs in a non-inertial frame is to use Hamilton's equations. We start by using the dispersion relation as a Hamiltonian.

$$H = g^{\mu\nu} k_\mu k_\nu \ , \tag{92.86}$$

Hamilton's equations give

$$\frac{dt}{d\tau} = -\frac{\partial H}{\partial \omega} = \frac{\partial H}{\partial k_0} = g^{\mu 0}k_\mu + g^{0\nu}k_\nu = 2g^{\mu 0}k_\mu \tag{92.87}$$

$$\frac{dx^i}{d\tau} = \frac{\partial H}{\partial k_i} = g^{\mu i}k_\mu + g^{i\nu}k_\nu = 2g^{\mu i}k_\mu \tag{92.88}$$

$$\frac{dk_\alpha}{d\tau} = -\frac{\partial H}{\partial x^\alpha} = -g^{\mu\nu}{}_{,\alpha}k_\mu k_\nu \tag{92.89}$$

$$\frac{d\omega}{d\tau} = \frac{\partial H}{\partial t} = g^{\mu\nu}{}_{,0}k_\mu k_\nu \tag{92.90}$$

$$\frac{dk_i}{d\tau} = -\frac{\partial H}{\partial x^i} = -g^{\mu\nu}{}_{,i}k_\mu k_\nu \tag{92.91}$$

where a comma denotes differentiation and τ has no physical significance.

Taking the ratio of (92.89) and (92.87) gives

$$\frac{dk_\mu}{dt} = -\frac{k_\alpha k_\beta}{2g^{\nu 0}k_\nu}g^{\alpha\beta}{}_{,\mu} \ . \tag{92.92}$$

Taking the ratio of (92.90) and (92.87) gives

$$\frac{d\omega}{dt} = \frac{k_\alpha k_\beta}{2g^{\nu 0}k_\nu}g^{\alpha\beta}{}_{,0} \ . \tag{92.93}$$

The group velocity of the wave is found by taking the ratio of (92.88) with respect to (92.87). This gives

$$v_i = \frac{dx^i}{dt} = \frac{g^{\mu i}k_\mu}{g^{\nu 0}k_\nu} \ . \tag{92.94}$$

To find the acceleration of the wave in a non-inertial frame, we take the time derivative of (92.94), and use (92.92) to give

$$\frac{dv_i}{dt} = \frac{k_\alpha k_\beta}{(g^{\nu 0}k_\nu)^2}(g^{\beta 0}g^{\alpha i}{}_{,0} - g^{\beta i}g^{\alpha 0}{}_{,0}) - \frac{g^{\gamma 0}g^{\mu i} - g^{\gamma i}g^{\mu 0}}{2(g^{\nu 0}k_\nu)^3}k_\alpha k_\beta k_\gamma g^{\alpha\beta}{}_{,\mu} \ . \tag{92.95}$$

To calculate the derivatives of the metric tensor, we use (92.45), (92.46), and (92.47) and dropping the primes, we get (ACTUALLY, I HAVE TO REDO THESE CALCULATIONS NOW)

$$\begin{aligned} g^{\alpha\beta}{}_{,\mu}(x) = (\overline{U}^0\overline{U}^0)_{,\mu}\Phi^{\alpha\beta}_{00} + \frac{1}{3}(\overline{U}\cdot\overline{U})_{,\mu}\phi^{\alpha\beta}_{00} + \frac{4}{3}(\overline{U}^0\overline{U}^i)_{,\mu}(\phi^{\alpha\beta}_{0i} + \phi^{\alpha\beta}_{i0}) \\ +(\overline{U}^i\ \overline{U}^j)_{,\mu}\Phi^{\alpha\beta}_{ij} + \frac{1}{3}\left(\overline{U}^i\ \overline{U}^j\frac{1+\overline{U}^0\overline{U}^0}{1+\overline{U}^0}\right)_{,\mu}\phi^{\alpha\beta}_{ij} \ , \end{aligned} \tag{92.96}$$

which can be rewritten

$$\begin{aligned} g^{\alpha\beta}{}_{,\mu}(x) \ &= (2\overline{U}^0\overline{U}^0_{,\mu})\Phi^{\alpha\beta}_{00} + \tfrac{2}{3}\overline{U}\cdot\overline{U}_{,\mu}\phi^{\alpha\beta}_{00} + \tfrac{4}{3}(\overline{U}^0\overline{U}^i_{,\mu} + \overline{U}^0_{,\mu}\overline{U}^i)(\phi^{\alpha\beta}_{0i} + \phi^{\alpha\beta}_{i0}) + (\overline{U}^i\ \overline{U}^j_{,\mu} + \overline{U}^i_{,\mu}\ \overline{U}^j)\Phi^{\alpha\beta}_{ij} \\ &\quad + \tfrac{1}{3}\left[\left(\overline{U}^i\ \overline{U}^j_{,\mu} + \overline{U}^i_{,\mu}\ \overline{U}^j\right)\tfrac{1+\overline{U}^0\overline{U}^0}{1+\overline{U}^0} + \overline{U}^i\ \overline{U}^j\tfrac{-1+2\overline{U}^0+\overline{U}^0\overline{U}^0}{(1+\overline{U}^0)^2}\overline{U}^0_{,\mu}\right]\phi^{\alpha\beta}_{ij} \ , \end{aligned} \tag{92.97}$$

or, using $\overline{U}^0\overline{U}^0 = 1 + \overline{U}\cdot\overline{U}$, we have

$$\begin{aligned} g^{\alpha\beta}{}_{,\mu}(x) \ &= (2\overline{U}\cdot\overline{U}_{,\mu})\Phi^{\alpha\beta}_{00} + \tfrac{2}{3}\overline{U}\cdot\overline{U}_{,\mu}\phi^{\alpha\beta}_{00} + \tfrac{4}{3}(\overline{U}^0\overline{U}^i_{,\mu} + \tfrac{\overline{U}\cdot\overline{U}_{,\mu}}{\overline{U}^0}\overline{U}^i)(\phi^{\alpha\beta}_{0i} + \phi^{\alpha\beta}_{i0}) + (\overline{U}^i\ \overline{U}^j_{,\mu} + \overline{U}^i_{,\mu}\ \overline{U}^j)\Phi^{\alpha\beta}_{ij} \\ &\quad + \tfrac{1}{3}\left[\left(\overline{U}^i\ \overline{U}^j_{,\mu} + \overline{U}^i_{,\mu}\ \overline{U}^j\right)\tfrac{1+\overline{U}^0\overline{U}^0}{1+\overline{U}^0} + \overline{U}^i\ \overline{U}^j\tfrac{-1+2\overline{U}^0+\overline{U}^0\overline{U}^0}{(1+\overline{U}^0)^2}\tfrac{\overline{U}\cdot\overline{U}_{,\mu}}{\overline{U}^0}\right]\phi^{\alpha\beta}_{ij} \ , \end{aligned} \tag{92.98}$$

which can be rewritten

$$g^{\alpha\beta}{}_{,\mu}(x) = A_k^{\alpha\beta} \frac{d\overline{U}^k}{dx^\mu} \ , \tag{92.99}$$

where

$$\begin{aligned} A_k^{\alpha\beta} \ &\equiv \left[2\Phi_{00}^{\alpha\beta} + \tfrac{2}{3}\phi_{00}^{\alpha\beta} + \tfrac{4}{3}(\phi_{0i}^{\alpha\beta} + \phi_{i0}^{\alpha\beta}) \tfrac{\overline{U}^i}{\overline{U}^0} + \tfrac{1}{3}\overline{U}^i \ \overline{U}^j \phi_{ij}^{\alpha\beta} \tfrac{-1+2\overline{U}^0 + \overline{U}^0\overline{U}^0}{(1+\overline{U}^0)^2} \right] \overline{U}^k \\ &\quad + \tfrac{4}{3}\overline{U}^0 (\phi_{0k}^{\alpha\beta} + \phi_{k0}^{\alpha\beta}) + (\Phi_{jk}^{\alpha\beta} + \Phi_{kj}^{\alpha\beta})\overline{U}^j + \tfrac{1}{3}\overline{U}^j (\phi_{jk}^{\alpha\beta} + \phi_{kj}^{\alpha\beta}) \tfrac{1+\overline{U}^0\overline{U}^0}{1+\overline{U}^0} \ . \end{aligned} \tag{92.100}$$

Then (92.93) and (92.95) become

$$\frac{d\omega}{dt} = \frac{k_\alpha k_\beta}{2g^{\nu 0}k_\nu} A_k^{\alpha\beta} \frac{d\overline{U}^k}{dt} \tag{92.101}$$

and

$$\frac{dv_i}{dt} = \frac{k_\alpha k_\beta}{(g^{\nu 0}k_\nu)^2} (g^{\beta 0} A_k^{\alpha i} \frac{d\overline{U}^k}{dt} - g^{\beta i} A_k^{\alpha 0} \frac{d\overline{U}^k}{dt}) - \frac{g^{\gamma 0}g^{\mu i} - g^{\gamma i}g^{\mu 0}}{2(g^{\nu 0}k_\nu)^3} k_\alpha k_\beta k_\gamma A_k^{\alpha\beta} \frac{d\overline{U}^k}{dx^\mu} \ . \tag{92.102}$$

Thus, we see how light will appear to bend in any frame in which the bulk of matter in the universe is accelerating.

To calculate (92.54), we need to solve (92.82) for $\psi_1(x', x)$. The general solution of (92.82) is a general solution of the homogeneous equation plus a particular solution of the inhomogeneous equation. We begin with the homogeneous equation. The homogeneous part of (92.82) can be written explicitly to give

$$\left(g^{00}\frac{\partial^2}{\partial t^2} + g^{11}\frac{\partial^2}{\partial x^2} + g^{22}\frac{\partial^2}{\partial y^2} + g^{33}\frac{\partial^2}{\partial z^2} - 2R^0{}_1{}^1{}_0 \right) \psi_1(x', x) = 0 \ . \tag{92.103}$$

Let

$$\psi_1(x', x) \equiv T(t - t')X(x - x')Y(y - y')Z(z - z') \ . \tag{92.104}$$

To make the notation easier, we temporarily drop the primed variables. Then, (92.103) becomes

$$g^{00}\frac{T''(t)}{T(t)} - 2R^0{}_1{}^1{}_0(t) + g^{11}\frac{X''(x)}{X(x)} + g^{22}\frac{Y''(y)}{Y(y)} + g^{33}\frac{Z''(z)}{Z(z)} = 0 \ . \tag{92.105}$$

The Robertson-Walker metric is spatially homogeneous, so we can separate variables in (92.105) to give

$$g^{00}\frac{T''(t)}{T(t)} - 2R^0{}_1{}^1{}_0(t) - g^{00}k^2 = 0 \ , \tag{92.106}$$

$$g^{11}\frac{X''(x)}{X(x)} + g^{11}k_x^2 = 0 \ , \tag{92.107}$$

$$g^{22}\frac{Y''(y)}{Y(y)} + g^{22}k_y^2 = 0 \ , \text{ and} \tag{92.108}$$

$$g^{33}\frac{Z''(z)}{Z(z)} + g^{33}k_z^2 = 0 \ , \tag{92.109}$$

where $g^{00}k^2 \equiv g^{11}k_x^2 + g^{22}k_y^2 + g^{33}k_z^2$. For a Robertson-Walker metric, we have[261, page 343]

$$R^0{}_1{}^1{}_0 = g^{00}g^{11}R_{0110} = -g^{00}g^{11}R_{0101} = -g^{00}g^{11}S\ddot{S}/\left(1 + \frac{k}{4}\delta_{kl}x^k x^l\right)^2 \ , \tag{92.110}$$

where $k/4$ is $+1/4$, $-1/4$, or 0 and S is the global scale factor. Thus, (92.106) becomes

$$g^{00}\frac{T''(t)}{T(t)} + 2g^{00}g^{11}S\ddot{S}/(1+\frac{k}{4}\delta_{kl}x^k x^l)^2 - g^{00}k^2 = 0 \ . \tag{92.111}$$

For the Robertson-Walker metric, we have $g_{11} = g_{22} = g_{33} = (1+\frac{k}{4}\delta_{kl}x^k x^l)^{-2}$ and $g^{11} = g^{22} = g^{33} = (1+\frac{k}{4}\delta_{kl}x^k x^l)^2$. Thus, (92.111) becomes

$$\frac{T''(t)}{T(t)} + 2S\ddot{S} - k^2 = 0 \ . \tag{92.112}$$

We can solve (92.107), (92.108), and (92.109) explicitly, and we can make a WKB approximation to (92.112) to give

$$\psi_1(x', x) = [A\omega(t-t')]^{-1/2} e^{\pm i\int^{t-t'}\omega(t)dt + ik_x(x-x') + ik_y(y-y') + ik_z(z-z')} \ , \tag{92.113}$$

where $\omega(t) \equiv (2S(t)\ddot{S}(t) - k^2)^{1/2}$ and A is an arbitrary constant.

We still have to find a particular solution to the inhomogeneous equation. I am not sure how to do that right now, so for now, I will try a different method.

Without the $S\ddot{S}$ term, we would have a normal retarded Green's function. That term acts like a mass term for the graviton. That term is very small. It corresponds to a mass whose Compton wavelength is the size of the universe. Although, it is not a constant because the size of the universe varies with time, it is slowly varying compared to the time scales we are considering. Thus, for our purposes, we can take that term to be constant, noticing that it plays the role of a mass term. Then we get a Green's function that is a retarded Yukawa potential[10, p. 620]. That gives[149, p. 185]

$$\psi_1(x', x) = \frac{\delta(t-t'+r)}{r} e^{-(2S\ddot{S})^{\frac{1}{2}} r} \ , \tag{92.114}$$

where $r \equiv |((x-x')^2 + (y-y')^2 + (z-z')^2)^{\frac{1}{2}}|$, and the speed of light c equals one. If the expansion of the universe is accelerating (as we think it is), then $\ddot{S}$ is positive, and that factor acts to cut off the integral in (92.54) at the size of the universe. However, since $\ddot{S}$ varies with cosmological time, and has been negative in the past, its effect is different in different eras.

Now we are ready to calculate (92.54). Define

$$f(t) = (\rho + p)\, U^0 \ . \tag{92.115}$$

Then (92.54) becomes

$$\Phi_0 = 2\kappa \int_\Omega \frac{\delta(t-t'+r)}{r} e^{-(2S\ddot{S})^{\frac{1}{2}} r} f(t)[-g(x)]^{1/2} d^4x \ . \tag{92.116}$$

Then we can take the metric in the unprimed system to be diagonal with $g^{01} = 0$ and $[-g(x)]^{1/2}d^4x = 4\pi r^2 dr dt$, and then (92.116) becomes

$$\Phi_0 = 2\kappa \int_0^\infty \frac{\delta(t-t'+r)}{r} e^{-(2S\ddot{S})^{\frac{1}{2}} r} f(t) 4\pi r^2 dr dt \ . \tag{92.117}$$

Performing the t integration gives

$$\Phi_0 = 8\pi\kappa \int_0^\infty e^{-(2S\ddot{S})^{\frac{1}{2}} r} f(t'-r) r dr \ , \tag{92.118}$$

In other words, when we perform the integration, the farther away we look, we are also looking backward in time. S and $\ddot{S}$ depend on time in a known way.

$$\Phi_0(t') = 8\pi\kappa \int_0^\infty e^{-(2S\ddot{S})^{\frac{1}{2}} r} \left[\rho(t'-r) + p(t'-r)\right] U^0 r dr \ . \tag{92.119}$$

92.5 Extension to mass and different spin

With the above, I have been able to do Sciama's 1953 calculation for a massless particle (the photon). In doing so, I have shown that it is not necessay to do the calculation in the frame of the body or particle. Now, I would like to extend this same method to other particles, such as the electron. Why do this? We already have the Dirac equation. What else do we need? My calculation here is based on Maxwell's equations, which are based on actual fields. Dirac's equation is based on getting a linear wave equation to give the correct relation between energy and momentum. We do not know what the elements in the Dirac spinor represent. It is really just an ad hoc equation.

I do not know how to add mass, but I can try to generalize the spin. We can be led by analogy with the Dirac equation in making this extension. A Dirac spinor has four components. There are two spin states for a spin-half particle, and we double that to include the antiparticle (positron). For the photon, we have three spin states for a spin-one particle, and we double that to get six. Thus, for a photon, my calculation above should have 6 variables and 6 equations instead of 12 and 12. So, Let's see if I can redo the calculation with 6 variables and 6 equations.

For the 6 equations, We still take as our independent dynamical equations the three of (92.6) that do not involve 1, then 2, and then 3; the and the three of (92.7) for α equal 1, 2, and then 3. For the 6 variables, we take ${F^2}_3$, ${F^3}_1$, ${F^1}_2$, ${F^0}_1$, ${F^0}_2$, ${F^0}_3$.

Similarly to (92.8, we have

$$F^{\alpha\mu} = g^{\mu\beta} {F^\alpha}_\beta \ , \tag{92.120}$$

and

$$F_{\mu\beta} = g_{\alpha\mu} {F^\alpha}_\beta \ . \tag{92.121}$$

Written out in detail, in matrix form, we have

$$\begin{pmatrix} F^{01} \\ F^{02} \\ F^{03} \\ \\ F^{23} \\ F^{31} \\ F^{12} \end{pmatrix} = \begin{pmatrix} g^{11} & g^{12} & g^{13} & 0 & 0 & 0 \\ g^{21} & g^{22} & g^{23} & 0 & 0 & 0 \\ g^{31} & g^{32} & g^{33} & 0 & 0 & 0 \\ \\ 0 & g^{30} & 0 & g^{33} & 0 & -g^{31} \\ 0 & 0 & g^{10} & -g^{12} & g^{11} & 0 \\ g^{20} & 0 & 0 & 0 & -g^{23} & g^{22} \end{pmatrix} \begin{pmatrix} {F^0}_1 \\ {F^0}_2 \\ {F^0}_3 \\ \\ {F^2}_3 \\ {F^3}_1 \\ {F^1}_2 \end{pmatrix} \tag{92.122}$$

and

$$\begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \\ \\ F_{23} \\ F_{31} \\ F_{12} \end{pmatrix} = \begin{pmatrix} g_{00} & 0 & 0 & 0 & g_{30} & -g_{20} \\ 0 & g_{00} & 0 & -g_{30} & 0 & g_{10} \\ 0 & 0 & g_{00} & g_{20} & -g_{10} & 0 \\ \\ 0 & 0 & g_{02} & g_{22} & -g_{12} & 0 \\ g_{03} & 0 & 0 & 0 & g_{33} & -g_{23} \\ 0 & g_{01} & 0 & -g_{31} & 0 & g_{11} \end{pmatrix} \begin{pmatrix} {F^0}_1 \\ {F^0}_2 \\ {F^0}_3 \\ \\ {F^2}_3 \\ {F^3}_1 \\ {F^1}_2 \end{pmatrix} \ . \tag{92.123}$$

Now, we write the first three equations in (92.23) as

$$\begin{pmatrix} -i\hat{\omega} & 0 & 0 & 0 & i\hat{k}_3 & -i\hat{k}_2 \\ 0 & -i\hat{\omega} & 0 & -i\hat{k}_3 & 0 & i\hat{k}_1 \\ 0 & 0 & -i\hat{\omega} & i\hat{k}_2 & -i\hat{k}_1 & 0 \end{pmatrix} \begin{pmatrix} F^{01} \\ F^{02} \\ F^{03} \\ \\ F^{23} \\ F^{31} \\ F^{12} \end{pmatrix} = 0 \ , \tag{92.124}$$

and the last three as

$$\begin{pmatrix} 0 & i\hat{k}_3 & -i\hat{k}_2 & -i\hat{\omega} & 0 & 0 \\ -i\hat{k}_3 & 0 & i\hat{k}_1 & 0 & -i\hat{\omega} & 0 \\ i\hat{k}_2 & -i\hat{k}_1 & 0 & 0 & 0 & -i\hat{\omega} \end{pmatrix} \begin{pmatrix} F_{01} \\ F_{02} \\ F_{03} \\ \\ F_{23} \\ F_{31} \\ F_{12} \end{pmatrix} = 0 \ . \tag{92.125}$$

Substituting gives

$$\begin{pmatrix} -i\hat{\omega} & 0 & 0 & 0 & i\hat{k}_3 & -i\hat{k}_2 \\ 0 & -i\hat{\omega} & 0 & -i\hat{k}_3 & 0 & i\hat{k}_1 \\ 0 & 0 & -i\hat{\omega} & i\hat{k}_2 & -i\hat{k}_1 & 0 \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} & g^{13} & 0 & 0 & 0 \\ g^{21} & g^{22} & g^{23} & 0 & 0 & 0 \\ g^{31} & g^{32} & g^{33} & 0 & 0 & 0 \\ \\ 0 & g^{30} & 0 & g^{33} & 0 & -g^{31} \\ 0 & 0 & g^{10} & -g^{12} & g^{11} & 0 \\ g^{20} & 0 & 0 & 0 & -g^{23} & g^{22} \end{pmatrix} \begin{pmatrix} F^0{}_1 \\ F^0{}_2 \\ F^0{}_3 \\ \\ F^2{}_3 \\ F^3{}_1 \\ F^1{}_2 \end{pmatrix} = 0 \tag{92.126}$$

and

$$\begin{pmatrix} 0 & i\hat{k}_3 & -i\hat{k}_2 & -i\hat{\omega} & 0 & 0 \\ -i\hat{k}_3 & 0 & i\hat{k}_1 & 0 & -i\hat{\omega} & 0 \\ i\hat{k}_2 & -i\hat{k}_1 & 0 & 0 & 0 & -i\hat{\omega} \end{pmatrix} \begin{pmatrix} g_{00} & 0 & 0 & 0 & g_{30} & -g_{20} \\ 0 & g_{00} & 0 & -g_{30} & 0 & g_{10} \\ 0 & 0 & g_{00} & g_{20} & -g_{10} & 0 \\ \\ 0 & 0 & g_{02} & g_{22} & -g_{12} & 0 \\ g_{03} & 0 & 0 & 0 & g_{33} & -g_{23} \\ 0 & g_{01} & 0 & -g_{31} & 0 & g_{11} \end{pmatrix} \begin{pmatrix} F^0{}_1 \\ F^0{}_2 \\ F^0{}_3 \\ \\ F^2{}_3 \\ F^3{}_1 \\ F^1{}_2 \end{pmatrix} = 0 \ . \tag{92.127}$$

Multiplying the matrices and combining gives a system of 6 equations with 6 variables, which can be expressed as a matrix equation. The equations now involve derivatives of the metric tensor, which was not true for the 12 by 12 version.

However, this gives a good idea for altering the Dirac equation. In curved spacetime, the Dirac equation does involve derivatives of the metric tensor, just like we will get here for Maxwell's equations. If I can do the opposite for the Dirac equation by doubling the number of variables and equations (from 4 to 8), then maybe I can get rid of the derivatives of the metric tensor. Then maybe the corresponding variables will correspond to real physical fields like electric and magnetic fields as with Maxwell's equations.

Another possibility is to do the same development for the Yang-Mills equations as a way of generalizing to mass and different spin.

92.6 Dirac equation without derivatives of the metric tensor

This section is based on the 1957 paper of Brill and Wheeler[198], which uses a $(+++-)$ signature, so some of the formulas may have to be changed for a different choice of signature.

We start with the Dirac equation in curved space[198, 262].

$$\gamma^\alpha \nabla_\alpha \psi + \mu\psi = 0 \ , \tag{92.128}$$

where $\mu \equiv mc/\hbar$, there is no confusion between this use of μ and μ used as a subscript or superscript,

$$\nabla_\alpha \psi = \frac{\partial \psi}{\partial x^\alpha} - \Gamma_\alpha \psi \ , \tag{92.129}$$

$$\Gamma_k = \frac{1}{4} g_{\mu\alpha} \left[\frac{\partial b_\nu{}^\beta}{\partial x^k} a^\alpha{}_\beta - \Gamma_{\nu k}{}^\alpha \right] S^{\mu\nu} + a_k \mathbf{1} \ , \tag{92.130}$$

$a^\alpha{}_\beta$ and $b_\nu{}^\beta$ give transformations between the x frame to a local Lorentzian frame $\tilde{x}$ such that

$$dx^\alpha = a^\alpha{}_\beta d\tilde{x}^\beta \ , \tag{92.131}$$

$$d\tilde{x}^\beta = b_\nu{}^\beta dx^\nu \ , \tag{92.132}$$

$$\Gamma_{\nu k}{}^\alpha = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{k\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^k} - \frac{\partial g_{\nu k}}{\partial x^\beta} \right) \ , \tag{92.133}$$

$$S^{ij} = \frac{1}{2}(\gamma^i \gamma^j - \gamma^j \gamma^i) \ , \tag{92.134}$$

a_k is arbitrary,

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ik} \mathbf{1} \ , \tag{92.135}$$

$\mathbf{1}$ is the unit matrix, and all subscripts and superscripts run from 1 to 4.

The existence of $\Gamma_{\nu k}{}^\alpha$ in the Dirac equation means that derivatives of the metric tensor will appear in the dispersion relation of electron waves. That can be avoided by introducing additional variables. Consider the following identity.

$$\left(\frac{\partial g_{k\beta}}{\partial x^\nu} \right) \psi = \frac{\partial}{\partial x^\nu}(g_{k\beta}\psi) - g_{k\beta} \frac{\partial \psi}{\partial x^\nu} \ . \tag{92.136}$$

Clearly, by introducing

$$\psi_{k\beta} = g_{k\beta}\psi \ , \tag{92.137}$$

as a new variable, and (92.137) as a new equation, it is possible to get a form of the Dirac equation that does not explicitly involve derivatives of the metric tensor.

However, the Dirac equation also contains the term $\partial b_\nu{}^\beta / \partial x^k$. To get rid of that term, we use the same trick. That is, we use the identity

$$\left(\frac{\partial b_\nu{}^\beta}{\partial x^k} \right) \psi = \frac{\partial}{\partial x^k}(b_\nu{}^\beta \psi) - b_\nu{}^\beta \frac{\partial \psi}{\partial x^k} \ , \tag{92.138}$$

introduce

$$\phi_\nu{}^\beta = b_\nu{}^\beta \psi \tag{92.139}$$

as a new variable, and add (92.139) as a new equation.

To help in the calculations, we use (92.131) and (92.132) to get the following two identities.

$$a^\alpha{}_\beta b_\nu{}^\beta = \delta^\alpha_\nu \ , \tag{92.140}$$

and

$$b_\alpha{}^\nu a^\alpha{}_\beta = \delta^\nu_\beta \ . \tag{92.141}$$

Leaving out the details, the Dirac equation can be written as

$$-\gamma^k (\frac{1}{2} \frac{\partial}{\partial x^k} + a_k)\psi + \mu\psi - A^{k\nu}{}_\beta \frac{\partial \phi_\nu{}^\beta}{\partial x^k} + B^{k\nu\mu} \frac{\partial \psi_{\nu\mu}}{\partial x^k} = 0 \ , \tag{92.142}$$

where

$$A^{k\nu}{}_\beta \equiv \frac{1}{4} \gamma^k S^{\mu\nu} a_{\mu\beta} \ , \tag{92.143}$$

and

$$B^{k\nu\mu} \equiv \frac{1}{4} \left(g^{\nu\mu}\gamma^k - g^{k\nu}\gamma^\mu - g^{k\mu}\gamma^\nu \right) \ . \tag{92.144}$$

To summarize where we are, we now have 3 variables (ψ, $\psi_{k\beta}$, and $\phi_\nu{}^\beta$) with 3 equations, (92.142), (92.137), and (92.139). However, all of the variables are 4-component spinors, and $\psi_{k\beta}$ and $\phi_\nu{}^\beta$ each designate 16 different spinors. That would suggest we really have $4+16\times4+16\times4 = 132$ variables with 132 equations. However, (92.137) shows that $\psi_{k\beta}$ really has only 10×4 independent components, and some reflection shows that $\phi_\nu{}^\beta$ also has only 10×4 independent components. Thus, we really have only $4+10\times4+10\times4 = 84$ independent variables and 84 equations.

We can write some of the equations in another form. From [198], we have

$$\gamma_i = b_i{}^\alpha \tilde{\gamma}_\alpha \ . \tag{92.145}$$

This equation, among others, allows us to rewrite (92.137) and (92.139) as

$$a^{\mu\beta}\psi_{\mu\nu} = b_\nu{}^\beta \psi = \phi_\nu{}^\beta \ . \tag{92.146}$$

It is not clear whether these forms are better or worse than the originals.

To calculate a dispersion relation, it might be necessary to calculate a 32×32 determinant. However, it might not be necessary. Since none of the coefficients in any of the equations contain derivatives of the metric tensor (or of anything else), the dispersion relation should be the same as though there were no gravitational field present. It may take tedious calculations to show that, however. Is this simply a consequence of being able to transform away gravitational fields by going to a freely falling frame? Maybe, but notice that here, we have not changed frames. We have simply changed dependent variables.

92.7 Rotating the coordinate axes

Some insight into the significance of the ωk_1 cross term can be seen by rotating the coordinates so that the coordinates are orthogonal. This can be done in an infinite number of ways, but two of these ways are instructive. These are

$$-\left[(-g^{00})^{\frac{1}{2}}\omega + \frac{g^{01}}{(-g^{00})^{\frac{1}{2}}}k_1\right]^2 + \left[(g^{11} - \frac{g^{01}g^{10}}{g^{00}})^{\frac{1}{2}}k_1\right]^2 + g^{22}k_2^2 + g^{33}k_3^2 = 0 \ , \tag{92.147}$$

and

$$-\left[(-g^{00} + \frac{g^{01}g^{10}}{g^{11}})^{\frac{1}{2}}\omega\right]^2 + \left[(g^{11})^{\frac{1}{2}}k_1 - \frac{g^{01}}{(g^{11})^{\frac{1}{2}}}\omega\right]^2 + g^{22}k_2^2 + g^{33}k_3^2 = 0 \ , \tag{92.148}$$

More generally, the rotated form can be written as

$$(\alpha\omega + \beta k_1)^2 + (\gamma\omega + \delta k_1)^2 + g^{22}k_2^2 + g^{33}k_3^2 = 0 \ , \tag{92.149}$$

where

$$\alpha = -\frac{g^{01}}{g^{11}}\beta \pm \frac{i\delta}{g^{11}}\left(g^{01}g^{10} - g^{00}g^{11}\right)^{\frac{1}{2}} \ , \tag{92.150}$$

$$\gamma = -\frac{g^{01}}{g^{11}}\delta \mp \frac{i\beta}{g^{11}}\left(g^{01}g^{10} - g^{00}g^{11}\right)^{\frac{1}{2}} \ , \tag{92.151}$$

and

$$\beta^2 + \delta^2 = g^{11} \ . \tag{92.152}$$

In performing this rotation of coordinates, we have rotated the ω and k_1 axes into each other. This means we have also rotated the t and x_1 axes into each other. This corresponds to a boost. That is, transforming to coordinate system that is moving at a constant velocity with respect to the first coordinate system.

An alternative representation of the rotation is

$$\beta = -\frac{g^{01}}{g^{00}}\alpha \pm \frac{i\gamma}{g^{00}}\left(g^{01}g^{10} - g^{00}g^{11}\right)^{\frac{1}{2}} \ , \tag{92.153}$$

$$\delta = -\frac{g^{01}}{g^{00}}\gamma \mp \frac{i\alpha}{g^{00}}\left(g^{01}g^{10} - g^{00}g^{11}\right)^{\frac{1}{2}} \ , \tag{92.154}$$

and

$$\alpha^2 + \gamma^2 = g^{00} \ . \tag{92.155}$$

When g^{01} is small, an approximate set of formulas show some symmetry:

$$\alpha \approx \pm i\sqrt{-g^{00}} \mp \frac{1}{2}i\frac{g^{01}}{\sqrt{g^{11}}} \ , \tag{92.156}$$

$$\beta \approx \left(-\frac{g^{11}}{g^{00}}\right)^{\frac{1}{4}}\sqrt{g^{01}} + \frac{1}{2}\frac{g^{01}}{\sqrt{-g^{00}}}(-1 \mp i) \ , \tag{92.157}$$

$$\gamma \approx \mp i\left(-\frac{g^{00}}{g^{11}}\right)^{\frac{1}{4}}\sqrt{g^{01}} + \frac{1}{2}\frac{g^{01}}{\sqrt{g^{11}}}(-1 \pm i) \ , \tag{92.158}$$

$$\delta \approx \sqrt{g^{11}} - \frac{1}{2}\frac{g^{01}}{\sqrt{-g^{00}}} \ , \tag{92.159}$$

92.8 A sparse universe

It may be useful to consider the limiting case where there is not much matter in the universe. If (92.24) were an integral rather than an integral equation, we might expect that $g^{\alpha\beta}$ will be small, which would require $g_{\alpha\beta}$ to be large. However, since (92.24) is an integral equation, the situation is more complicated. When there is not much matter in the universe, the surface term in (92.24) may not be negligible.

Chapter 93

Quantum theory - philosophies versus interpretations versus models[1]

abstract

"Models" is a more accurate description for what is usually called "interpretations" in quantum theory.

93.1 Introduction

Discussions of quantum theory center around interpretations of what it means. This is because, although the equations (Schrödinger, Dirac, and Klein-Gordon) are generally accepted and well-known, the meaning of the solution, ψ, is not generally agreed upon, and therefore needs an "interpretation."

Although these discussions usually lead to some enlightenment, they have not lead to general agreement, partly because not enough distinction is made between interpretations and models. The present discussion cannot hope to include or replace all of the discussions that have already taken place, but can at least address the distinction between interpretations and models to try to add some more clarity to the discussion.

To help do that, I will be discusssing one other point as well, because this point is generally not discussed at all. Although the discussions of interpretations are meant to apply to quantum theory as a whole, all of the examples are from elementary quantum mechanics, ignoring quantum field theory, quantum mechanics on a curved background, and quantum electrodynamics (QED).

There are at least two reasons for this. First, it is probably generally thought that if we cannot understand ordinary quantum mechanics, then how can we possibly understand the more complicated situations? Second, ordinary quantum mechanics already has the main interpretation difficulties. Once we understand those, we will understand the more complicated cases. Here, I will discuss the validity of these assumptions.

There are two ways that quantum mechanics can be formulated. In the first, we talk about the state of the system. In the second, we talk about the amplitude for a process. The former does not satisfy the positivist philosophy (discussed below), while the latter does.

[1]10-15 July, 22 December 2011 - 26 March 2012 - 10 July 2014

93.2 Philosophies

There are two main philosophies, positivist and realist, which represent the main extremes. I will not discuss other variations.

93.2.1 Positivist philosophy

The positivist philosophy has been promoted by Ernst Mach, Niels Bohr, Werner Heisenberg, and others. The positivist philosophy is represented by several different statements. Some of these are:

- "It makes no sense to talk about what cannot be measured." [268, Wolfson and Burgstaller, 2000, Course Guidebook, Part 2, Lecture 19, p. 36]
- "what cannot be measured does not exist."[269, Bohm]
- "what cannot be measured – does not exist."[270, Beller, 1996, p. 220] [271, Beller, 1999, p. 202]
- "In constructing physical theories, one has to use only observable quantities."[272, Golshani,2011]
- "it is reasonable to consider only those quantities in a theory that can be measured, [273, Heisenberg, in Heisenberg, 1971]
- "But you don't seriously believe that only observable quantities should be considered in a physical theory?" [273, Einstein, in Heisenberg, 1971]
- "Shut up and calculate."[274, Mermin, 2004][275, wikipedia, 2011]

The main interpretation associated with this philosophy is the Copenhagen interpretation.

93.2.2 Realist philosophy

The realist philosophy represents the other extreme. It was promoted by Albert Einstein, Erwin Schrödinger, and Louis deBroglie. It is also represented by several statements, some of which are:

- "What is really happening?"
- "What holds the world together when nobody is watching?"
- "I like to believe that the moon is still there even if we don't look at it." [276, Einstein, as paraphrased by Knierim] [277, Einstein, as paraphrased by Mermin, 1985]
- "Do you really believe that the moon exists only when you look at it?" [278, Einstein, paraphrased in Pais, 1979, p. 907]
- "Quantum mechanics is certaintly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the 'old one'. I, at any rate, am convinced that He is not playing at dice." [279, Einstein, in "The Born-Einstein Letters" (with comments by M. Born), p. 88].

This philosophy requires models of what is actually happening.

93.3 Interpretations

Originally, various interpretations of quantum theory were devised to interpret the meaning of the wave function (or state vector) ψ.

Eventually, these various interpretations attempted to explain what was really happening, and thus became more than interpretations, but models of what was really happening.

Even the Copenhagen interpretation, which claimed not to be a model of what was really happening, had some aspects of a model, because it also dealt with some quantities which could not be measured, and assertions which could not be verified by measurement.

The various interpretations, are therefore discussed below under "models."

However, there are still some matters of interpretation that have not been resolved. Some of these are:

1. "Does the wave function represent reality or knowledge?"

2. "How do we explain wave-function collapse?"

3. "How do we deal with the difficulty that, although we know how a closed system evolves, we do not know how to deal with a system during the process of measurement."

4. "How do we treat the whole universe quantum mechanically?"

93.4 Models

What are usually called interpretations are really models, with the possible exception of the Copenhagen interpretation, which is based of the positivist philosophy, which essentially says that we need no models. Still, for the present discussion, I will include it under models, and discuss to what extent it is or is not a model and to what extent it really is a positivist model (or interpretation).

93.4.1 Copenhagen model

Here is a statement representative of the Copenhagen model:

- "A state is a complete description of the observable properties of a physical system" [280, wikipedia Glossary of quantum mechanics, 2011]

The first thing to notice is that there is no experiment that can decide the truth of the statement. Therefore, this statement violates the positivist philosophy because it talks about something that is not measurable.

93.4.2 Many worlds model

The many worlds model is based on taking any of the equations (Schrödinger, Dirac, and Klein-Gordon) as being true not for just an ensemble, but for each individual system (that is, for each member of the ensemble).

93.4.3 Transactional model

The transactional model (Cramer) tries to do the same thing without assuming many worlds.

93.4.4 Relational model

The relational model [281] may have some advantages.

93.4.5 Ensemble model

The ensemble model [282] may be close to a correct model except that it does not believe there is anything beyond the ensemble.

93.5 What do we know?

There are really only two categories about quantum theory that we can really say are correct and that work, in spite of the usual assertion that "Quantum theory is one of the most successful theories we have."

1. If we calculate the amplitude for a process, the corresponding probability agrees completely with measurements made on an ensemble of identically prepared states. This includes various scattering and phase-interference effects.

2. Calculated energy levels agree with measured energy-level differences.

Beyond these two, most everything else is part of various models.

93.6 What do we not know?

1. There is no evidence that a wave function (or state vector) represents anything real.

2. There is no evidence that solutions for ψ of any of the equations (Schrödinger, Dirac, and Klein-Gordon) represent anything other that a method for calculating the amplitude for a process or calculating energy levels.

3. There is no evidence that solutions for ψ of any of the equations (Schrödinger, Dirac, and Klein-Gordon) apply to anything but an ensemble except in a probabilistic sense or to calculate energy levels.

93.7 Where does this leave us?

Because there is no evidence that solutions for ψ of any of the equations (Schrödinger, Dirac, and Klein-Gordon) apply to anything but an ensemble, any effort to justify treating ψ as though it did apply to an individual system other than in a probabilistic sense vanishes. Thus, there is no justification for either the many-worlds model or the transactional model.

In fact, there is evidence that ψ does not apply to an individual system. It is known, at least for electromagnetic fields, that the fields fluctuate. Such fluctuations are not predicted by the usual calculation of electromagnetic wave functions.

The discussion of myths and facts in quantum mechanics by Nikolić [283] is relevant to the present discussion.

We should also discuss "completeness," "reality," "locality," and "causality."

Chapter 94

A wave function allows forecasts, but is not a real field[1]

abstract

Wave functions serve very well to calculate energy levels and to forecast the probable outcome of experiments. There is no evidence to support the assertion that a wave function represents a real field. Wave propagation in turbulent fluids (which is well understood) gives an example of a theory that applies to the statistical behavior of an ensemble, but does not apply to an individual system, except in a statistical sense. I suggest that quantum theory applies to ensembles for a similar reason.

94.1 Introduction

There is still no agreement on the proper interpretation of quantum theory. Part of the controversy is whether a wave function represents a real field or whether it represents knowledge.

Either of those interpretations leads to difficulties. If the wave function represents a real field, then we are led to the problems of wave function collapse (including the Einstein-Rosen-Podolsky problem) and either a many-worlds scenario or advanced potentials and backward time travel with Cramer's Transactional Interpretation.

On the other hand, if the wave function represents knowledge, whose knowledge? This then leads to the possibility of bringing the mental state of experimenters into the foundations of physics.

Because neither of those two scenarios is attractive, a third viewpoint (in which a wave function represents neither a real field nor someone's knowledge) may help. Notice that all of the usual difficulties with quantum theory disappear if the wave function is neither a real field nor someone's knowledge. For an example of a well-understood theory that applies only to ensembles, but not to individual systems (except in a probabilistic sense) we turn to turbulent fluids.

94.2 Wave propagation in turbulent fluids

The Navier-Stokes equations give an accurate description of fluid flow by approximating the molecular reality of the system by a continuous fluid. The approximation is accurate for both liquids and gases in describing the fluid for scales of motion much larger than the mean-free path between molecular collisions and for time scales much larger than the average time between collisions.

[1] 29 July 2011 - 26 Mar 2012

However, there is a practical problem in that an accurate calculation of motion on all scales (for which the approximation is valid) is necessary even to be able to make calculations on any scales for many situations because the nonlinearity of the equations couples different scales of motion. It is really not practical to calculate the very small scales of motion, even if we were interested in the details of the small scales.

However, because the small scales of motion affect the large scales, it is often necessary to approximate the small scales of motion by assuming some model for how the small scales depend on the larger scales on the average. Such models are often called turbulence models or closure models, and so-called second-order closure models are common.

The result is that when including turbulence, only averages of the fluid flow on any scales are possible to calculate, and even those calculations are only approximate. What we end up with are calculations of ensemble averages for the fluid. The calculations apply to only an ensemble of fluids, and in comparing calculations with experiment, it is necessary to run the experiment many times for an ensemble of systems.

We understand perfectly how this situation arises. We know that the underlying physical situation is a system of colliding molecules. We know the limitations of the Navier-Stokes equations, and we know how ensemble averages entered into the situation. We also know that the resulting turbulence equations apply only to an ensemble, and not to an individual system, except in a probabilistic sense.

Applying the Navier-Stokes equations to a turbulent fluid gives a wave equation in terms of a set of dependent variables that are ensemble averages of various quantities that include pressure, three components of fluid velocity, density, and temperature, in addition to ensemble averages of products of the deviations of these quantities from the ensemble averages. In addition, each of these quantities consists of a background value plus a wave-associated value. Examples of such calculations are given by [258, 259, Jones and Hooke, 1986a,b].

As an example in a particularly simple case. Let U and W be the eastward and upward components of the fluid velocity. Then we have $U = \overline{U} + u$ and $W = \overline{W} + w$ for the ensemble average plus deviations from the ensemble. We also define one of the components of the Reynolds stress as $\overline{uw}$.

We now take each of these quantities to have a background value (with subscript 0) plus a wave-associated value (with subscript 1). Thus, we have

$$\overline{U}(t, \mathbf{x}) = U_0(\mathbf{x}) + u_1(\mathbf{x}) \exp\left(i\omega t - i\mathbf{k}\cdot\mathbf{x}\right) , \tag{94.1}$$

$$\overline{W}(t, \mathbf{x}) = W_0(\mathbf{x}) + w_1(\mathbf{x}) \exp\left(i\omega t - i\mathbf{k}\cdot\mathbf{x}\right) , \tag{94.2}$$

$$\overline{uw}(t, \mathbf{x}) = \overline{uw}_0(\mathbf{x}) + \overline{uw}_1(\mathbf{x}) \exp\left(i\omega t - i\mathbf{k}\cdot\mathbf{x}\right) , \tag{94.3}$$

and similarly for other variables, including pressure, p. The above example is for plane waves, but usually we have a general wave function $\psi(t, \mathbf{x})$, such as

$$\psi(t, \mathbf{x}) = \begin{pmatrix} u_1(t, \mathbf{x}) \\ w_1(t, \mathbf{x}) \\ \overline{uw}_1(t, \mathbf{x}) \\ p_1(t, \mathbf{x}) \end{pmatrix} . \tag{94.4}$$

Notice that each element of the wave function is an ensemble average. In calculating the propagation of this wave function, there is the possibility for refraction, diffraction, scattering, and wave interference of these waves. However, the results apply only to an ensemble, not to an individual member of the ensemble. To compare with measurement, it is necessary to take averages over an ensemble of similar systems.

94.3 Quantum theory

In quantum theory, we also have equations (Schrödinger, Dirac, or Klein-Gordon) that apply to an ensemble of systems. We do not know why the equations apply to only an ensemble, but we know from practice that they do. To compare with experiment, it is always necessary to compare with an ensemble of identically prepared systems. The equations do not apply to a single system except in a probabilistic sense. (An exception is in calculating energy levels in some time-independent systems.)

Difficulties arise if one tries to apply the equations to individual systems, other than in a probabilistic sense. When that is tried, we end up with all kinds of wierdness. Many worlds, collapsing wave functions, and waves propagating back in time, to name a few. Notice, that there is absolutely no evidence to suggest that these equations should apply to any situation other than to an ensemble. And yet, people try to apply these equations to individual systems, with the resulting wierdness.

The desire to apply our equations to a single system is easily explained. We want a theory that describes the world in a realistic way. That is, a theory that describes real fields or real particle motions. We do not want some probabilistic theory. We would like to believe that we already have such a theory, in spite of all the evidence telling us otherwise.

94.4 How good is the analogy?

In turbulent fluids, the calculations give ensemble averages for the kinetic energy of the fluid flow as a function of position, and covariances of various fluid properties. In quantum theory the absolute square of the wave function gives the probability of detecting the particle in question. In this sense, quantum theory and the physics of turbulent fluids are not similar. That is, the analogy is not perfect. If the analogy were perfect, we could look to turbulence as a model for quantum theory.

Another difference is that in wave propagation in turbulent fluids, the elements of the wave function are already ensemble averages, whereas in quantum theory, it is necessary to calculate $\psi^*\psi$ to get an ensemble average. However, in another context (nonlinear gravity waves on the ocean surface)[284, 285], second-order waves look very much like quantum theory, including conservation of energy (as frequency ω) and conservation of momentum (as wavenumber $\mathbf{k}$).

As mentioned above, a time-independent system is an exception to the rule that quantum theory gives only probabilistic answers. Why is this? Mathematically, we know why that is true, and is easily explained.

Another consideration is that if the analogy between turbulence theory and quantum theory were to hold, could turbulence theory give exact rather than probabilistic answers in time-independent systems? At first glance, it would seem that the answer to that question would have to be "No" because, as mentioned above, the elements of the wave function are already ensemble-averages. However, on second thought, a turbulent fluid that is somehow bounded would give solutions that have discrete frequencies and discrete wavenumbers as in cavity modes. These might be valid for individual members of the ensemble. Just as in atomic wave functions, the wave functions would apply only to the ensemble, but the eigenvalues for frequency and wavenumber eigenmodes would apply to each member of the ensemble.

94.5 A future theory?

It is clear that the present theory is not complete in that it gives only ensemble averages and energy levels. Maybe that it is the best we can do. A more complete theory that allows us to know more about a system may not be possible.

However, there is the possibility of making a model of the underlying behavior that would be consistent with the present quantum theory. It might not be possible to verify that such a model is the correct one, but if it is self-consistent and consistent with known experiments, it might be the best we can do, and might be a satisfactory resolution of the present difficult situation.

In addition, a successful resolution of the difficulties with quantum theory may lead to some progress in trying to resolve the difficulties with bringing gravitation and quantum theory together. As readers of this book may be aware, the main tactic to resolving gravitation and quantum theory is to replace inertial frames as a background for quantum theory by the gravitational interaction that gives rise to the inertial frames. We can essentially consider the gravitational interaction to be a gravitational induction interaction with every other particle in the universe. In a sense, these individual gravitational interactions (a bombardment from gravitons) might be analogous to the collisions of molecules in a turbulent fluid. Then, the smooth inertial frame approximation might be analogous to the Navier-Stokes equations for a fluid.

Apparently, I have just come back to the "ensemble interpretation" [282] of quantum theory. The more I think about it, it is the only interpretation that avoids all of the problems with the other interpretations. There is a difference with the "ensemble interpretation," however. As long as we realize that there is a deeper reality beneath the wave function, we can look for a model of that deeper reality that is self-consistent and consistent with the measurements.

The wave function is objective. It does not depend on "us." For a given physical situation, there is a correct calculation of the amplitude for the process. Whether "we" or some people actually make that calculation is irrelevant.

Chapter 95

Gravitational vector potential revisited[1]

abstract

The apparent wave properties of particles may simply be the result of trying to reconcile a tensor gravitational field with a vector electromagnetic field.

95.1 introduction

It is usual to think that gravitation is not important in atomic, nuclear, or particle physics because the gravitational force between a proton an electron is so much smaller (by a factor of 10^{40}) than the electrostatic force between them. However, we know now, from General Relativity, that inertia is also a gravitational force, and that in an atom, for example, the electrostatic attraction between an electron and the nucleus is exactly balanced by centrifugal force (which is an inertial force, and therefore, a gravitational force).

However, the centrifugal force does not enter directly, as it would in Newtonian physics, when applied to planets in the Solar system, for example. Instead, it enters indirectly through gradient terms in the Dirac equation or the Schrödinger equation. Thus, the gravitational part of micro physics is completely hidden in the formality of the theory, in terms of the inertial frame that is implicit in the theory. Because the gravitational sources for the local inertial field are not local, it is easy to forget that an important part of quantum theory is dependent on the gravitational interaction that gives local inertia. In an attempt to make the gravitational interaction more explicit, I looked at an electron in a gravitational and an electromagnetic field.

95.1.1 Background

Chapter 27 pointed out that particle 4-momentum and electromagnetic 4-vector potential always appear together in the Klein-Gordon equation and the Dirac equation. I wondered then, if perhaps momentum might play the same role as a vector potential. To lend support to this idea, I noticed that the geodesic equation, when written in a coordinate system fixed to the body, involves only derivatives of the metric tensor, just as the Lorentz force involves only derivatives of the electromagnetic 4-vector potential.

[1] 8 Aug - 16 Sep 2011

95.1.2 Gravitational vector potential (GVP)

In fact, it is possible, in the rest frame of the particle, to write the geodesic equation in the same form as a Lorentz force, with an explicit formula for a gravitational 4-vector potential. I then defined this to be a vector potential, with the corresponding vector transformation properties, and transformed it to a general frame. The result, from chapter 27 gives a GVP as

$$g_\mu = \frac{\pi_\mu}{m} - \frac{1}{2m^2} g^{\alpha\beta} \pi_\alpha \pi_\beta \frac{\partial \tau}{\partial x^\mu} \ , \tag{95.1}$$

where π_μ is a covariant vector field (which equals the mechanical 4-momentum on the trajectory of the particle) and satisfies the constraint

$$g^{\mu\nu} \pi_\mu \frac{\partial}{\partial x^\nu} (g^{\alpha\beta} \pi_\alpha \pi_\beta) = 0 \ , \tag{95.2}$$

τ is a scalar field (which equals the proper time on the trajectory of the particle) and satisfies the constraint

$$g^{\mu\nu} \pi_\mu \frac{\partial \tau}{\partial x^\nu} = m \ , \tag{95.3}$$

and m is the mass of the particle.

The "gravitational vector potential" g_μ can be combined with the electromagnetic vector potential A_μ to give a total vector potential q_μ.

$$q_\mu = m g_\mu + e A_\mu \ , \tag{95.4}$$

where e is the charge on the particle, that when put into the Lorentz force equation

$$(q_{\mu,\nu} - q_{\nu,\mu}) \dot{x}^\nu = 0 \ , \tag{95.5}$$

yields the geodesic equation for the gravitational part of the interaction plus the Lorentz force equation for the electromagnetic part of interaction.

Does this mean that a gravitational field is really a vector field rather than a tensor field? No, first because the above calculation does not take into account sources of the gravitational field. Second, because there are cases (discussed in chapter 27) where that procedure does not work. However, in many situations, it is not possible to tell (locally) that the gravitational field is not a vector field. It requires vector and scalar fields π and τ that must satisfy certain constraints. In addition, those fields must satisfy other conditions imposed by the physics of the situation.

95.1.3 Condition for the validity of a gravitational vector potential

The above development of a gravitational vector potential applies not just to a single particle with a single trajectory, but to a whole family of trajectories. We could call such a family of trajectories a timelike geodesic congruence. The particle in each trajectory will experience the same gravitational vector potential. (That is a g_μ that depends on position and time in the same way.) Notice that a single trajectory does not determine a congruence. As a simple example, for the case of no electromagnetic field, all of the trajectories will be straight lines in an inertial frame. Given one such straight line, it is a member of a congruence of trajectories that are parallel to it. It is also a member of a congruence of trajectories that all go through a given point. Those two congruences would not have the same gravitational vector potential.

That is why the gravitational interaction is not a vector interaction. Different congruences will see different gravitational vector potentials. Thus, the condition for the validity of a gravitational vector potential is that it can apply to only one congruence (that is, to one family of trajectories).

I had already noticed in chapter 27 some parallels between the "gravitational vector potential" (GVP) and wave functions. I wondered, when I first derived this "gravitational vector potential" whether there was some connection with wave functions. That is, notice that if we take a solution of the Schrödinger equation, it corresponds to a family of trajectories of electrons. Such a family of trajectories forms a congruence, and an electron on each of those trajectories may see the same gravitational vector potential. Thus, a solution of the Schrödinger equation may always correspond to a congruence of trajectories that have the same gravitational vector potential.

The converse may also be true. That is, any congruence of trajectories for a given physical situation that see the same gravitational vector potential may also correspond to a solution of the Schrödinger equation for the same physical situation. Thus, there may be a 1-to-1 correspondance between solutions of the Schrödinger equation and the validity of the gravitational vector potential. The same may also be true for the Dirac equation. In other words, a GVP may exist if and only if standard quantum theory applies.

To show that, I don't have to show that I can derive the Dirac equation from the GVP (as I tried to do in one of the sections below). I only have to show a 1-to-1 relationship between a GVP and ψ.

When I tried to do that in the following sections, I was not able to do it. I get conflicting results depending on whether I to use τ or q_μ to make the 1-to-1 relationship between a GVP and ψ. If I use τ, then that leads to (95.43) with (95.42) to define the 1-to-1 relationship, and that leads to (95.46), which requires $q_\mu = \frac{i\hbar}{2}\frac{\psi_{,\mu}}{\psi}$ to agree with the Dirac equation.

However, if we use q_μ to make the 1-to-1 relationship between a GVP and ψ, then that leads to $q_\mu = i\hbar\frac{\psi_{,\mu}}{\psi}$. There is a disagreement by a factor of two.

To find out the source of this disagreement, we go back to the geodesic equation in the frame of the electron, we notice that there are two terms. (Both of these terms have derivatives of the metric tensor, and come from $\Gamma_{\mu 00}$). The first term involves a derivative with respect to time, and this corresponds to the inertia term. The second term involves a spatial derivative, has a factor of a half, and this corresponds to a local gravitational field.

We know that local gravitational fields are negligible (by a factor of 10^{40}) in micro physics, so any alteration of the local gravitational field term would not be noticed in micro physics. For example, if we were to double that term in the geodesic equation, micro physics would not notice the difference. Doing so, would eliminate the τ term in the gravitational vector potential. Although doing so, would mean that the gravitational vector potential would no longer be exact, it would also eliminate the conflict between choosing τ or q_μ to make the 1-to-1 relationship between a GVP and ψ. We could choose q_μ to make the 1-to-1 relationship between a GVP and ψ. That would give $q_\mu = i\hbar\frac{\psi_{,\mu}}{\psi}$, and there would no longer be a disagreement by a factor of two.

This allows us to say that an approximate gravitational vector potential is valid if and only if standard quantum theory is valid. (I have not actually proved the second part of that here, but the examples referred to in chapter 27 show why that should be true.)

Does this mean that the Dirac equation is simply a way to guarantee the existence of a gravitational vector potential? When I wondered about that in chapter 27, it was the gradient term (the τ term) in the GVP that reminded me of the Dirac equation and the Schrödinger equation. Now that we have gotten rid of the τ term, the similarity of the GVP with the Dirac equation and the Schrödinger equation is much less.

The above is wrong. Having a solution for π_μ allows at most being able to calculate a WKB approximation for ψ. The above "change of variable" between π_μ and ψ is valid only for plane waves. Still, is it possible that quantum theory is simply an artifact of trying to reconcile a tensor gravitational field with a vector electromagnetic field?

However, in regions where there is a large gravitational field, such as near a black hole, it might not be possible to neglect the gravitational term. In that case, it might be necessary to alter quantum theory to handle that situation.

95.2 Necessity of a wave equation

Does the interaction of a tensor gravitational field with a vector electromagnetic field and the apparent gravitational vector potential require apparent wave properties of particles? To answer that question, I try to see if the above constraints lead to a wave equation.

We begin by considering (95.2), which requires that the quantity $g^{\alpha\beta}\pi_\alpha\pi_\beta$ have no variation in the π_μ direction. That is, that $g^{\alpha\beta}\pi_\alpha\pi_\beta$ remain constant in the π_μ direction. On the trajectory of the particle, we know that constant to be m^2, so we can take it to be that everywhere. That is, we take

$$g^{\alpha\beta}\pi_\alpha\pi_\beta = m^2 \ . \tag{95.6}$$

We know that an equivalent way to guarantee (95.6) to hold for a linear equation is

$$g^{\mu\nu}\gamma_\nu\pi_\mu = \gamma^\mu\pi_\mu = m \ , \tag{95.7}$$

where γ_ν and γ^μ are the Dirac gamma martices in covariant and contravariant form. From (95.1) and (95.4), we have

$$\pi_\mu = q_\mu - eA_\mu + \frac{1}{2m}g^{\alpha\beta}\pi_\alpha\pi_\beta\frac{\partial\tau}{\partial x^\mu} \ . \tag{95.8}$$

Using (95.6) gives

$$\pi_\mu = q_\mu - eA_\mu + \frac{m}{2}\frac{\partial\tau}{\partial x^\mu} \ . \tag{95.9}$$

We now make a change of variable from τ to ψ.

$$\frac{\partial\tau}{\partial x^\mu}\psi = \frac{i}{\omega_c}\frac{\partial\psi}{\partial x^\mu} = \frac{i\hbar}{m}\frac{\partial\psi}{\partial x^\mu} \ , \tag{95.10}$$

where

$$\omega_c \equiv \frac{mc^2}{\hbar} \tag{95.11}$$

is the rest frequency of the particle. We multiply (95.9) by ψ to give

$$\pi_\mu\psi = \left(q_\mu - eA_\mu + \frac{m}{2}\frac{\partial\tau}{\partial x^\mu}\right)\psi \ . \tag{95.12}$$

We substitute (95.10) into (95.12) to give

$$\pi_\mu\psi = \left(q_\mu - eA_\mu + \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu}\right)\psi \ . \tag{95.13}$$

Since (95.7) is valid, it also is valid when multiplied by ψ. That is,

$$g^{\mu\nu}\gamma_\nu\pi_\mu\psi = \gamma^\mu\pi_\mu\psi = m\psi \ . \tag{95.14}$$

Substituting (95.13) into (95.14) gives

$$g^{\mu\nu}\gamma_\nu\left(q_\mu - eA_\mu + \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu}\right)\psi = \gamma^\mu\left(q_\mu - eA_\mu + \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu}\right)\psi = m\psi \ . \tag{95.15}$$

This is not yet the Dirac equation, but we are getting closer. We still have not used one equation, namely (95.3), which we multiply by ψ to give

$$g^{\mu\nu}\pi_\mu\frac{\partial\tau}{\partial x^\nu}\psi = m\psi \ . \tag{95.16}$$

We now substitute (95.10) into (95.16) to give

$$g^{\mu\nu}\pi_\mu \frac{i\hbar}{m}\frac{\partial}{\partial x^\nu}\psi = m\psi \ . \tag{95.17}$$

At this point, I am not sure what to do. I know from comparing with the Dirac equation, how to turn (95.15) into the Dirac equation, but that is cheating. I am here trying to show that the constraints in the gravitational vector potential require a wave equation. It seems that I am almost there, but not quite. I have an additional equation, but I am not sure how to use it.

95.3 Necessity of a wave equation - starting over

The above has a lot of mistakes, so I am starting over. Here is the general idea. We have a scalar field τ and a vector field π_μ, that make up the gravitational vector potential. These fields seem to have a lot of the same properties as a wave function. Is it possible that they are somehow equivalent to a wave function? Maybe, but I need to develop these ideas carefully.

First, I will start by making a change in variable from π_μ to ψ as follows.

$$\psi = e^{-\frac{i}{\hbar}\int(\pi_\mu + eA_\mu)\mathrm{d}x^\mu} e^{\int \Gamma_\mu \mathrm{d}x^\mu} \ , \tag{95.18}$$

where Γ_μ is the spinor connection. This really is just a change of variable. We had a vector function π_μ, and we change to a scalar function ψ. There might be some questions about whether the function is integrable and whether we get a unique, single-valued function. I suspect that those conditions are already satisfied, or we add conditions to make it OK. Equation (95.18) is equivalent to

$$\ln\psi = -\frac{i}{\hbar}\int(\pi_\mu + eA_\mu + i\hbar\Gamma_\mu)\mathrm{d}x^\mu \ , \tag{95.19}$$

or

$$\frac{\partial}{\partial x^\mu}\ln\psi = -\frac{i}{\hbar}(\pi_\mu + eA_\mu + i\hbar\Gamma_\mu) \ , \tag{95.20}$$

or,

$$\pi_\mu = i\hbar\frac{\partial}{\partial x^\mu}\ln\psi - eA_\mu - i\hbar\Gamma_\mu = \frac{i\hbar}{\psi}\frac{\partial}{\partial x^\mu}\psi - eA_\mu - i\hbar\Gamma_\mu \ , \tag{95.21}$$

or,

$$\pi_\mu\psi = i\hbar\frac{\partial}{\partial x^\mu}\psi - eA_\mu - i\hbar\Gamma_\mu\psi = (i\hbar\frac{\partial}{\partial x^\mu} - eA_\mu - i\hbar\Gamma_\mu)\psi \ . \tag{95.22}$$

Or,

$$\pi_\mu\psi = i\hbar\left(\frac{\partial}{\partial x^\mu} - \Gamma_\mu\right)\psi - eA_\mu\psi = \left[i\hbar\left(\frac{\partial}{\partial x^\mu} - \Gamma_\mu\right) - eA_\mu\right]\psi \ , \tag{95.23}$$

where Γ_μ is the spinor connection. I don't think I am cheating here. This really is just a change of variable.

Now it gets slightly more complicated. Since the gravitational vector potential was based on the geodesic equation, it does not contain spin. I think I could have included spin by using precession and geodesic deviation. However, instead, I will make a slight change here. Up until now ψ was just a scalar function, but now I want to include spin, and make ψ a Dirac spinor. I realize that in so doing, the above math steps would no longer be valid, but let us simply take (95.23) to define our change of variable. Now let us take (95.6).

$$g^{\alpha\beta}\pi_\alpha\pi_\beta = m^2 \ . \tag{95.24}$$

Equation (95.24) can be written

$$2g^{\alpha\beta}\mathbf{1}\pi_\alpha\pi_\beta = 2m^2\mathbf{1} \ , \tag{95.25}$$

where **1** is a unit matrix. Equation (95.25) is equivalent to

$$(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha)\pi_\alpha\pi_\beta = 2m^2\mathbf{1} \ , \tag{95.26}$$

where γ^α are Dirac γ matrices. If we now multiply (95.26) by any column vector ψ, we have

$$(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha)\pi_\alpha\pi_\beta\psi = 2m^2\psi \ . \tag{95.27}$$

Equation (95.27) is equivalent to

$$\gamma^\alpha\pi_\alpha\psi = m\psi \ . \tag{95.28}$$

There is one more step to get the Dirac equation. We substitute (95.23) into (95.28) to give

$$\gamma^\mu\left[i\hbar\left(\frac{\partial}{\partial x^\mu} - \Gamma_\mu\right) - eA_\mu\right]\psi = m\psi \ . \tag{95.29}$$

This is the Dirac equation, but is the derivation valid? Is (95.23) just a simple change of variable? I think it is. Let us assume that we have a solution for the vector field π_μ, which we found by solving the geodesic equation (95.30). Then we make the change of variable (95.23).

95.4 Geodesic equation

Suppose I first start with the equation that determines the vector field π_μ,

$$(q_{\mu,\nu} - q_{\nu,\mu})\pi^\nu = 0 \ , \tag{95.30}$$

solve it, and then make a change of variable(95.23).

From chapter 27, we have

$$(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = \frac{1}{m}g_{\mu\alpha}\pi^\alpha{}_{,\nu}\pi^\nu + \frac{1}{2m}\pi^\alpha\pi^\beta(2g_{\mu\alpha}{}_{,\beta} - g_{\alpha\beta}{}_{,\mu}) \ , \tag{95.31}$$

where I have used both constraints. Or,

$$(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = \frac{1}{m}g_{\mu\alpha}\pi^\alpha{}_{,\nu}\pi^\nu + \frac{1}{2m}\pi^\alpha\pi^\beta(g_{\mu\alpha}{}_{,\beta} + g_{\mu\beta}{}_{,\alpha} - g_{\alpha\beta}{}_{,\mu}) \ . \tag{95.32}$$

Or,

$$m(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = g_{\mu\alpha}\pi^\alpha{}_{,\nu}\pi^\nu + \pi^\alpha\pi^\beta\Gamma_{\mu\alpha\beta} \ . \tag{95.33}$$

Or,

$$m(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = g_{\mu\alpha}\pi^\alpha{}_{,\nu}\pi^\nu + \pi^\nu\pi^\beta\Gamma_{\mu\nu\beta} \ . \tag{95.34}$$

Or,

$$m(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = g_{\mu\alpha}\left(\pi^\alpha{}_{,\nu} + \Gamma^\alpha{}_{\nu\beta}\pi^\beta\right)\pi^\nu \ . \tag{95.35}$$

Or,

$$m(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = g_{\mu\alpha}\left(\pi^\alpha{}_{,\nu} + \Gamma^\alpha{}_{\beta\nu}\pi^\beta\right)\pi^\nu \ . \tag{95.36}$$

Or,

$$m(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = g_{\mu\alpha}\pi^\alpha{}_{;\nu}\pi^\nu \ . \tag{95.37}$$

Or,

$$m(g_{\mu,\nu} - g_{\nu,\mu})\pi^\nu = \pi_{\mu;\nu}\pi^\nu \ . \tag{95.38}$$

Finally,

$$(q_{\mu,\nu} - q_{\nu,\mu})\pi^\nu = \left[\pi_{\mu;\nu} + e(A_{\mu;\nu} - A_{\nu;\mu})\right]\pi^\nu = 0 \ . \tag{95.39}$$

Or,

$$(q_{\mu,\nu} - q_{\nu,\mu})\dot{x}^\nu = \pi_{\mu;\nu}\dot{x}^\nu + (A_{\mu;\nu} - A_{\nu;\mu})J^\nu = 0 \ . \tag{95.40}$$

95.5 Klein-Gordon equation

Although an electron has spin, so that the Klein-Gordon equation does not apply to it, I start first with the Klein-Gordon equation because the derivation of the gravitational vector potential did not include spin. Notice that the geodesic equation does not include any effect from spin.

For a free particle, the wave function (un-normalized),when evaluated on the trajectory of the particle, is given by

$$\psi = e^{-i\omega_c\tau} \ , \tag{95.41}$$

where

$$\omega_c \equiv \frac{mc^2}{\hbar} \tag{95.42}$$

is the rest frequency of the particle, and τ is the proper time of the particle. If we generalize τ to be a scalar field, then (95.41) can apply anywhere, not just on the trajectory of the particle. From (95.41), we can write

$$\tau = i\frac{\ln\psi}{\omega_c} \ . \tag{95.43}$$

From (95.1), (95.4), and (95.43), we have

$$\begin{aligned}
\pi_\mu &= mg_\mu + \frac{1}{2m}g^{\alpha\beta}\pi_\alpha\pi_\beta\frac{\partial\tau}{\partial x^\mu} \\
&= q_\mu - eA_\mu + \frac{i}{2m\omega_c\psi}g^{\alpha\beta}\pi_\alpha\pi_\beta\frac{\partial\psi}{\partial x^\mu} \\
&= q_\mu - eA_\mu + \frac{i\hbar}{2m^2\psi}g^{\alpha\beta}\pi_\alpha\pi_\beta\frac{\partial\psi}{\partial x^\mu} \\
&= q_\mu - eA_\mu + \frac{i\hbar}{2\psi}\frac{\partial\psi}{\partial x^\mu} \ ,
\end{aligned} \tag{95.44}$$

where I have used

$$g^{\alpha\beta}\pi_\alpha\pi_\beta = m^2 \ . \tag{95.45}$$

From (95.44), we have

$$\pi_\mu\psi = (q_\mu - eA_\mu + \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})\psi \ . \tag{95.46}$$

Although this is not yet finished, we shall come back to it later.

An interesting calculation of gravitomagnetism in quantum mechanics is given in a 2010 paper by Adler and Chen[286] without spin and in a 2012 paper by Adler, Chen, and Varani[287] with spin.

95.6 Dirac equation

Similarly to (95.41), we have for a free Dirac particle (un-normalized),

$$\psi = ue^{-i\omega_c\tau} \ , \tag{95.47}$$

where u is a column vector, normalized so that $u^\dagger u = 1$, and $u^\dagger$ is a row vector. Multiplying gives

$$u^\dagger\psi = e^{-i\omega_c\tau} \ , \tag{95.48}$$

so that

$$\tau = i\frac{\ln(u^\dagger\psi)}{\omega_c} \ . \tag{95.49}$$

So now we have

$$\pi_\mu(u^\dagger\psi) = (q_\mu - eA_\mu + \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})(u^\dagger\psi) \ . \tag{95.50}$$

For a free particle, $u^\dagger$ and u are constant, so we have

$$\pi_\mu\psi = (q_\mu - eA_\mu + \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})\psi \ , \tag{95.51}$$

which is the correct form in the general case (when the particle is not free).

Equation (95.51) does not lead to Dirac's equation, but we can write (95.51) in the form

$$q_\mu\psi = (\pi_\mu + eA_\mu - \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})\psi = (p_\mu - \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})\psi = (i\hbar\frac{\partial}{\partial x^\mu} - \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})\psi = (\frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})\psi \ . \tag{95.52}$$

Substituting (95.52) into (95.51) gives

$$\pi_\mu\psi = (\frac{i\hbar}{2}\frac{\partial}{\partial x^\mu} - eA_\mu + \frac{i\hbar}{2}\frac{\partial}{\partial x^\mu})\psi = (i\hbar\frac{\partial}{\partial x^\mu} - eA_\mu)\psi \ . \tag{95.53}$$

This last step is actually circular, because (95.53) was used to derive (95.52). Finally, we can write the Dirac equation.

$$g^{\mu\nu}\gamma_\nu\pi_\mu\psi = m\psi \ , \tag{95.54}$$

where $\pi_\mu\psi$ is given by (95.53). Actually, this is correct in flat space-time. In general, there is another term. The actual Dirac equation is (95.54), with $\pi_\mu\psi$ given by

$$\pi_\mu\psi = \left[i\hbar\left(\frac{\partial}{\partial x^\mu} - \Gamma_\mu\right) - eA_\mu\right]\psi \ , \tag{95.55}$$

where Γ_μ is the spinor connection[198, Brill and Wheeler, 1957]. Clearly, I can adjust the formula for q_μ slightly to include the spinor connection.

95.7 Added 11 March 2015

If we have solved the Dirac equation for the wave function, then it is possible to calculate the corresponding gravitational vector potential. For many cases, such as a hydrogen atom, this works fine. However, for the 2-slit diffraction experiment or diffraction by a crystal, for example, it must be applied to each separate segment, rather that the whole wave function, just as in the WKB approximation. In a sense, this gives a mechanism for the deBroglie-Bohm pilot wave theory, but it shows that it does not apply to the whole wave function.

Chapter 96

Inertial frames[1]

abstract

In trying to reconcile General Relativity and quantum theory, it is necessary to consider that inertial frames provided by the gravitational field give the background on which quantum theory is based. We can get insight to that process by considering several definitions of what an inertial frame is. In addition, we get insight into unifying gravitation with the other three interactions by comparing those different definitions with analogous definitions for the other three interactions.

The various definitions of an inertial frame are (1) the frame in which the gravitational field (as defined by the geodesic equations) is zero, (2) the frame in which the gravitational field (as defined by the geodesic equations) excluding the scalar gravitational field of local bodies is zero, (3) the frame in which the gravitational field (as defined by the geodesic equations) excluding the scalar gravitational field of all bodies is zero, and (4) the frame in which the gravitational field (as defined by the geodesic equations) excluding the scalar gravitational field of all bodies and the gravitational field of local bodies is zero.

96.1 Introduction

In trying to reconcile General Relativity and quantum theory, it is necessary to consider that inertial frames provided by the gravitational field give the background on which quantum theory is based.[2] We can get insight to that process by considering several definitions of what an inertial frame is. In addition, we get insight into unifying gravitation with the other three interactions by comparing those different definitions with analogous definitions for the other three interactions.

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Then I discuss each of these in more detail in separate sections.

[1] 13 Jan 2012

[2] In General Relativity, the metric (and therefore the gravitational field and inertial frames) is determined by the distribution of matter (including dark matter) plus initial conditions and boundary conditions. Even for Minkowski space (which is a solution of Einstein's field equations), this is true, although in that case, the metric is determined completely by boundary conditions, since there is no matter. There is the possibility of theories for which this is not true, but I do not think they are seriously considered.

I also need to discuss the possibility of treating the problem in a frame-independent way. This would be like the ideas that Donald Lynden-Bel had where the kinetic energy term was also relative. It might be possibility to do that by choosing a frame with its origin at a chosen particle. Then looking at the results for the distance to each of the other particles. Then we repeat that, taking each particle in turn as the origin. Then we combine everything into a single Lagrangian.

96.2 Frame in which the gravitational field is zero

In General relativity, this frame is referred to as an inertial frame. This is a frame of freely falling particle in the absence of all non-gravitational forces. In this frame, the connection Γ is zero. The geodesic equation can be expressed as $\pi_{\mu;\nu}\pi^{\nu} = 0$.

Quantum theory and EM theory are then built in this frame, and moved to any arbitrary frame.

96.3 Frame in which the gravitational field excluding the scalar gravitational field of local bodies is zero

Before General Relativity, there was an earlier definition of an inertial frame. This was defined as the frame in which Newton's laws (including Newton's law of gravity) works. This is almost this frame, because the gravitational field of distant matter would cancel if the universe is considered homogeneous.

96.4 Frame in which the gravitational field excluding the scalar gravitational field of all bodies is zero

Before General Relativity, there was an earlier definition of an inertial frame. This was defined as the frame in which Newton's laws (including Newton's law of gravity) works. This is this frame, which probably differs not much from the frame described in the previous section.

96.5 Frame in which the gravitational field excluding the scalar gravitational field of all bodies and the gravitational field of local bodies is zero

This is similar to the frame described in the previous section, except that we have excluded frame dragging of local matter.

Then I discuss analogous definitions for each of the other three interactions, starting with EM. I also point out that gravitation is the only interaction for which we have a theory in the absence of all other interactions. The other three have an inertial frame as a background. We do not know what Maxwell's equations would look like, for example, in the absence of gravitation. So $SU(3) \times SU(2) \times U(1)$ is not the real unification of those three interactions. It is something like $SU(3) \times SU(2) \times U(1)$ divided by Minkowski space.

96.6 Electromagnetic theory

Induction in EM gives inertia to electrically charged particles in a charged universe.[12, Sciama, 1953] Therefore, we can get frames analogous to inertial frames just by considering EM interactions.

Since the electromagnetic tensor $F_{\mu\nu}$ is a tensor, if it is zero in one frame, then it will be zero in all frames. Therefore, finding a frame in which the electromagnetic force is zero is not the same as finding a frame in which the electromagnetic field is zero. It is a frame in which the Lorentz force is zero.

96.6.1 Frame in which the electromagnetic force is zero

This is a frame in which the Lorentz force is zero. It is not a frame in which either E or B is necessarily zero.

96.6.2 Frame in which the electromagnetic force excluding the scalar electromagnetic force of local bodies is zero

96.6.3 Frame in which the electromagnetic force excluding the scalar electromagnetic force of all bodies is zero

96.6.4 Frame in which the electromagnetic force excluding the scalar electromagnetic force of all bodies and the electromagnetic force of local bodies is zero

96.7 Weak and strong nuclear interactions

96.8 all interactions

Chapter 97

Fine-grained and coarse-grained histories[1]

abstract

Gell-Mann and Hartle talk about fine-grained histories and coarse-grained histories. As an example, they refer to a fluid described in terms of its molecules as a fine-grained history and a fluid described by the Navier-Stokes equations as a course-grained history.

It seems that Newton's law of gravitation is a fine-grained law. It seems that General Relativity is a coarse-grained law. Or is it? It applies to bodies in the solar system, also. That would make it a fine-grained law. I'm not sure how this works.

97.1 Introduction

Gell-Mann and Hartle talk about fine-grained histories and coarse-grained histories. As an example, they refer to a fluid described in terms of its molecules as a fine-grained history and a fluid described by the Navier-Stokes equations as a course-grained history.

It seems that Newton's law of gravitation is a fine-grained law. It seems that General Relativity is a coarse-grained law. Or is it? It applies to bodies in the solar system, also. That would make it a fine-grained law. I'm not sure how this works.

97.2 The universe as a turbulent fluid

General Relativity treats the universe as a perfect fluid following the perfect gas law. The equations are nonlinear. That gives the possibility of turbulence, with eddies of various sizes. We have large eddies (galaxies), medium eddies (our solar system), and small eddies(Earth and other planets). I can probably use a second-order closure model as I did in those turbulence papers in 1986[258, 259, Jones and Hooke, 1986a,b]. This would work.

I also need to further expand that work I did on the 5×5 nonlinear wave model that I started awhile ago. (See Chapter 98).

[1] 13 Jan 2012

Chapter 98

Nonlinear acoustic-gravity waves in a fluid[1]

abstract

In trying to reconcile General Relativity and quantum theory, it is useful to look at General Relativity as a system for nonlinear fluid dynamics. We already have a model for doing that in the atmosphere, which I do here.

98.1 Introduction

Here, we will be comparing the calculation of the full nonlinear Navier-Stokes equations with a set of linearized equations, as was done in the three papers by Jones [288, 289, 290]. In doing that, we shall still be subtracting a set of background equations, but not requiring that differences are small.

There are various ways for dealing with nonlinearities in wave propagation in fluids. For example, [291, 292, Andrews and McIntyre (1978)] give one method. In the following, I give another.

98.1.1 Gradient correction to the refractive index (Physicist's versus mathematician's point of view) - 1979 or 1980

Mathematician - Solve by method of characteristics. The differential equation for any variable can be used to get the full-wave or WKB solution and the solution for the other variables from it. No matter which variable is chosen as the one to work with, the full-wave solutions will be equivalent. We can conjecture that all WKB approximations will agree within the validity of the WKB approximation. More generally, we can express a set of coupled first-order equations as a matrix equation. To change to a different set of dependent variables, we can do a matrix transformation of those coupled first-order equations. The Hamiltonian is the determinant of the matrix for the coupled equations. It doesn't matter which set of variables we choose.

Physicist - There must be a unique quantity such as energy or phase describing the whole wave that should be used, perhaps certain conserved quantities - Lagrangian phase function, action, Hamiltonian.

[1]Dec 2011 - 14 September 2012

Mathematician again - If we take the full set of equations, including enough to give the solution to all of the variables, is there still something that can be found unique about the whole set of equations to point to a unique set of characteristics? Are there conserved quantities? Are there general properties of the set of equations that we can discover from purely mathematical means without regard to their physical interpretation? Under what conditions do the mathematical and physical uniqueness agree?

Ways to proceed

1. Take the set of differential equations, linearize them, and make transformations to try to find a unique basis in which setting the determinant of the matrix of equations to zero gives a unique dispersion relation. (Hamiltonian equals the determinant of the matrix of equations, Hamilton's formulation)

2. Try to construct a Lagrangian for the wave based on phase. Calculate the action ($= S$). Set the variation of S to zero to define the path. (Consider non-linear equations.)

Physical energy and momentum and energy-momentum tensor (characterizes actual physical energy and momentum transported by the wave) versus canonical energy and momentum and canonical energy-momentum tensor (characterizes phase function):

Let E, P, T = energy, momentum, and energy-momentum tensor. (physical or canonical)

There is an E, P, T for the wave, for the background, and for the whole system. There is also an average E, P, T and an instantaneous E, P, T. (There are many ways to do the averaging.) Consider non-linear equations, try not to use perturbation methods, consider general interactions.

Consider the wave and the medium. If we can neglect the effect of the wave on the propagation characteristics of the medium, then we can treat the medium as a background. Otherwise, we must consider the medium as a dynamical quantity, and consider dynamical interactions between two systems.

It is not always possible to separate the background and the wave.

The whole relationship of the mathematician's point of view and the physical point of view is related to the definition of group velocity.

Mathematical definition (could also call this the canonical definition - from phase interference (gives the wave packet velocity.)

physical definition - propagation velocity of energy.

There is really no reason why the two should agree.

Potential momentum and energy are useful maybe only if the background field is not dynamic (is determined externally).

It is probably just a fluke that the Poynting velocity and the wave packet velocity agree ever.

Is there potential angular momentum?

Given the system of equations fully describing all of the significant variables in the system, can we discover a unique mathematical structure to the system?

Can we write a path integral form for the above equations? We can if we can calculate the infinitesimal particle propagator for the system. See the papers by Hoyle and Narlikar in 1972[35, 103] and the paper on product integrals by Hamilton and Schulman in 1971[293].

98.1.2 Non-linear effects on the uniqueness of the dispersion relation - March 1986

As is well known, when the dispersion relation for acoustic-gravity waves is calculated by normal methods, it is not unique. That is, the dispersion relation one obtains depends on which wave variable (pressure, density, temperature, velocity, etc.) one examines because the differential equation is different for each wave variable. The solutions for the various wave variables are all internally consistent, however.

Because gravity-wave propagation is a characteristic of the wave, and not just of the individual variables, we would expect the dispersion relation to be also a characteristic of the wave. Therefore, we expect that there should be some unique, but not arbitrary, way to specify the dispersion relation.

In spite of recognizing that there should be a unique, correct, but not arbitrary way to specify how the dispersion relation should be calculated, no one has yet demonstrated that he knows the correct criterion to calculate the dispersion relation. In the report I wrote 2 years ago [1984] (NOAA TM ERL WPL-112)[294], I argued that the correct criterion should be that when the differential equations are written as a system of first-order differential equations in matrix form, that the matrices should be Hermitian. Although my arguments supporting that hypothesis were reasonably convincing, they were not definitive. When my criterion led to an effective Brunt-Väisälä frequency different from the usual one, my convictions were not strong enough to try to publish the result in a journal.

The non-uniqueness of the dispersion relation for the linearized inviscid Navier-Stokes equations results from making a linear transformation to a new set of dependent variables in which the transformation depends on position. Such a transformation brings in terms that contain products of the new dependent variables with gradients of the background quantities. Such terms exist only because the equations have been linearized about some background.

The original non-linear equations have no “background.” When we make such transformations, we are, in effect, choosing a different way to separate the “background.” Seen in this light, the non-uniqueness of the dispersion relation appears to be closely associated with the non-linear aspect of the equations, or at least with the linearization process.

If the non-uniqueness is associated with the linearization process, then it is probably necessary to avoid linearization until it is necessary. It may very well be that results depend critically on whether we linearize first (as I did in my report) or transform variables first. If that is so, then it is probably better to transform variables first.

Possibly we can get insight into the non-uniqueness problem if we consider the original, non-linear set of equations. Although the inviscid Navier-Stokes equations are non-linear, they are quasi-linear. That is, they are linear in the derivatives. In addition, there are no terms in the non-linear equations that are not proportional to a derivative.

Abarbanel and Gottlieb (J. Comp. Phys. 41, pp. 1-33, 1981)[295] made transformations on the dependent variables to symmetrize all of the matrix coefficients of the derivative terms in the non-linear equations. However, they did not do it correctly. There are two ways to do it correctly. One way is to make a general transformation. For a homogeneous, quasi-linear system, that corresponds to a linear transformation in which the elements of the transformation are partial derivatives of functions. The other way is to make a linear transformation. In that case, derivatives of the transformation matrix will be generated and must be added to the system. Unfortunately, the transformations considered by Abarbanel and Gottlieb are linear transformations that satisfy neither criterion.

It is probably worth while to make a general transformation on the non-linear system and do it correctly. After the equations have been symmetrized, there are further conditions on the matrix coefficients so that the system will be Hermitian after linearization.

There is another way to use the non-linear aspect of the system of equations to try to get a unique dispersion relation. Although the equations are non-linear, it may be possible to write them in such a way that first-order perturbations in the wave variables introduce only second-order perturbations in the dispersion relation. That would be as close to a true linearization of the system of equations as would be possible.

I don't know if this is possible. To do it might require transforming the independent variables also, maybe even transforming to an accelerated coordinate system. Such a transformation has been done for General Relativity, so it seems that it should be possible to do it for the Navier-Stokes equations.

98.1.3 The uniqueness of the dispersion relation in ray tracing - Dirac equation as an example - March 1986

There are two ways to calculate the dispersion relation for a system of first-order differential equations. The one we are familiar with is to substitute $-ik$ for gradient and a similar substitution for time derivatives and set the determinant of the matrix to zero.

A second method I have seen only on the Dirac equation. The Dirac equation is a system of four first-order differential equations written in matrix form. The equation has only 4 matrices, coefficients of the time and spatial derivatives. The multiplication table for these matrices is such that when the Dirac equation is squared in a particular way the dispersion relation appears automatically.

The advantage of using that method to calculate the dispersion relation is that the dispersion relation depends only on the multiplication table for the matrices, not on the particular matrix representation being used. If the Dirac equation is changed to a different set of dependent variables through a unitary transformation, the matrix representation for the equation will change, but the multiplication table will not change, and therefore, the dispersion relation will not change. Thus, by treating the Dirac equation in terms of the multiplication table of the matrices and not in terms of the matrix representation allows one to derive general results about the dispersion relation that are independent of the representation.

For several years I have wondered whether I could do that for the linearized inviscid Navier-Stokes equations. If I could, then that should help find the unique, correct, but not arbitrary way to specify how the dispersion relation should be calculated. This year I found out how to do that.

When the linearized inviscid Navier-Stokes equations are written for only one spatial dimension, we have a system of three first-order differential equations. If we cube the equation in a particular way, then the dispersion relation is recovered.

For two spatial dimensions, we have a system of four first-order equations. If we take the fourth power of the equation in just the right way, the dispersion relation for two dimensions is recovered. I am confident that raising the 3-dimensional version of the equation to the fifth power would yield the corresponding dispersion relation for three dimensions. (I suspect that raising any system of n first-order equations to the nth power will give the dispersion relation, although I don't know how to prove it. That the system of four first-order equations known collectively as the Dirac equation needs only to be squared to give the dispersion relation probably results from the special symmetry of that equation.)

This discovery does not solve the problem of the non-uniqueness of the dispersion relation; it only makes looking for a solution easier. The kinds of transformations which we have been considering include those in which the transformation has non-zero spatial derivatives. In fact, it is the spatial derivatives of the transformation matrix that demonstrate the difficulty. Thus, the transformations we must consider are not unitary, and thus change the dispersion relation.

Even so, this discovery should make it possible to treat the dispersion relation problem in a more organized way.

Notice that this is a general problem in wave propagation. Once the problem has been solved for the inviscid Navier-Stokes equations, it will be solved for any system of differential equations (at least those that are quasi-linear).

Note added in September 2012: I seem to no longer have the any record of how to cube the equations in a particular way to get the dispersion relation for one dimension or how to take the fourth power of the equations to get the dispersion relation for two dimensions. However, I have found out that I was able to do that only for pure acoustic waves, that is, where we neglected all terms except for derivative terms.

I have, however discovered how to square the equations to get the dispersion relation in 1 dimension. To show this, we start with equation (16) from [288, Jones (2001)].

$$\begin{pmatrix} -\hat{\omega}/C & 2i\frac{\Omega_z}{C} & -2i\frac{\Omega_y}{C} & \frac{i}{\rho C^2}\frac{\partial p}{\partial x} & \hat{k}_x - i\Gamma_x \\ -2i\frac{\Omega_z}{C} & -\hat{\omega}/C & 2i\frac{\Omega_x}{C} & \frac{i}{\rho C^2}\frac{\partial p}{\partial y} & \hat{k}_y - i\Gamma_y \\ 2i\frac{\Omega_y}{C} & -2i\frac{\Omega_x}{C} & -\hat{\omega}/C & \frac{i}{\rho C^2}\frac{\partial p}{\partial z} & \hat{k}_z - i\Gamma_z \\ -\frac{i}{\rho}\frac{\partial \rho_\theta}{\partial x} & -\frac{i}{\rho}\frac{\partial \rho_\theta}{\partial y} & -\frac{i}{\rho}\frac{\partial \rho_\theta}{\partial z} & -\hat{\omega}/C & 0 \\ \hat{k}_x + i\Gamma_x & \hat{k}_y + i\Gamma_y & \hat{k}_z + i\Gamma_z & 0 & -\hat{\omega}/C \end{pmatrix} \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta \rho_\theta}{\rho_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} +$$

$$\begin{pmatrix} -\frac{i}{C}\frac{\partial U_x}{\partial x} & -\frac{i}{C}\frac{\partial U_x}{\partial y} & -\frac{i}{C}\frac{\partial U_x}{\partial z} & 0 & 0 \\ -\frac{i}{C}\frac{\partial U_y}{\partial x} & -\frac{i}{C}\frac{\partial U_y}{\partial y} & -\frac{i}{C}\frac{\partial U_y}{\partial z} & 0 & 0 \\ -\frac{i}{C}\frac{\partial U_z}{\partial x} & -\frac{i}{C}\frac{\partial U_z}{\partial y} & -\frac{i}{C}\frac{\partial U_z}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta \rho_\theta}{\rho_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} +$$

$$\begin{pmatrix} \frac{i}{2\rho C}\frac{D\rho}{Dt} & 0 & 0 & 0 & -\frac{i}{C}\frac{\partial C}{\partial x} \\ 0 & \frac{i}{2\rho C}\frac{D\rho}{Dt} & 0 & 0 & -\frac{i}{C}\frac{\partial C}{\partial y} \\ 0 & 0 & \frac{i}{2\rho C}\frac{D\rho}{Dt} & 0 & -\frac{i}{C}\frac{\partial C}{\partial z} \\ 0 & 0 & 0 & -\frac{i}{2\rho C}\frac{D\rho}{Dt} & 0 \\ 0 & 0 & 0 & \frac{i}{\rho C}\frac{D\rho}{Dt} & \frac{i}{2\rho C}\frac{D\rho}{Dt} + \frac{i}{C^2}\frac{DC}{Dt} \end{pmatrix} \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta \rho_\theta}{\rho_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} = 0\ , \quad (98.1)$$

where I have reversed the order of the matrices, and have made the corrections according to the errata for that paper. For pure acoustic waves, we neglect the latter two matrices to give

$$\begin{pmatrix} -\hat{\omega}/C & 2i\frac{\Omega_z}{C} & -2i\frac{\Omega_y}{C} & \frac{i}{\rho C^2}\frac{\partial p}{\partial x} & \hat{k}_x - i\Gamma_x \\ -2i\frac{\Omega_z}{C} & -\hat{\omega}/C & 2i\frac{\Omega_x}{C} & \frac{i}{\rho C^2}\frac{\partial p}{\partial y} & \hat{k}_y - i\Gamma_y \\ 2i\frac{\Omega_y}{C} & -2i\frac{\Omega_x}{C} & -\hat{\omega}/C & \frac{i}{\rho C^2}\frac{\partial p}{\partial z} & \hat{k}_z - i\Gamma_z \\ -\frac{i}{\rho}\frac{\partial \rho_\theta}{\partial x} & -\frac{i}{\rho}\frac{\partial \rho_\theta}{\partial y} & -\frac{i}{\rho}\frac{\partial \rho_\theta}{\partial z} & -\hat{\omega}/C & 0 \\ \hat{k}_x + i\Gamma_x & \hat{k}_y + i\Gamma_y & \hat{k}_z + i\Gamma_z & 0 & -\hat{\omega}/C \end{pmatrix} \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta \rho_\theta}{\rho_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} = 0\ . \quad (98.2)$$

Further, for pure acoustic waves, we neglect all terms but derivative terms to give

$$\begin{pmatrix} -\hat{\omega}/C & 0 & 0 & 0 & \hat{k}_x \\ 0 & -\hat{\omega}/C & 0 & 0 & \hat{k}_y \\ 0 & 0 & -\hat{\omega}/C & 0 & \hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega}/C & 0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & -\hat{\omega}/C \end{pmatrix} \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta \rho_\theta}{\rho_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} = 0\ . \quad (98.3)$$

In this form, the fourth element of the column vector is now uncoupled from the other variables, so we can disregard it along with the fourth row and fourth column of the matrix to give

$$\begin{pmatrix} -\hat{\omega}/C & 0 & 0 & \hat{k}_x \\ 0 & -\hat{\omega}/C & 0 & \hat{k}_y \\ 0 & 0 & -\hat{\omega}/C & \hat{k}_z \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & -\hat{\omega}/C \end{pmatrix} \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2} \frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2} \frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2} \frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2} \frac{\delta p}{\rho_s C_s C} \end{pmatrix} = 0 \ . \tag{98.4}$$

For one dimension, we neglect the middle two rows and middle two columns to give

$$\begin{pmatrix} -\hat{\omega}/C & \hat{k}_x \\ \hat{k}_x & -\hat{\omega}/C \end{pmatrix} \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2} \frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2} \frac{\delta p}{\rho_s C_s C} \end{pmatrix} = 0 \ . \tag{98.5}$$

Equation (98.5) can be written

$$\left(-\frac{\hat{\omega}}{C} + \alpha \hat{k}_x\right) \psi = 0 \ , \tag{98.6}$$

where

$$\alpha \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{98.7}$$

and

$$\psi \equiv \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2} \frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2} \frac{\delta p}{\rho_s C_s C} \end{pmatrix} \ . \tag{98.8}$$

Multiplying (98.6) by $(\frac{\hat{\omega}}{C} + \alpha \hat{k}_x)$ gives

$$\left(\frac{\hat{\omega}}{C} + \alpha \hat{k}_x\right) \left(-\frac{\hat{\omega}}{C} + \alpha \hat{k}_x\right) \psi = \left[-\left(\frac{\hat{\omega}}{C}\right)^2 + \left(\hat{k}_x\right)^2 + \alpha \left(\frac{\hat{\omega}}{C}\hat{k}_x - \hat{k}_x \frac{\hat{\omega}}{C}\right)\right] \psi = 0 \ . \tag{98.9}$$

since $\alpha^2 = 1$. Normally, the term $\left(\frac{\hat{\omega}}{C}\hat{k}_x - \hat{k}_x \frac{\hat{\omega}}{C}\right)$ in (98.9) is not zero if the fluid velocity depends on x, but since we are considering only pure acoustic waves, we can neglect that term to give

$$\left[-\left(\frac{\hat{\omega}}{C}\right)^2 + \left(\hat{k}_x\right)^2 \right] \psi = 0 \ , \tag{98.10}$$

which gives a dispersion relation in the same way as we get a dispersion relation for the Dirac equation. By the way, this method of getting the dispersion relation for the Dirac equation works only for the free particle, so that is equivalent to neglecting gradient terms here. I am still trying to extend this calculation to two or three dimensions, so far, with no success.

For two dimensions, we have

$$\left(-\frac{\hat{\omega}}{C} + \alpha \hat{k}_x + \beta \hat{k}_y\right) \psi = 0 \ , \tag{98.11}$$

where

$$\alpha \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \tag{98.12}$$

$$\beta \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \tag{98.13}$$

and

$$\psi \equiv \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} . \tag{98.14}$$

For three dimensions, we have

$$\left(-\frac{\hat{\omega}}{C} + \alpha\hat{k}_x + \beta\hat{k}_y + \gamma\hat{k}_z\right)\psi = 0 , \tag{98.15}$$

where

$$\alpha \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \tag{98.16}$$

$$\beta \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \tag{98.17}$$

$$\gamma \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{98.18}$$

and

$$\psi \equiv \begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} . \tag{98.19}$$

Multiplying (98.4) or (98.15) on the left by

$$\begin{pmatrix} \left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}_y^2 - \hat{k}_z^2 & \hat{k}_x\hat{k}_y & \hat{k}_x\hat{k}_z & 2\hat{k}_x\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_x \\ \hat{k}_y\hat{k}_x & \left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}_x^2 - \hat{k}_z^2 & \hat{k}_y\hat{k}_z & 2\hat{k}_y\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_y \\ \hat{k}_z\hat{k}_x & \hat{k}_z\hat{k}_y & \left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}_x^2 - \hat{k}_y^2 & 2\hat{k}_z\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_z \\ 2\frac{\hat{\omega}}{C}\hat{k}_x - \hat{k}_x\frac{\hat{\omega}}{C} & 2\frac{\hat{\omega}}{C}\hat{k}_y - \hat{k}_y\frac{\hat{\omega}}{C} & 2\frac{\hat{\omega}}{C}\hat{k}_z - \hat{k}_z\frac{\hat{\omega}}{C} & \left(\frac{\hat{\omega}}{C}\right)^2 \end{pmatrix} \tag{98.20}$$

gives

$$\begin{pmatrix} \left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}_y^2 - \hat{k}_z^2 & \hat{k}_x\hat{k}_y & \hat{k}_x\hat{k}_z & 2\hat{k}_x\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_x \\ \hat{k}_y\hat{k}_x & \left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}_x^2 - \hat{k}_z^2 & \hat{k}_y\hat{k}_z & 2\hat{k}_y\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_y \\ \hat{k}_z\hat{k}_x & \hat{k}_z\hat{k}_y & \left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}_x^2 - \hat{k}_y^2 & 2\hat{k}_z\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_z \\ 2\frac{\hat{\omega}}{C}\hat{k}_x - \hat{k}_x\frac{\hat{\omega}}{C} & 2\frac{\hat{\omega}}{C}\hat{k}_y - \hat{k}_y\frac{\hat{\omega}}{C} & 2\frac{\hat{\omega}}{C}\hat{k}_z - \hat{k}_z\frac{\hat{\omega}}{C} & \left(\frac{\hat{\omega}}{C}\right)^2 \end{pmatrix} \begin{pmatrix} -\frac{\hat{\omega}}{C} & 0 & 0 & \hat{k}_x \\ 0 & -\frac{\hat{\omega}}{C} & 0 & \hat{k}_y \\ 0 & 0 & -\frac{\hat{\omega}}{C} & \hat{k}_z \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & -\frac{\hat{\omega}}{C} \end{pmatrix} \psi$$

$$= -\left(\left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}^2\right)\frac{\hat{\omega}}{C}\psi$$

$$+$$

$$\begin{pmatrix} [[\hat{k}_x,\frac{\hat{\omega}}{C}],\hat{k}_x] & [\hat{k}_x,\frac{\hat{\omega}}{C}]\hat{k}_y - \hat{k}_x[\hat{k}_y,\frac{\hat{\omega}}{C}] & [\hat{k}_x,\frac{\hat{\omega}}{C}]\hat{k}_z - \hat{k}_x[\hat{k}_z,\frac{\hat{\omega}}{C}] & -[\hat{k}_x,\left(\frac{\hat{\omega}}{C}\right)^2] - [\hat{k}_x,\frac{\hat{\omega}}{C}]\frac{\hat{\omega}}{C} \\ [\hat{k}_y,\frac{\hat{\omega}}{C}]\hat{k}_x - \hat{k}_y[\hat{k}_x,\frac{\hat{\omega}}{C}] & [[\hat{k}_y,\frac{\hat{\omega}}{C}],\hat{k}_y] & [\hat{k}_y,\frac{\hat{\omega}}{C}]\hat{k}_z - \hat{k}_y[\hat{k}_z,\frac{\hat{\omega}}{C}] & -[\hat{k}_y,\left(\frac{\hat{\omega}}{C}\right)^2] - [\hat{k}_y,\frac{\hat{\omega}}{C}]\frac{\hat{\omega}}{C} \\ [\hat{k}_z,\frac{\hat{\omega}}{C}]\hat{k}_x - \hat{k}_z[\hat{k}_x,\frac{\hat{\omega}}{C}] & [\hat{k}_z,\frac{\hat{\omega}}{C}]\hat{k}_y - \hat{k}_z[\hat{k}_y,\frac{\hat{\omega}}{C}] & [[\hat{k}_z,\frac{\hat{\omega}}{C}],\hat{k}_z] & -[\hat{k}_z,\left(\frac{\hat{\omega}}{C}\right)^2] - [\hat{k}_z,\frac{\hat{\omega}}{C}]\frac{\hat{\omega}}{C} \\ [[\hat{k}_x,\frac{\hat{\omega}}{C}],\frac{\hat{\omega}}{C}] & [[\hat{k}_y,\frac{\hat{\omega}}{C}],\frac{\hat{\omega}}{C}] & [[\hat{k}_z,\frac{\hat{\omega}}{C}],\frac{\hat{\omega}}{C}] & -[\hat{k}^2,\frac{\hat{\omega}}{C}] - [\hat{k}_x,\frac{\hat{\omega}}{C}]\hat{k}_x - [\hat{k}_y,\frac{\hat{\omega}}{C}]\hat{k}_y - [\hat{k}_z,\frac{\hat{\omega}}{C}]\hat{k}_z \end{pmatrix}\psi$$
$$= 0 \ , \tag{98.21}$$

where the commutators are given by

$$[\hat{k}_x,\frac{\hat{\omega}}{C}] \equiv \hat{k}_x\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_x = -i\frac{\partial}{\partial x}\left(\frac{1}{C}\right)\hat{\omega} + \frac{i}{C}\frac{\partial U_x}{\partial x}\hat{k}_x + \frac{i}{C}\frac{\partial U_y}{\partial x}\hat{k}_y + \frac{i}{C}\frac{\partial U_z}{\partial x}\hat{k}_z \ , \tag{98.22}$$

$$[\hat{k}_y,\frac{\hat{\omega}}{C}] \equiv \hat{k}_y\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_y = -i\frac{\partial}{\partial y}\left(\frac{1}{C}\right)\hat{\omega} + \frac{i}{C}\frac{\partial U_x}{\partial y}\hat{k}_x + \frac{i}{C}\frac{\partial U_y}{\partial y}\hat{k}_y + \frac{i}{C}\frac{\partial U_z}{\partial y}\hat{k}_z \ , \tag{98.23}$$

$$[\hat{k}_z,\frac{\hat{\omega}}{C}] \equiv \hat{k}_z\frac{\hat{\omega}}{C} - \frac{\hat{\omega}}{C}\hat{k}_z = -i\frac{\partial}{\partial z}\left(\frac{1}{C}\right)\hat{\omega} + \frac{i}{C}\frac{\partial U_x}{\partial z}\hat{k}_x + \frac{i}{C}\frac{\partial U_y}{\partial z}\hat{k}_y + \frac{i}{C}\frac{\partial U_z}{\partial z}\hat{k}_z \ . \tag{98.24}$$

If we consider the case where the sound speed C and fluid velocity $\mathbf{U}$ are independent of position and time, then all of the commutators will be zero, so that (98.21) becomes

$$-\left(\left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}^2\right)\frac{\hat{\omega}}{C}\psi = 0 \ . \tag{98.25}$$

On the other hand, another interpretation for neglecting the commutators is that it gives the dispersion relation we would have gotten simply by taking the determinant of the matrix in (98.4).

We can write the multiplier matrix in (98.20) as

$$\left(\frac{\hat{\omega}}{C} + \alpha\hat{k}_x + \beta\hat{k}_y + \gamma\hat{k}_z\right)^2 - \hat{k}^2 - \frac{1}{2}\alpha\left\{\hat{k}_x,\frac{\hat{\omega}}{C}\right\} - \frac{1}{2}\beta\left\{\hat{k}_y,\frac{\hat{\omega}}{C}\right\} - \frac{1}{2}\gamma\left\{\hat{k}_z,\frac{\hat{\omega}}{C}\right\}$$
$$+\frac{3}{2}\alpha_A\left[\hat{k}_x,\frac{\hat{\omega}}{C}\right] + \frac{3}{2}\beta_A\left[\hat{k}_y,\frac{\hat{\omega}}{C}\right] + \frac{3}{2}\gamma_A\left[\hat{k}_z,\frac{\hat{\omega}}{C}\right] \ , \tag{98.26}$$

where $\hat{k}^2 \equiv \hat{k}_x^2 + \hat{k}_y^2 + \hat{k}_z^2$,

$$\alpha_A \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \ , \tag{98.27}$$

$$\beta_A \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \ , \tag{98.28}$$

$$\gamma_A \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \tag{98.29}$$

and the anticommutators are defined as

$$\{\hat{k}_x,\frac{\hat{\omega}}{C}\} \equiv \hat{k}_x\frac{\hat{\omega}}{C} + \frac{\hat{\omega}}{C}\hat{k}_x \ , \tag{98.30}$$

$$\{\hat{k}_y, \frac{\hat{\omega}}{C}\} \equiv \hat{k}_y \frac{\hat{\omega}}{C} + \frac{\hat{\omega}}{C}\hat{k}_y \ , \tag{98.31}$$

$$\{\hat{k}_z, \frac{\hat{\omega}}{C}\} \equiv \hat{k}_z \frac{\hat{\omega}}{C} + \frac{\hat{\omega}}{C}\hat{k}_z \ . \tag{98.32}$$

Neglecting the commutators, we multiply (98.15) on the left by (98.26) to give

$$\begin{aligned}&\left[\left(\frac{\hat{\omega}}{C} + \alpha\hat{k}_x + \beta\hat{k}_y + \gamma\hat{k}_z\right)^2 - \hat{k}^2 - \frac{1}{2}\alpha\left\{\hat{k}_x, \frac{\hat{\omega}}{C}\right\} - \frac{1}{2}\beta\left\{\hat{k}_y, \frac{\hat{\omega}}{C}\right\} - \frac{1}{2}\gamma\left\{\hat{k}_z, \frac{\hat{\omega}}{C}\right\}\right]\left(\frac{-\hat{\omega}}{C} + \alpha\hat{k}_x + \beta\hat{k}_y + \gamma\hat{k}_z\right)\psi \\ &= -\left(\left(\frac{\hat{\omega}}{C}\right)^2 - \hat{k}^2\right)\frac{\hat{\omega}}{C}\psi = 0 \ ,\end{aligned} \tag{98.33}$$

where I have used the following identities

$$\alpha^3 - \alpha = 0 \ , \tag{98.34}$$

$$\beta^3 - \beta = 0 \ , \tag{98.35}$$

$$\gamma^3 - \gamma = 0 \ , \tag{98.36}$$

$$\alpha^2\beta + \{\alpha, \beta\}\alpha - \beta = 0 \ , \tag{98.37}$$

$$\alpha^2\gamma + \{\alpha, \gamma\}\alpha - \gamma = 0 \ , \tag{98.38}$$

$$\beta^2\alpha + \{\alpha, \beta\}\beta - \alpha = 0 \ , \tag{98.39}$$

$$\beta^2\gamma + \{\beta, \gamma\}\beta - \gamma = 0 \ , \tag{98.40}$$

$$\gamma^2\alpha + \{\alpha, \gamma\}\gamma - \alpha = 0 \ , \tag{98.41}$$

$$\gamma^2\beta + \{\gamma, \beta\}\gamma - \beta = 0 \ , \tag{98.42}$$

and

$$\{\alpha, \beta\}\gamma = \{\gamma, \alpha\}\beta = \{\gamma, \beta\}\alpha = 0 \ . \tag{98.43}$$

Can we do the same for the full Navier-Stokes equations (98.1)? I am reasonably sure we can, at least if we neglect the third matrix in that equation, because it is equivalent to calculating the determinant of the sum of the first two matrices, which I have already done. Let us here write down that equation, neglecting the third matrix.

$$\begin{aligned}&\begin{pmatrix} -\hat{\omega}/C & 2i\frac{\Omega_z}{C} & -2i\frac{\Omega_y}{C} & \frac{i}{C^2}\tilde{g}_x & \hat{k}_x - i\Gamma_x \\ -2i\frac{\Omega_z}{C} & -\hat{\omega}/C & 2i\frac{\Omega_x}{C} & \frac{i}{C^2}\tilde{g}_y & \hat{k}_y - i\Gamma_y \\ 2i\frac{\Omega_y}{C} & -2i\frac{\Omega_x}{C} & -\hat{\omega}/C & \frac{i}{C^2}\tilde{g}_z & \hat{k}_z - i\Gamma_z \\ -\frac{i}{\rho}\frac{\partial\rho_\theta}{\partial x} & -\frac{i}{\rho}\frac{\partial\rho_\theta}{\partial y} & -\frac{i}{\rho}\frac{\partial\rho_\theta}{\partial z} & -\hat{\omega}/C & 0 \\ \hat{k}_x + i\Gamma_x & \hat{k}_y + i\Gamma_y & \hat{k}_z + i\Gamma_z & 0 & -\hat{\omega}/C \end{pmatrix}\begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta\rho_\theta}{\rho_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} + \\ &\begin{pmatrix} -\frac{i}{C}\frac{\partial U_x}{\partial x} & -\frac{i}{C}\frac{\partial U_x}{\partial y} & -\frac{i}{C}\frac{\partial U_x}{\partial z} & 0 & 0 \\ -\frac{i}{C}\frac{\partial U_y}{\partial x} & -\frac{i}{C}\frac{\partial U_y}{\partial y} & -\frac{i}{C}\frac{\partial U_y}{\partial z} & 0 & 0 \\ -\frac{i}{C}\frac{\partial U_z}{\partial x} & -\frac{i}{C}\frac{\partial U_z}{\partial y} & -\frac{i}{C}\frac{\partial U_z}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_x}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_y}{C_s} \\ (\frac{\rho}{\rho_s})^{1/2}\frac{\delta U_z}{C_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta\rho_\theta}{\rho_s} \\ (\frac{\rho}{\rho_s})^{-1/2}\frac{\delta p}{\rho_s C_s C} \end{pmatrix} = 0 \ .\end{aligned} \tag{98.44}$$

It is also looking like I should use these equations for my nonlinear calculations instead of the equations I did use, because these work better with the α, β, and γ matrices.

98.2 Navier-Stokes equations

We start with the momentum equations

$$\frac{D\mathbf{U}}{Dt} + \frac{\nabla p}{\rho} - \mathbf{g} + 2\mathbf{\Omega} \times \mathbf{U} = 0 \ , \tag{98.45}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \tag{98.46}$$

is the intrinsic derivative (the derivative with respect to a frame moving with the fluid), $\mathbf{U}$ is the fluid velocity, p is pressure, ρ is density, $\mathbf{g}$ is the acceleration due to the Earth's gravitational field, and $\mathbf{\Omega}$ is the Earth's angular velocity.

We consider a general equation of state

$$p = p(\rho, S, s) \ , \tag{98.47}$$

where S is entropy and s is fluid composition.

Taking the intrinsic derivative of (98.47) gives

$$\frac{Dp}{Dt} = \left(\frac{\partial p}{\partial \rho}\right)_{S,s} \frac{D\rho}{Dt} + \left(\frac{\partial p}{\partial S}\right)_{\rho,s} \frac{DS}{Dt} + \left(\frac{\partial p}{\partial s}\right)_{\rho,S} \frac{Ds}{Dt} \ . \tag{98.48}$$

Because parcel motion is an adiabatic process,

$$\frac{DS}{Dt} = 0 \ , \tag{98.49}$$

and because the composition remains constant for a parcel motion,

$$\frac{Ds}{Dt} = 0 \ , \tag{98.50}$$

and because [207, Weinberg, 1962]

$$C^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S,s} \tag{98.51}$$

is the square of the sound speed at the place where the pressure is p, we can write (98.48) as

$$\frac{Dp}{Dt} = C^2 \frac{D\rho}{Dt} \ , \tag{98.52}$$

or,

$$\frac{D\rho}{Dt} - \frac{1}{C^2}\frac{Dp}{Dt} = 0 \ . \tag{98.53}$$

Equation (98.53) is one form of the adiabatic equation of state, in that it gives a relationship between density ρ and presure p of a parcel under an adiabatic change.

Potential density of a parcel of fluid is defined as the density that parcel of fluid would have if brought to some standard pressure, say p_a, keeping the entropy constant. (The composition of the parcel will, of course, remain unchanged as well.) Thus, we have

$$p_a = p(\rho_{\mathrm{pot}}, S, s) \ . \tag{98.54}$$

The pressure p_a in (98.54) is a constant, since it is the chosen standard pressure. Normally, it is taken to be the pressure at sea level. We could take it to be at sea level here, and we would eventually get the same results for the dispersion relation, but with more effort. It can be chosen to be the pressure anywhere arbitrarily.

Here, I shall choose the standard pressure p_a to be the same as the background pressure p_0 at the point where calculations of the dispersion relation are being made. To avoid confusion with the usual sea-level definition of potential density, we use $\tilde{\rho}_{\text{pot}}$ as the notation for our local definition of potential density from here on. To reflect these choices, we rewrite (98.54) as

$$p_a = p(\tilde{\rho}_{\text{pot}}, S, s) \ . \tag{98.55}$$

and

$$p_a = p_0 = p(\rho_0, S_0, s_0) = p(\tilde{\rho}_{\text{pot}0}, S_0, s_0) \ . \tag{98.56}$$

Taking the intrinsic derivative of (98.55) gives

$$\frac{Dp_a}{Dt} = 0 = \left(\frac{\partial p}{\partial \rho}\right)_{S,s} \frac{D\tilde{\rho}_{\text{pot}}}{Dt} + \left(\frac{\partial p}{\partial S}\right)_{\rho,s} \frac{DS}{Dt} + \left(\frac{\partial p}{\partial s}\right)_{\rho,S} \frac{Ds}{Dt} \tag{98.57}$$

Using (98.49) and (98.50) in (98.57) gives

$$\frac{D\tilde{\rho}_{\text{pot}}}{Dt} = 0 \ . \tag{98.58}$$

Equation (98.58) is another version of the equation of adiabatic state. It shows that the potential density $\tilde{\rho}_{\text{pot}}$ does not change during an adiabatic change.

Taking the intrinsic derivative of (98.56) gives

$$\frac{Dp_a}{Dt} = 0 = \left(\frac{\partial p_0}{\partial \rho}\right)_{S,s} \frac{D\tilde{\rho}_{\text{pot}0}}{Dt} + \left(\frac{\partial p_0}{\partial S}\right)_{\rho,s} \frac{DS_0}{Dt} + \left(\frac{\partial p_0}{\partial s}\right)_{\rho,S} \frac{Ds_0}{Dt} \ , \tag{98.59}$$

which requires that

$$\frac{D\tilde{\rho}_{\text{pot}0}}{Dt} = 0 \ , \tag{98.60}$$

because (98.49) and (98.50) apply to the background as well as the total. We need to find a formula for $\tilde{\rho}_{\text{pot}0}$. The equation of state for the background is

$$p_0 = p(\rho_0, S_0, s_0) \ . \tag{98.61}$$

The standard pressure is

$$p_a = p(\tilde{\rho}_{\text{pot}0}, S_0, s_0) \ . \tag{98.62}$$

Taking the gradient of (98.61) minus the gradient of (98.62) has some cancellations to give

$$\nabla p_0 = C_0^2 (\nabla \rho_0 - \nabla \tilde{\rho}_{\text{pot}0}) \ , \tag{98.63}$$

where[207, Weinberg, 1962]

$$C_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S,s}\Bigg|_{p=p_0} = \left(\frac{\partial p_0}{\partial \rho}\right)_{S,s} \tag{98.64}$$

is the square of the sound speed at the place where the pressure is p_0.

Equation (98.63) can be written

$$\nabla \tilde{\rho}_{\text{pot}0} = \nabla \rho_0 - \frac{1}{C_0^2} \nabla p_0 \ . \tag{98.65}$$

Equation (98.65) gives a formula for the potential density for the background.

The continuity equation is

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{U} = 0\ . \tag{98.66}$$

Taking the difference of (98.66) and (98.53), gives

$$\frac{1}{C^2}\frac{Dp}{Dt} + \rho\nabla\cdot\mathbf{U} = 0\ . \tag{98.67}$$

Equations (98.45), (98.58), (98.50), and (98.67) define the system of equations.

98.3 Matrix form of the equations[2]

We can write (98.45), (98.58), (98.50), and (98.67) as a matrix equation as

$$\mathbf{M}\begin{pmatrix} U \\ V \\ W \\ \tilde{\rho}\text{pot} \\ s \\ p \end{pmatrix} + i\begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\ , \tag{98.68}$$

where $\mathbf{M}$ is the 6×6 matrix given by

$$\mathbf{M} = \begin{pmatrix} -\hat{\omega} & 2i\Omega_z & -2i\Omega_y & 0 & 0 & \rho^{-1}\hat{k}_x \\ -2i\Omega_z & -\hat{\omega} & 2i\Omega_x & 0 & 0 & \rho^{-1}\hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega} & 0 & 0 & \rho^{-1}\hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega} & 0 \\ \rho\hat{k}_x & \rho\hat{k}_y & \rho\hat{k}_z & 0 & 0 & -C^{-2}\hat{\omega} \end{pmatrix}\ , \tag{98.69}$$

where $\hat{\omega} \equiv iD/Dt \equiv i\partial/\partial t + i\mathbf{U}\cdot\nabla$ and $\hat{\mathbf{k}} \equiv -i\nabla$. The first three rows in (98.68) are equivalent to (98.45), the fourth row is equivalent to (98.58), the fifth row is equivalent to (98.50), and the sixth row is equivalent to (98.67). Often, fluid composition is represented by several parameters. In that case, the fifth row in (98.68) and the fifth row and fifth column in (98.69) would be expanded accordingly. Equation (98.68) is still fully nonlinear, because there are terms in the matrix M in (98.69) that depend on fluid variables other than just the background.

The purpose here is to put the nonlinear Navier-Stokes equations in a form that is analogous to the usual linearized equations that are used for a wave equation. To do that, we need to subtract a background, so that we are looking at perturbation equations without assuming that the perturbations are small. We start with the same equations as above, but for the background. That is, we have

$$\mathbf{M_0}\begin{pmatrix} U_0 \\ V_0 \\ W_0 \\ \tilde{\rho}\text{pot0} \\ s_0 \\ p_0 \end{pmatrix} + i\begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\ , \tag{98.70}$$

[2] mostly written in 2006, finished in 2012

where $\mathbf{M}_0$ is the 6×6 matrix given by

$$\mathbf{M_0} = \begin{pmatrix} -\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & {\rho_0}^{-1}\hat{k}_x \\ -2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & {\rho_0}^{-1}\hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & {\rho_0}^{-1}\hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\ \rho_0\hat{k}_x & \rho_0\hat{k}_y & \rho_0\hat{k}_z & 0 & 0 & -C_0^{-2}\hat{\omega}_0 \end{pmatrix}, \tag{98.71}$$

where $\hat{\omega}_0 \equiv iD_0/D_0t \equiv i\partial/\partial t + i\mathbf{U_0} \cdot \nabla$.

There is another possibility for the matrix form that allows all elements of the matrix to have the same units. This form is illustrated in[288, Jones, 2001, equation (16)] . We shall pursue that below.

98.4 Alternative matrix form of the equations

We have a choice of whether to first subtract the background equations from the non-background equations (represented as matrix equations) and then scale the dependent variables by a factor of $\rho_0^{\pm\frac{1}{2}}$, or reverse the order of that procedure. That is, we can first scale the dependent variables by a factor of $\rho^{\pm\frac{1}{2}}$ or $\rho_0^{\pm\frac{1}{2}}$ and then subtract the background equations from the non-background equations. We shall do that in this section.

We first scale the dependent variables in (98.68) by $\rho^{\pm\frac{1}{2}}$. We also multiply the first three equations (rows) of (98.68) by $\rho^{\frac{1}{2}}$ and the last three equations (rows) of (98.68) by $\rho^{-\frac{1}{2}}$. This gives

$$\begin{pmatrix} -\hat{\omega} & 2i\Omega_z & -2i\Omega_y & 0 & 0 & \hat{k}_x \\ -2i\Omega_z & -\hat{\omega} & 2i\Omega_x & 0 & 0 & \hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega} & 0 & 0 & \hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega} & 0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & 0 & -C^{-2}\hat{\omega} \end{pmatrix} \begin{pmatrix} \rho^{1/2}U \\ \rho^{1/2}V \\ \rho^{1/2}W \\ \rho^{-1/2}\tilde{\rho}_{\rm pot} \\ \rho^{-1/2}s \\ \rho^{-1/2}p \end{pmatrix} +$$

$$\begin{pmatrix} \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial x} \\ 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial y} \\ 0 & 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial z} \\ 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 \\ \frac{i}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial x} & \frac{i}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial y} & \frac{i}{2}\frac{1}{\rho}\frac{\partial\rho}{\partial z} & 0 & 0 & -C^{-2}\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} \end{pmatrix} \begin{pmatrix} \rho^{1/2}U \\ \rho^{1/2}V \\ \rho^{1/2}W \\ \rho^{-1/2}\tilde{\rho}_{\rm pot} \\ \rho^{-1/2}s \\ \rho^{-1/2}p \end{pmatrix}$$

$$+i\rho^{\frac{1}{2}} \begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\ . \tag{98.72}$$

Next, we scale the dependent variables in (98.70) by $\rho_0^{\pm\frac{1}{2}}$. We also multiply the first three equations (rows) of (98.70) by $\rho_0^{\frac{1}{2}}$ and the last three equations (rows) of (98.70) by $\rho_0^{-\frac{1}{2}}$. This gives

$$\begin{pmatrix} -\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & \hat{k}_x \\ -2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & \hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & \hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & 0 & -C_0^{-2}\hat{\omega}_0 \end{pmatrix} \begin{pmatrix} \rho_0^{1/2}U_0 \\ \rho_0^{1/2}V_0 \\ \rho_0^{1/2}W_0 \\ \rho_0^{-1/2}\tilde{\rho}_{\text{pot}0} \\ \rho_0^{-1/2}s_0 \\ \rho_0^{-1/2}p_0 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} \\ 0 & \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} \\ 0 & 0 & \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} \\ 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 \\ \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} & 0 & 0 & -C_0^{-2}\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} \end{pmatrix} \begin{pmatrix} \rho_0^{1/2}U_0 \\ \rho_0^{1/2}V_0 \\ \rho_0^{1/2}W_0 \\ \rho_0^{-1/2}\tilde{\rho}_{\text{pot}0} \\ \rho_0^{-1/2}s_0 \\ \rho_0^{-1/2}p_0 \end{pmatrix}$$

$$+i\rho_0^{\frac{1}{2}} \begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \; . \tag{98.73}$$

One advantage to performing the scaling before subtracting is that the matrices in both (98.72) and (98.73) are symmetric in the derivative operators $\hat{\omega}$ or $\hat{\omega}_0$ and $\hat{\mathbf{k}}$, exhibiting that the equations are symmetric hyperbolic, which implies that Cauchy data will be propagated causally[41, 296, Courant and Hilbert, 1962; Garabedian, 1964]. A disadvantage is that the difference in the two equations will have a term that involves the gravitational field $\mathbf{g}$. Another disadvantage may be that subtracting the equations is really complicated.

Therefore, we do not pursue this method further. Actually, there is a possibility, which we pursue below. We change the last element in the column vector in (98.72) by a factor of C and we multiply the last of the 5 equations (bottom row of the matrix) by a factor of C. This gives

$$\begin{pmatrix} -\hat{\omega} & 2i\Omega_z & -2i\Omega_y & 0 & 0 & C\hat{k}_x \\ -2i\Omega_z & -\hat{\omega} & 2i\Omega_x & 0 & 0 & C\hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega} & 0 & 0 & C\hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega} & 0 \\ C\hat{k}_x & C\hat{k}_y & C\hat{k}_z & 0 & 0 & -\hat{\omega} \end{pmatrix} \begin{pmatrix} \rho^{1/2}U \\ \rho^{1/2}V \\ \rho^{1/2}W \\ \rho^{-1/2}\tilde{\rho}_{\text{pot}} \\ \rho^{-1/2}s \\ \rho^{-1/2}p/C \end{pmatrix} +$$

$$\begin{pmatrix} \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial x} - i\frac{\partial C}{\partial x} \\ 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial y} - i\frac{\partial C}{\partial y} \\ 0 & 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial z} - i\frac{\partial C}{\partial z} \\ 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 \\ \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial x} & \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial y} & \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial z} & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} - \frac{i}{C}\frac{DC}{Dt} \end{pmatrix} \begin{pmatrix} \rho^{1/2}U \\ \rho^{1/2}V \\ \rho^{1/2}W \\ \rho^{-1/2}\tilde{\rho}_{\text{pot}} \\ \rho^{-1/2}s \\ \rho^{-1/2}p/C \end{pmatrix}$$

$$
+i\rho^{\frac{1}{2}}\begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\ . \tag{98.74}
$$

Now, to solve the problem mentioned earlier, we multiply everything by $\rho^{-1/2}$. This gives

$$
\rho^{-1/2}\begin{pmatrix} -\hat{\omega} & 2i\Omega_z & -2i\Omega_y & 0 & 0 & C\hat{k}_x \\ -2i\Omega_z & -\hat{\omega} & 2i\Omega_x & 0 & 0 & C\hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega} & 0 & 0 & C\hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega} & 0 \\ C\hat{k}_x & C\hat{k}_y & C\hat{k}_z & 0 & 0 & -\hat{\omega} \end{pmatrix}\begin{pmatrix} \rho^{1/2}U \\ \rho^{1/2}V \\ \rho^{1/2}W \\ \rho^{-1/2}\tilde{\rho}_{\rm pot} \\ \rho^{-1/2}s \\ \rho^{-1/2}p/C \end{pmatrix} +
$$

$$
\rho^{-1/2}\begin{pmatrix} \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial x} - i\frac{\partial C}{\partial x} \\ 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial y} - i\frac{\partial C}{\partial y} \\ 0 & 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial z} - i\frac{\partial C}{\partial z} \\ 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 \\ \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial x} & \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial y} & \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial z} & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} - \frac{i}{C}\frac{DC}{Dt} \end{pmatrix}\begin{pmatrix} \rho^{1/2}U \\ \rho^{1/2}V \\ \rho^{1/2}W \\ \rho^{-1/2}\tilde{\rho}_{\rm pot} \\ \rho^{-1/2}s \\ \rho^{-1/2}p/C \end{pmatrix}
$$

$$
+i\begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\ . \tag{98.75}
$$

We now write (98.75) as

$$
\mathbf{M}\begin{pmatrix} \rho^{1/2}U \\ \rho^{1/2}V \\ \rho^{1/2}W \\ \rho^{-1/2}\tilde{\rho}_{\rm pot} \\ \rho^{-1/2}s \\ \rho^{-1/2}p/C \end{pmatrix} + i\begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\ , \tag{98.76}
$$

where $\mathbf{M}$ is the 6×6 matrix given by

$$
\mathbf{M} \ = \ \rho^{-1/2}\begin{pmatrix} -\hat{\omega} & 2i\Omega_z & -2i\Omega_y & 0 & 0 & C\hat{k}_x \\ -2i\Omega_z & -\hat{\omega} & 2i\Omega_x & 0 & 0 & C\hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega} & 0 & 0 & C\hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega} & 0 \\ C\hat{k}_x & C\hat{k}_y & C\hat{k}_z & 0 & 0 & -\hat{\omega} \end{pmatrix}
$$

$$
+ \quad \rho^{-1/2} \begin{pmatrix} \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial x} - i\frac{\partial C}{\partial x} \\ 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial y} - i\frac{\partial C}{\partial y} \\ 0 & 0 & \frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 & -\frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial z} - i\frac{\partial C}{\partial z} \\ 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} & 0 \\ \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial x} & \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial y} & \frac{i}{2}C\frac{1}{\rho}\frac{\partial\rho}{\partial z} & 0 & 0 & -\frac{i}{2}\frac{1}{\rho}\frac{D\rho}{Dt} - \frac{i}{C}\frac{DC}{Dt} \end{pmatrix} . \tag{98.77}
$$

For the background, we have

$$
\mathbf{M_0} \begin{pmatrix} \rho_0^{1/2} U_0 \\ \rho_0^{1/2} V_0 \\ \rho_0^{1/2} W_0 \\ \rho_0^{-1/2} \tilde{p}_{\mathrm{pot}0} \\ \rho_0^{-1/2} s_0 \\ \rho_0^{-1/2} p_0/C_0 \end{pmatrix} + i \begin{pmatrix} g_x \\ g_y \\ g_z \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \ , \tag{98.78}
$$

where $\mathbf{M_0}$ is the 6×6 matrix given by

$$
\begin{aligned}
\mathbf{M_0} \quad &= \quad \rho_0^{-1/2} \begin{pmatrix} -\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & C_0\hat{k}_x \\ -2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & C_0\hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & C_0\hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\ C_0\hat{k}_x & C_0\hat{k}_y & C_0\hat{k}_z & 0 & 0 & -\hat{\omega}_0 \end{pmatrix} \\
&+ \quad \rho_0^{-1/2} \begin{pmatrix} \frac{i}{2}\frac{1}{\rho}\frac{D_0\rho_0}{Dt} & 0 & 0 & 0 & 0 & -\frac{i}{2}C_0\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} - i\frac{\partial C_0}{\partial x} \\ 0 & \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{Dt} & 0 & 0 & 0 & -\frac{i}{2}C_0\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} - i\frac{\partial C_0}{\partial y} \\ 0 & 0 & \frac{i}{2}\frac{1}{\rho}\frac{D_0\rho_0}{Dt} & 0 & 0 & -\frac{i}{2}C_0\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} - i\frac{\partial C_0}{\partial z} \\ 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{Dt} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{Dt} & 0 \\ \frac{i}{2}C_0\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} & \frac{i}{2}C_0\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} & \frac{i}{2}C_0\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{Dt} - \frac{i}{C_0}\frac{D_0C_0}{Dt} \end{pmatrix} .
\end{aligned}
$$

I shall pursue doing the nonlinear calculations using these equations sometime in the future.

98.5 Subtracting the background in Section 98.3

Using an obvious notation, we can write the difference of (98.68) and (98.70) as

$$
M\psi - M_0\psi_0 = M\psi - (M - \Delta M)(\psi - \Delta\psi) = M_0\Delta\psi + (\Delta M)\psi \ . \tag{98.80}
$$

The $M_0\Delta\psi$ term is clear, and easy to calculate, but the $(\Delta M)\psi$ term will take some work to calculate.

Taking the difference between (98.69) and (98.71) gives

$$
\Delta\mathbf{M} = \begin{pmatrix} -\Delta\hat{\omega} & 0 & 0 & 0 & 0 & \Delta(\rho^{-1})\hat{k}_x \\ 0 & -\Delta\hat{\omega} & 0 & 0 & 0 & \Delta(\rho^{-1})\hat{k}_y \\ 0 & 0 & -\Delta\hat{\omega} & 0 & 0 & \Delta(\rho^{-1})\hat{k}_z \\ 0 & 0 & 0 & -\Delta\hat{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta\hat{\omega} & 0 \\ \Delta\rho\hat{k}_x & \Delta\rho\hat{k}_y & \Delta\rho\hat{k}_z & 0 & 0 & -C^{-2}\hat{\omega} + C_0^{-2}\hat{\omega}_0 \end{pmatrix} , \tag{98.81}
$$

where $\Delta\hat{\omega} = \hat{\omega} - \hat{\omega}_0 = i\mathbf{\Delta U} \cdot \nabla$, $\Delta\mathbf{U} = \mathbf{U} - \mathbf{U}_0$, $\Delta\rho = \rho - \rho_0$, $\Delta(\rho^{-1}) = \rho^{-1} - \rho_0^{-1} = -\Delta\rho/(\rho\rho_0)$, and $-C^{-2}\hat{\omega} + C_0^{-2}\hat{\omega}_0 = C^{-2}C_0^{-2}\Delta(C^2)\hat{\omega} - C_0^{-2}\Delta\hat{\omega}$. So,

$$\Delta\mathbf{M}\psi = \begin{pmatrix} -i\Delta\mathbf{U}\cdot\nabla U + \Delta(\rho^{-1})\hat{k}_x p \\ -i\Delta\mathbf{U}\cdot\nabla V + \Delta(\rho^{-1})\hat{k}_y p \\ -i\Delta\mathbf{U}\cdot\nabla W + \Delta(\rho^{-1})\hat{k}_z p \\ -i\Delta\mathbf{U}\cdot\nabla\tilde{\rho}_{\text{pot}} \\ -i\Delta\mathbf{U}\cdot\nabla s \\ \Delta\rho\hat{k}_x U + \Delta\rho\hat{k}_y V + \Delta\rho\hat{k}_z W - C_0^{-2} i\Delta\mathbf{U}\cdot\nabla p - \Delta(C^{-2})\hat{\omega}p \end{pmatrix}, \tag{98.82}$$

Or,

$$\Delta\mathbf{M}\psi = \begin{pmatrix} -i\Delta U\frac{\partial U}{\partial x} - i\Delta V\frac{\partial U}{\partial y} - i\Delta W\frac{\partial U}{\partial z} + i(\rho_0)^{-1}\tilde{g}_x\Delta\rho \\ -i\Delta U\frac{\partial V}{\partial x} - i\Delta V\frac{\partial V}{\partial y} - i\Delta W\frac{\partial V}{\partial z} + i(\rho_0)^{-1}\tilde{g}_y\Delta\rho \\ -i\Delta U\frac{\partial W}{\partial x} - i\Delta V\frac{\partial W}{\partial y} - i\Delta W\frac{\partial W}{\partial z} + i(\rho_0)^{-1}\tilde{g}_z\Delta\rho \\ -i\Delta U\partial\tilde{\rho}_{\text{pot}}/\partial x - i\Delta V\partial\tilde{\rho}_{\text{pot}}/\partial y - i\Delta W\partial\tilde{\rho}_{\text{pot}}/\partial z \\ -i\Delta U\frac{\partial s}{\partial x} - i\Delta V\frac{\partial s}{\partial y} - i\Delta W\frac{\partial s}{\partial z} \\ -i\Delta\rho\frac{\partial U}{\partial x} - i\Delta\rho\frac{\partial V}{\partial y} - i\Delta\rho\frac{\partial W}{\partial z} - i\frac{\Delta U}{C_0^2}\rho_0\tilde{g}_{1x} - i\frac{\Delta V}{C_0^2}\rho_0\tilde{g}_{1y} - i\frac{\Delta W}{C_0^2}\rho_0\tilde{g}_{1z} + i\frac{\Delta(C^2)}{C_0^2}\frac{D\rho}{Dt} \end{pmatrix}, \tag{98.83}$$

where $\tilde{\mathbf{g}} \equiv \nabla p/\rho$ and $\tilde{\mathbf{g}}_1 \equiv \nabla p/\rho_0$ are effective accelerations due to gravity [including (minus) the acceleration of the fluid flow] and I have used (98.52).

We need to calculate $\Delta\rho = \rho - \rho_0$ and $\Delta(C^2) = C^2 - C_0^2$ in terms of $\Delta p = p - p_0$, $\Delta\tilde{\rho}_{\text{pot}} = \tilde{\rho}_{\text{pot}} - \tilde{\rho}_{\text{pot}0}$, and $\Delta s = s - s_0$, without assuming small changes. To do this, we define

$$\begin{aligned}\overline{C^2} \equiv \overline{\left(\frac{\partial p}{\partial\rho}\right)}_{S,s} \equiv{}& \frac{p(\rho,S,s) - p(\rho_0,S,s)}{3(\rho-\rho_0)} + \frac{p(\rho,S,s_0) - p(\rho_0,S,s_0)}{6(\rho-\rho_0)} \\ &+ \frac{p(\rho,S_0,s) - p(\rho_0,S_0,s)}{6(\rho-\rho_0)} + \frac{p(\rho,S_0,s_0) - p(\rho_0,S_0,s_0)}{3(\rho-\rho_0)}\end{aligned}, \tag{98.84}$$

$$\begin{aligned}\overline{\left(\frac{\partial p}{\partial S}\right)}_{\rho,s} \equiv{}& \frac{p(\rho,S,s) - p(\rho,S_0,s)}{3(S-S_0)} + \frac{p(\rho,S,s_0) - p(\rho,S_0,s_0)}{6(S-S_0)} \\ &+ \frac{p(\rho_0,S,s) - p(\rho_0,S_0,s)}{6(S-S_0)} + \frac{p(\rho_0,S,s_0) - p(\rho_0,S_0,s_0)}{3(S-S_0)}\end{aligned}, \tag{98.85}$$

and

$$\begin{aligned}\overline{\left(\frac{\partial p}{\partial s}\right)}_{\rho,S} \equiv{}& \frac{p(\rho,S,s) - p(\rho,S,s_0)}{3(s-s_0)} + \frac{p(\rho,S_0,s) - p(\rho,S_0,s_0)}{6((s-s_0)} \\ &+ \frac{p(\rho_0,S,s) - p(\rho_0,S,s_0)}{6((s-s_0)} + \frac{p(\rho_0,S_0,s) - p(\rho_0,S_0,s_0)}{3((s-s_0)}\end{aligned}. \tag{98.86}$$

Then from (98.47) and (98.61) we can write

$$\Delta p = p - p_0 = \overline{C^2}\Delta\rho + \overline{\left(\frac{\partial p}{\partial S}\right)}_{\rho,s}\Delta S + \overline{\left(\frac{\partial p}{\partial s}\right)}_{\rho,S}\Delta s \tag{98.87}$$

If we define

$$\begin{aligned}\overline{\overline{C^2}} \equiv \overline{\overline{\left(\frac{\partial p}{\partial\rho}\right)}}_{S,s} \equiv{}& \frac{p(\tilde{\rho}_{\text{pot}},S,s) - p(\tilde{\rho}_{\text{pot}0},S,s)}{3(\tilde{\rho}_{\text{pot}} - \tilde{\rho}_{\text{pot}0})} + \frac{p(\tilde{\rho}_{\text{pot}},S,s_0) - p(\tilde{\rho}_{\text{pot}0},S,s_0)}{6(\tilde{\rho}_{\text{pot}} - \tilde{\rho}_{\text{pot}0})} \\ &+ \frac{p(\tilde{\rho}_{\text{pot}},S_0,s) - p(\tilde{\rho}_{\text{pot}0},S_0,s)}{6(\tilde{\rho}_{\text{pot}} - \tilde{\rho}_{\text{pot}0})} + \frac{p(\tilde{\rho}_{\text{pot}},S_0,s_0) - p(\tilde{\rho}_{\text{pot}0},S_0,s_0)}{3(\tilde{\rho}_{\text{pot}} - \tilde{\rho}_{\text{pot}0})}\end{aligned}, \tag{98.88}$$

$$\overline{\overline{\left(\frac{\partial p}{\partial S}\right)}}_{\rho,s} \equiv \frac{p(\tilde{\rho}_{\text{pot}},S,s)-p(\tilde{\rho}_{\text{pot}},S_0,s)}{3(S-S_0)}+\frac{p(\tilde{\rho}_{\text{pot}},S,s_0)-p(\tilde{\rho}_{\text{pot}},S_0,s_0)}{6(S-S_0)} \\ +\frac{p(\tilde{\rho}_{\text{pot0}},S,s)-p(\tilde{\rho}_{\text{pot0}},S_0,s)}{6(S-S_0)}+\frac{p(\tilde{\rho}_{\text{pot0}},S,s_0)-p(\tilde{\rho}_{\text{pot0}},S_0,s_0)}{3(S-S_0)} \quad , \quad (98.89)$$

and

$$\overline{\overline{\left(\frac{\partial p}{\partial s}\right)}}_{\rho,S} \equiv \frac{p(\tilde{\rho}_{\text{pot}},S,s)-p(\tilde{\rho}_{\text{pot}},S,s_0)}{3(s-s_0)}+\frac{p(\tilde{\rho}_{\text{pot}},S_0,s)-p(\tilde{\rho}_{\text{pot}},S_0,s_0)}{6((s-s_0)} \\ +\frac{p(\tilde{\rho}_{\text{pot0}},S,s)-p(\tilde{\rho}_{\text{pot0}},S,s_0)}{6((s-s_0)}+\frac{p(\tilde{\rho}_{\text{pot0}},S_0,s)-p(\tilde{\rho}_{\text{pot0}},S_0,s_0)}{3((s-s_0)} \quad , \quad (98.90)$$

then we have

$$\overline{\overline{C^2}}\Delta\tilde{\rho}_{\text{pot}}+\overline{\overline{\left(\frac{\partial p}{\partial S}\right)}}_{\rho,s}\Delta S+\overline{\overline{\left(\frac{\partial p}{\partial s}\right)}}_{\rho,S}\Delta s = p(\tilde{\rho}_{\text{pot}},S,s)-p(\tilde{\rho}_{\text{pot0}},S_0,s_0)=p_a-p_a=0\ . \quad (98.91)$$

Thus, we have

$$\Delta S = \frac{-\overline{\overline{C^2}}\Delta\tilde{\rho}_{\text{pot}}-\overline{\overline{(\partial p/\partial s)}}_{\rho,S}\Delta s}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\ . \quad (98.92)$$

Substituting (98.92) into (98.87 gives

$$\Delta p = \overline{C^2}\Delta\rho - \frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\overline{\overline{C^2}}\Delta\tilde{\rho}_{\text{pot}} - \left[\frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\overline{\overline{(\partial p/\partial s)}}_{\rho,S}-\overline{\left(\frac{\partial p}{\partial s}\right)}_{\rho,S}\right]\Delta s \quad (98.93)$$

From (98.93), we have

$$\Delta\tilde{\rho}_{\text{pot}} = \frac{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}{\overline{(\partial p/\partial S)}_{\rho,s}}\left(\frac{\overline{C^2}}{\overline{\overline{C^2}}}\Delta\rho-\frac{\Delta p}{\overline{\overline{C^2}}}\right)-\frac{1}{\overline{\overline{C^2}}}\left[\overline{\overline{(\partial p/\partial s)}}_{\rho,S}-\frac{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}{\overline{(\partial p/\partial S)}_{\rho,s}}\overline{\left(\frac{\partial p}{\partial s}\right)}_{\rho,S}\right]\Delta s\ . \quad (98.94)$$

Also from (98.93), we have

$$\Delta\rho = \mu\Delta\tilde{\rho}_{\text{pot}}+\Delta p/\overline{C^2}+\nu\Delta s\ , \quad (98.95)$$

where

$$\mu \equiv \frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\overline{\overline{C^2}}/\overline{C^2}\ . \quad (98.96)$$

and

$$\nu \equiv \left[\frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\overline{\overline{(\partial p/\partial s)}}_{\rho,S}-\overline{\left(\frac{\partial p}{\partial s}\right)}_{\rho,S}\right]/\overline{C^2}\ , \quad (98.97)$$

To get $\Delta C^2 = C^2 - C_0^2$, we define

$$C^2 = C^2(\rho,S,s)\ . \quad (98.98)$$

and

$$C_0^2 = C^2(\rho_0,S_0,s_0)\ . \quad (98.99)$$

Now we define

$$\begin{aligned}\overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s} \equiv{}& \frac{C^2(\rho,S,s)-C^2(\rho_0,S,s)}{3(\rho-\rho_0)} + \frac{C^2(\rho,S,s_0)-C^2(\rho_0,S,s_0)}{6(\rho-\rho_0)}\\ &+\frac{C^2(\rho,S_0,s)-C^2(\rho_0,S_0,s)}{6(\rho-\rho_0)} + \frac{C^2(\rho,S_0,s_0)-C^2(\rho_0,S_0,s_0)}{3(\rho-\rho_0)}\end{aligned} \quad , \tag{98.100}$$

$$\begin{aligned}\overline{\left(\frac{\partial C^2}{\partial S}\right)}_{\rho,s} \equiv{}& \frac{C^2(\rho,S,s)-C^2(\rho,S_0,s)}{3(S-S_0)} + \frac{C^2(\rho,S,s_0)-C^2(\rho,S_0,s_0)}{6(S-S_0)}\\ &+\frac{C^2(\rho_0,S,s)-C^2(\rho_0,S_0,s)}{6(S-S_0)} + \frac{C^2(\rho_0,S,s_0)-C^2(\rho_0,S_0,s_0)}{3(S-S_0)}\end{aligned} \quad , \tag{98.101}$$

and

$$\begin{aligned}\overline{\left(\frac{\partial C^2}{\partial s}\right)}_{\rho,S} \equiv{}& \frac{C^2(\rho,S,s)-C^2(\rho,S,s_0)}{3(s-s_0)} + \frac{C^2(\rho,S_0,s)-C^2(\rho,S_0,s_0)}{6((s-s_0)}\\ &+\frac{C^2(\rho_0,S,s)-C^2(\rho_0,S,s_0)}{6((s-s_0)} + \frac{C^2(\rho_0,S_0,s)-C^2(\rho_0,S_0,s_0)}{3((s-s_0)}\end{aligned} \quad . \tag{98.102}$$

Then from (98.98) and (98.99) we can write

$$\Delta C^2 = C^2 - C_0^2 = \overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}\Delta\rho + \overline{\left(\frac{\partial C^2}{\partial S}\right)}_{\rho,s}\Delta S + \overline{\left(\frac{\partial C^2}{\partial s}\right)}_{\rho,S}\Delta s \tag{98.103}$$

Or, using (98.95) and (98.92) gives

$$\begin{aligned}\Delta C^2 ={}& \overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}\left(\frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\overline{\overline{C^2}}/\overline{C^2}\Delta\tilde{\rho}_{\text{pot}} + \Delta p/\overline{C^2} + \nu\Delta s\right)\\ &+\overline{\left(\frac{\partial C^2}{\partial S}\right)}_{\rho,s}\frac{-\overline{\overline{C^2}}\Delta\tilde{\rho}_{\text{pot}} - \overline{\overline{(\partial p/\partial s)}}_{\rho,S}\Delta s}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}} + \overline{\left(\frac{\partial C^2}{\partial s}\right)}_{\rho,S}\Delta s\end{aligned} \tag{98.104}$$

Or,

$$\begin{aligned}\Delta C^2 ={}& \overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}\frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\overline{\overline{C^2}}/\overline{C^2}\Delta\tilde{\rho}_{\text{pot}} + \overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}\Delta p/\overline{C^2} + \overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}\nu\Delta s\\ &-\overline{\left(\frac{\partial C^2}{\partial S}\right)}_{\rho,s}\frac{\overline{\overline{C^2}}\Delta\tilde{\rho}_{\text{pot}}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}} - \overline{\left(\frac{\partial C^2}{\partial S}\right)}_{\rho,s}\frac{\overline{\overline{(\partial p/\partial s)}}_{\rho,S}\Delta s}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}} + \overline{\left(\frac{\partial C^2}{\partial s}\right)}_{\rho,S}\Delta s\end{aligned} \tag{98.105}$$

Or,

$$\begin{aligned}\Delta C^2 ={}& \left[\overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}\frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\overline{\overline{C^2}}/\overline{C^2} - \overline{\left(\frac{\partial C^2}{\partial S}\right)}_{\rho,s}\frac{\overline{\overline{C^2}}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}}\right]\Delta\tilde{\rho}_{\text{pot}}\\ &+\left[\overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}\nu - \overline{\left(\frac{\partial C^2}{\partial S}\right)}_{\rho,s}\frac{\overline{\overline{(\partial p/\partial s)}}_{\rho,S}}{\overline{\overline{(\partial p/\partial S)}}_{\rho,s}} + \overline{\left(\frac{\partial C^2}{\partial s}\right)}_{\rho,S}\right]\Delta s\\ &+\left[\overline{\left(\frac{\partial C^2}{\partial \rho}\right)}_{S,s}/\overline{C^2}\right]\Delta p\end{aligned} \tag{98.106}$$

Or,

$$\Delta C^2 = \alpha \Delta\tilde{\rho}_{\text{pot}} + \beta\Delta s + \gamma\Delta p \tag{98.107}$$

as a shorthand notation, where

$$\alpha = \overline{\left(\frac{\partial C^2}{\partial \rho}\right)_{S,s}} \frac{\overline{(\partial p/\partial S)}_{\rho,s}}{\overline{\overline{\overline{(\partial p/\partial S)}}}_{\rho,s}} \overline{\overline{C^2}}/\overline{C^2} - \overline{\left(\frac{\partial C^2}{\partial S}\right)_{\rho,s}} \frac{\overline{\overline{C^2}}}{\overline{\overline{\overline{(\partial p/\partial S)}}}_{\rho,s}} , \tag{98.108}$$

$$\beta = \overline{\left(\frac{\partial C^2}{\partial \rho}\right)_{S,s}} \nu - \overline{\left(\frac{\partial C^2}{\partial S}\right)_{\rho,s}} \frac{\overline{\overline{(\partial p/\partial s)}}_{\rho,S}}{\overline{\overline{\overline{(\partial p/\partial S)}}}_{\rho,s}} + \overline{\left(\frac{\partial C^2}{\partial s}\right)_{\rho,S}} , \tag{98.109}$$

and

$$\gamma = \overline{\left(\frac{\partial C^2}{\partial \rho}\right)_{S,s}} /\overline{C^2} . \tag{98.110}$$

Substituting (98.95) and (98.107) into (98.83) gives

$$\Delta\mathbf{M}\psi = \begin{pmatrix} -i\Delta U\frac{\partial U}{\partial x} - i\Delta V\frac{\partial U}{\partial y} - i\Delta W\frac{\partial U}{\partial z} + \frac{i}{\rho_0}\left(\mu\Delta\tilde{\rho}_{\text{pot}} + \Delta p/\overline{C^2} + \nu\Delta s\right)\tilde{g}_x \\ -i\Delta U\frac{\partial V}{\partial x} - i\Delta V\frac{\partial V}{\partial y} - i\Delta W\frac{\partial V}{\partial z} + \frac{i}{\rho_0}\left(\mu\Delta\tilde{\rho}_{\text{pot}} + \Delta p/\overline{C^2} + \nu\Delta s\right)\tilde{g}_y \\ -i\Delta U\frac{\partial W}{\partial x} - i\Delta V\frac{\partial W}{\partial y} - i\Delta W\frac{\partial W}{\partial z} + \frac{i}{\rho_0}\left(\mu\Delta\tilde{\rho}_{\text{pot}} + \Delta p/\overline{C^2} + \nu\Delta s\right)\tilde{g}_z \\ -i\Delta U\partial\tilde{\rho}_{\text{pot}}/\partial x - i\Delta V\partial\tilde{\rho}_{\text{pot}}/\partial y - i\Delta W\partial\tilde{\rho}_{\text{pot}}/\partial z \\ -i\Delta U\frac{\partial s}{\partial x} - i\Delta V\frac{\partial s}{\partial y} - i\Delta W\frac{\partial s}{\partial z} \\ -i\nabla\cdot\mathbf{U}\left(\mu\Delta\tilde{\rho}_{\text{pot}} + \frac{\Delta p}{C^2} + \nu\Delta s\right) - i\frac{\Delta U}{C_0^2}\rho_0\tilde{g}_{1x} - i\frac{\Delta V}{C_0^2}\rho_0\tilde{g}_{1y} - i\frac{\Delta W}{C_0^2}\rho_0\tilde{g}_{1z} + i\frac{\alpha\Delta\tilde{\rho}_{\text{pot}}+\beta\Delta s+\gamma\Delta p}{C_0^2}\frac{D\rho}{Dt} \end{pmatrix} , \tag{98.111}$$

Or,

$$\Delta\mathbf{M}\psi = \begin{pmatrix} -i\Delta U\frac{\partial U}{\partial x} - i\Delta V\frac{\partial U}{\partial y} - i\Delta W\frac{\partial U}{\partial z} + i(\rho_0)^{-1}\tilde{g}_x\mu\Delta\tilde{\rho}_{\text{pot}} + \tilde{g}_x\frac{i\nu\Delta s}{\rho_0} + i(\rho_0)^{-1}\tilde{g}_x\frac{1}{\overline{\overline{C^2}}}\Delta p \\ -i\Delta U\frac{\partial V}{\partial x} - i\Delta V\frac{\partial V}{\partial y} - i\Delta W\frac{\partial V}{\partial z} + i(\rho_0)^{-1}\tilde{g}_y\mu\Delta\tilde{\rho}_{\text{pot}} + \tilde{g}_y\frac{i\nu\Delta s}{\rho_0} + i(\rho_0)^{-1}\tilde{g}_y\frac{1}{\overline{\overline{C^2}}}\Delta p \\ -i\Delta U\frac{\partial W}{\partial x} - i\Delta V\frac{\partial W}{\partial y} - i\Delta W\frac{\partial W}{\partial z} + i(\rho_0)^{-1}\tilde{g}_z\mu\Delta\tilde{\rho}_{\text{pot}} + \tilde{g}_z\frac{i\nu\Delta s}{\rho_0} + i(\rho_0)^{-1}\tilde{g}_z\frac{1}{\overline{\overline{C^2}}}\Delta p \\ -i\Delta U\partial\tilde{\rho}_{\text{pot}}/\partial x - i\Delta V\partial\tilde{\rho}_{\text{pot}}/\partial y - i\Delta W\partial\tilde{\rho}_{\text{pot}}/\partial z \\ -i\Delta U\frac{\partial s}{\partial x} - i\Delta V\frac{\partial s}{\partial y} - i\Delta W\frac{\partial s}{\partial z} \\ -i\frac{\Delta U}{C_0^2}\rho_0\tilde{g}_{1x} - i\frac{\Delta V}{C_0^2}\rho_0\tilde{g}_{1y} - i\frac{\Delta W}{C_0^2}\rho_0\tilde{g}_{1z} + i\frac{D\rho}{Dt}\frac{\mu\Delta\tilde{\rho}_{\text{pot}}}{\rho} + \frac{i\alpha\Delta\tilde{\rho}_{\text{pot}}}{C_0^2}\frac{D\rho}{Dt} + i\frac{D\rho}{Dt}\frac{\nu\Delta s}{\rho} + \frac{i\beta\Delta s}{C_0^2}\frac{D\rho}{Dt} + i\frac{D\rho}{Dt}\frac{\Delta p}{\rho\overline{C^2}} + \frac{i\gamma\Delta p}{C_0^2}\frac{D\rho}{Dt} \end{pmatrix} \tag{98.112}$$

Or,

$$\Delta\mathbf{M}\psi = \begin{pmatrix} -i\frac{\partial U}{\partial x} & -i\frac{\partial U}{\partial y} & -i\frac{\partial U}{\partial z} & \frac{i\mu}{\rho_0}\tilde{g}_x & \frac{i\nu}{\rho_0}\tilde{g}_x & \frac{i}{\rho_0\overline{\overline{C^2}}}\tilde{g}_x \\ -i\frac{\partial V}{\partial x} & -i\frac{\partial V}{\partial y} & -i\frac{\partial V}{\partial z} & \frac{i\mu}{\rho_0}\tilde{g}_y & \frac{i\nu}{\rho_0}\tilde{g}_y & \frac{i}{\rho_0\overline{\overline{C^2}}}\tilde{g}_y \\ -i\frac{\partial W}{\partial x} & -i\frac{\partial W}{\partial y} & -i\frac{\partial W}{\partial z} & \frac{i\mu}{\rho_0}\tilde{g}_z & \frac{i\nu}{\rho_0}\tilde{g}_z & \frac{i}{\rho_0\overline{\overline{C^2}}}\tilde{g}_z \\ -i\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial x} & -i\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial y} & -i\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial z} & 0 & 0 & 0 \\ -i\frac{\partial s}{\partial x} & -i\frac{\partial s}{\partial y} & -i\frac{\partial s}{\partial z} & 0 & 0 & 0 \\ -i\frac{1}{C_0^2}\rho_0\tilde{g}_{1x} & -i\frac{1}{C_0^2}\rho_0\tilde{g}_{1y} & -i\frac{1}{C_0^2}\rho_0\tilde{g}_{1z} & i\frac{\mu}{\rho}\frac{D\rho}{Dt} + \frac{i\alpha}{C_0^2}\frac{D\rho}{Dt} & i\frac{\nu}{\rho}\frac{D\rho}{Dt} + \frac{i\beta}{C_0^2}\frac{D\rho}{Dt} & \frac{i}{\rho\overline{C^2}}\frac{D\rho}{Dt} + \frac{i\gamma}{C_0^2}\frac{D\rho}{Dt} \end{pmatrix} \begin{pmatrix} \Delta U \\ \Delta V \\ \Delta W \\ \Delta\tilde{\rho}_{\text{pot}} \\ \Delta s \\ \Delta p \end{pmatrix} \tag{98.113}$$

Subtracting (98.70) from (98.68) , and using (98.80), (98.71), and (98.113) gives

$$
\begin{pmatrix}
-\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & {\rho_0}^{-1}\hat{k}_x \\
-2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & {\rho_0}^{-1}\hat{k}_y \\
2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & {\rho_0}^{-1}\hat{k}_z \\
0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\
\rho_0\hat{k}_x & \rho_0\hat{k}_y & \rho_0\hat{k}_z & 0 & 0 & -C_0^{-2}\hat{\omega}_0
\end{pmatrix}
\begin{pmatrix}
\Delta U \\ \Delta V \\ \Delta W \\ \Delta\tilde{\rho}_{\text{pot}} \\ \Delta s \\ \Delta p
\end{pmatrix}
+
$$

$$
\begin{pmatrix}
-i\frac{\partial U}{\partial x} & -i\frac{\partial U}{\partial y} & -i\frac{\partial U}{\partial z} & \frac{i\mu}{\rho_0}\tilde{g}_x & \frac{i\nu}{\rho_0}\tilde{g}_x & \frac{i}{\rho_0\overline{C^2}}\tilde{g}_x \\
-i\frac{\partial V}{\partial x} & -i\frac{\partial V}{\partial y} & -i\frac{\partial V}{\partial z} & \frac{i\mu}{\rho_0}\tilde{g}_y & \frac{i\nu}{\rho_0}\tilde{g}_y & \frac{i}{\rho_0\overline{C^2}}\tilde{g}_y \\
-i\frac{\partial W}{\partial x} & -i\frac{\partial W}{\partial y} & -i\frac{\partial W}{\partial z} & \frac{i\mu}{\rho_0}\tilde{g}_z & \frac{i\nu}{\rho_0}\tilde{g}_z & \frac{i}{\rho_0\overline{C^2}}\tilde{g}_z \\
-i\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial x} & -i\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial y} & -i\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial z} & 0 & 0 & 0 \\
-i\frac{\partial s}{\partial x} & -i\frac{\partial s}{\partial y} & -i\frac{\partial s}{\partial z} & 0 & 0 & 0 \\
-i\frac{1}{C_0^2}\rho_0\tilde{g}_{1x} & -i\frac{1}{C_0^2}\rho_0\tilde{g}_{1y} & -i\frac{1}{C_0^2}\rho_0\tilde{g}_{1z} & i\frac{\mu}{\rho}\frac{D\rho}{Dt}+\frac{i\alpha}{C_0^2}\frac{D\rho}{Dt} & i\frac{\nu}{\rho}\frac{D\rho}{Dt}+\frac{i\beta}{C_0^2}\frac{D\rho}{Dt} & \frac{i}{\rho\overline{C^2}}\frac{D\rho}{Dt}+\frac{i\gamma}{C_0^2}\frac{D\rho}{Dt}
\end{pmatrix}
\begin{pmatrix}
\Delta U \\ \Delta V \\ \Delta W \\ \Delta\tilde{\rho}_{\text{pot}} \\ \Delta s \\ \Delta p
\end{pmatrix}
= 0 \tag{98.114}
$$

Equation (98.114) is equivalent to the full non-linear Navier-Stokes equations. No approximations have been made. The appearance of linearity is an illusion, because the second of the two matrices in (98.114) contains variables that are not background variables. The composition variable s in practice may involve several variables. Thus, row 5 and column 5 in (98.114) may have to be expanded to more than one row and column in general. The extension is straightforward, however.

At this point, we need to scale the dependent variables to avoid the appearance of exponential growth of the waves in an atmosphere in which the density decreases approximately exponentially with height. The appropriate scaling factors are $\rho_0^{\pm\frac{1}{2}}$. We also multiply the first three equations (rows) by $\rho_0^{\frac{1}{2}}$ and the last three equations (rows) by $\rho_0^{-\frac{1}{2}}$. This gives

$$
\begin{aligned}
&\begin{pmatrix}
-\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & \hat{k}_x \\
-2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & \hat{k}_y \\
2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & \hat{k}_z \\
0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\
\hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & 0 & -C_0^{-2}\hat{\omega}_0
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\
\rho_0^{1/2}\Delta V \\
\rho_0^{1/2}\Delta W \\
\rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\
\rho_0^{-1/2}\Delta s \\
\rho_0^{-1/2}\Delta p
\end{pmatrix} + \\
&\begin{pmatrix}
-i\frac{\partial U}{\partial x} & -i\frac{\partial U}{\partial y} & -i\frac{\partial U}{\partial z} & (i\mu)\tilde{g}_x & (i\nu)\tilde{g}_x & \frac{i}{\overline{C}^2}\tilde{g}_x \\
-i\frac{\partial V}{\partial x} & -i\frac{\partial V}{\partial y} & -i\frac{\partial V}{\partial z} & (i\mu)\tilde{g}_y & (i\nu)\tilde{g}_y & \frac{i}{\overline{C}^2}\tilde{g}_y \\
-i\frac{\partial W}{\partial x} & -i\frac{\partial W}{\partial y} & -i\frac{\partial W}{\partial z} & (i\mu)\tilde{g}_z & (i\nu)\tilde{g}_z & \frac{i}{\overline{C}^2}\tilde{g}_z \\
-\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\mathrm{pot}}}{\partial x} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\mathrm{pot}}}{\partial y} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\mathrm{pot}}}{\partial z} & 0 & 0 & 0 \\
-\frac{i}{\rho_0}\frac{\partial s}{\partial x} & -\frac{i}{\rho_0}\frac{\partial s}{\partial y} & -\frac{i}{\rho_0}\frac{\partial s}{\partial z} & 0 & 0 & 0 \\
-\frac{i}{C_0^2}\tilde{g}_{1x} & -\frac{i}{C_0^2}\tilde{g}_{1y} & -\frac{i}{C_0^2}\tilde{g}_{1z} & \frac{i\mu}{\rho}\frac{D\rho}{Dt} + \frac{i\alpha}{C_0^2}\frac{D\rho}{Dt} & \frac{i\nu}{\rho}\frac{D\rho}{Dt} + \frac{i\beta}{C_0^2}\frac{D\rho}{Dt} & \frac{i}{\rho\overline{C}^2}\frac{D\rho}{Dt} + \frac{i\gamma}{C_0^2}\frac{D\rho}{Dt}
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\
\rho_0^{1/2}\Delta V \\
\rho_0^{1/2}\Delta W \\
\rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\
\rho_0^{-1/2}\Delta s \\
\rho_0^{-1/2}\Delta p
\end{pmatrix} \\
&+\begin{pmatrix}
\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} \\
0 & \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} \\
0 & 0 & \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} \\
0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 \\
\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} & 0 & 0 & -C_0^{-2}\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t}
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\
\rho_0^{1/2}\Delta V \\
\rho_0^{1/2}\Delta W \\
\rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\
\rho_0^{-1/2}\Delta s \\
\rho_0^{-1/2}\Delta p
\end{pmatrix} \\
&= 0 \qquad (98.115)
\end{aligned}
$$

Equation (98.115) can be rewritten as

$$
\begin{pmatrix}
-\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & \hat{k}_x \\
-2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & \hat{k}_y \\
2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & \hat{k}_z \\
0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\
\hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & 0 & -C_0^{-2}\hat{\omega}_0
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
+
$$

$$
\begin{pmatrix}
-i\frac{\partial U}{\partial x} & -i\frac{\partial U}{\partial y} & -i\frac{\partial U}{\partial z} & (i\mu)\tilde{g}_x & (i\nu)\tilde{g}_x & \frac{i}{\overline{C^2}}\tilde{g}_x \\
-i\frac{\partial V}{\partial x} & -i\frac{\partial V}{\partial y} & -i\frac{\partial V}{\partial z} & (i\mu)\tilde{g}_y & (i\nu)\tilde{g}_y & \frac{i}{\overline{C^2}}\tilde{g}_y \\
-i\frac{\partial W}{\partial x} & -i\frac{\partial W}{\partial y} & -i\frac{\partial W}{\partial z} & (i\mu)\tilde{g}_z & (i\nu)\tilde{g}_z & \frac{i}{\overline{C^2}}\tilde{g}_z \\
-\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\mathrm{pot}}}{\partial x} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\mathrm{pot}}}{\partial y} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\mathrm{pot}}}{\partial z} & 0 & 0 & 0 \\
-\frac{i}{\rho_0}\frac{\partial s}{\partial x} & -\frac{i}{\rho_0}\frac{\partial s}{\partial y} & -\frac{i}{\rho_0}\frac{\partial s}{\partial z} & 0 & 0 & 0 \\
-\frac{i}{C_0^2}\tilde{g}_{1x} & -\frac{i}{C_0^2}\tilde{g}_{1y} & -\frac{i}{C_0^2}\tilde{g}_{1z} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
$$

$$
+
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} \\
0 & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} \\
0 & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
+
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{i\mu}{\rho}\frac{D\rho}{Dt} + \frac{i\alpha}{C_0^2}\frac{D\rho}{Dt} & \frac{i\nu}{\rho}\frac{D\rho}{Dt} + \frac{i\beta}{C_0^2}\frac{D\rho}{Dt} & \frac{i}{\rho\overline{C^2}}\frac{D\rho}{Dt} + \frac{i\gamma}{C_0^2}\frac{D\rho}{Dt}
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
+
$$

$$
\begin{pmatrix}
\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t} & 0 \\
0 & 0 & 0 & 0 & 0 & -C_0^{-2}\frac{i}{2}\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0t}
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\mathrm{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
$$

$$
= 0 \tag{98.116}
$$

Equation (98.116) can be rewritten

$$\begin{pmatrix} -\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & \hat{k}_x \\ -2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & \hat{k}_y \\ 2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & \hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & 0 & -C_0^{-2}\hat{\omega}_0 \end{pmatrix} \begin{pmatrix} \rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p \end{pmatrix}$$

$$+ \begin{pmatrix} -i\frac{\partial U}{\partial x} & -i\frac{\partial U}{\partial y} & -i\frac{\partial U}{\partial z} & (i\mu)\tilde{g}_x & (i\nu)\tilde{g}_x & \frac{i}{C^2}\tilde{g}_x \\ -i\frac{\partial V}{\partial x} & -i\frac{\partial V}{\partial y} & -i\frac{\partial V}{\partial z} & (i\mu)\tilde{g}_y & (i\nu)\tilde{g}_y & \frac{i}{C^2}\tilde{g}_y \\ -i\frac{\partial W}{\partial x} & -i\frac{\partial W}{\partial y} & -i\frac{\partial W}{\partial z} & (i\mu)\tilde{g}_z & (i\nu)\tilde{g}_z & \frac{i}{C^2}\tilde{g}_z \\ -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial x} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial y} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial z} & 0 & 0 & 0 \\ -\frac{i}{\rho_0}\frac{\partial s}{\partial x} & -\frac{i}{\rho_0}\frac{\partial s}{\partial y} & -\frac{i}{\rho_0}\frac{\partial s}{\partial z} & 0 & 0 & 0 \\ -\frac{i}{C_0^2}\tilde{g}_{1x} & -\frac{i}{C_0^2}\tilde{g}_{1y} & -\frac{i}{C_0^2}\tilde{g}_{1z} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} \\ 0 & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} \\ 0 & 0 & 0 & 0 & 0 & -\frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial y} & \frac{i}{2}\frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p \end{pmatrix}$$

$$+\frac{1}{\rho}\frac{D\rho}{Dt} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mu + \frac{i\rho}{C_0^2}\alpha & i\nu + \frac{i\rho}{C_0^2}\beta & \frac{i}{C^2} + \frac{i\rho}{C_0^2}\gamma \end{pmatrix} \begin{pmatrix} \rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p \end{pmatrix}$$

$$+\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0 t} \begin{pmatrix} \frac{i}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{i}{2}C_0^{-2} \end{pmatrix} \begin{pmatrix} \rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p \end{pmatrix}$$

$$= 0 \quad , \quad (98.117)$$

Using (98.110) allows (98.117) to be rewritten as

$$
\begin{pmatrix}
-\hat{\omega}_0 & 2i\Omega_z & -2i\Omega_y & 0 & 0 & \hat{k}_x \\
-2i\Omega_z & -\hat{\omega}_0 & 2i\Omega_x & 0 & 0 & \hat{k}_y \\
2i\Omega_y & -2i\Omega_x & -\hat{\omega}_0 & 0 & 0 & \hat{k}_z \\
0 & 0 & 0 & -\hat{\omega}_0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\hat{\omega}_0 & 0 \\
\hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & 0 & -C_0^{-2}\hat{\omega}_0
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
$$

$$
+\begin{pmatrix}
-i\frac{\partial U}{\partial x} & -i\frac{\partial U}{\partial y} & -i\frac{\partial U}{\partial z} & (i\mu)\tilde{g}_x & (i\nu)\tilde{g}_x & -i\Gamma_{2x} \\
-i\frac{\partial V}{\partial x} & -i\frac{\partial V}{\partial y} & -i\frac{\partial V}{\partial z} & (i\mu)\tilde{g}_y & (i\nu)\tilde{g}_y & -i\Gamma_{2y} \\
-i\frac{\partial W}{\partial x} & -i\frac{\partial W}{\partial y} & -i\frac{\partial W}{\partial z} & (i\mu)\tilde{g}_z & (i\nu)\tilde{g}_z & -i\Gamma_{2z} \\
-\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial x} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial y} & -\frac{i}{\rho_0}\frac{\partial\tilde{\rho}_{\text{pot}}}{\partial z} & 0 & 0 & 0 \\
-\frac{i}{\rho_0}\frac{\partial s}{\partial x} & -\frac{i}{\rho_0}\frac{\partial s}{\partial y} & -\frac{i}{\rho_0}\frac{\partial s}{\partial z} & 0 & 0 & 0 \\
i\Gamma_{1x} & i\Gamma_{1y} & i\Gamma_{1z} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
$$

$$
+\frac{1}{\rho}\frac{D\rho}{Dt}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\mu + \frac{i\rho}{C_0^2}\alpha & i\nu + \frac{i\rho}{C_0^2}\beta & \frac{i}{C^2} + \frac{i\rho}{C_0^2 C^2}\overline{\left(\frac{\partial C^2}{\partial\rho}\right)}_{S,s}
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
$$

$$
+\frac{1}{\rho_0}\frac{D_0\rho_0}{D_0 t}
\begin{pmatrix}
\frac{i}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{i}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{i}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{i}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{i}{2}C_0^{-2}
\end{pmatrix}
\begin{pmatrix}
\rho_0^{1/2}\Delta U \\ \rho_0^{1/2}\Delta V \\ \rho_0^{1/2}\Delta W \\ \rho_0^{-1/2}\Delta\tilde{\rho}_{\text{pot}} \\ \rho_0^{-1/2}\Delta s \\ \rho_0^{-1/2}\Delta p
\end{pmatrix}
$$

$$
= 0 \quad , \tag{98.118}
$$

where

$$
\mathbf{\Gamma}_1 \equiv \frac{\nabla\rho_0}{2\rho_0} - \frac{\tilde{\mathbf{g}}_1}{C_0^2} = \frac{\nabla\rho_0}{2\rho_0} - \frac{\nabla p}{\rho_0 C_0^2} \tag{98.119}
$$

and

$$
\mathbf{\Gamma}_2 \equiv \frac{\nabla\rho_0}{2\rho_0} - \frac{\tilde{\mathbf{g}}}{C^2} = \frac{\nabla\rho_0}{2\rho_0} - \frac{\nabla p}{\rho C^2} \tag{98.120}
$$

are vector generalizations of Eckart's coefficient.[297, Gossard & Hooke, 1975 p. 90]

98.6 Comparison with the linearized equations

Equation (98.118) is exactly equivalent to the non-linear Navier-Stokes equations. There are no approximations. There are no limits on how large ΔU, etc. can be. Although the equations appear to be linear, the equations are non-linear because the matrices contain variables other than the background variables. However, they are quasi-linear (because they are linear in the derivatives). Also, that the matrices are symmetric in the derivative operators $\hat{\omega}_0$ and $\hat{\mathbf{k}}$, exhibits that the equations are symmetric hyperbolic, which implies that Cauchy data will be propagated causally[41, 296, Courant and Hilbert, 1962; Garabedian, 1964].

It is useful, however, to compare (98.118) with the corresponding equations of the linear system. For that purpose, we shall compare with equations (6) and (7) in the paper by Jones (2005)[289]. These are

$$\mathbf{M}\begin{pmatrix} \rho^{1/2}u \\ \rho^{1/2}v \\ \rho^{1/2}w \\ \rho^{-1/2}\delta\rho_{pot} \\ \rho^{-1/2}\delta p \end{pmatrix} = 0\ , \tag{98.121}$$

where $\mathbf{M}$ is the 5×5 matrix given by

$$\begin{aligned}\mathbf{M} = &\begin{pmatrix} -\hat{\omega} & 2i\tilde{\Omega}_z & -2i\tilde{\Omega}_y & 0 & \hat{k}_x \\ -2i\tilde{\Omega}_z & -\hat{\omega} & 2i\tilde{\Omega}_x & 0 & \hat{k}_y \\ 2i\tilde{\Omega}_y & -2i\tilde{\Omega}_x & -\hat{\omega} & 0 & \hat{k}_z \\ 0 & 0 & 0 & -\hat{\omega} & 0 \\ \hat{k}_x & \hat{k}_y & \hat{k}_z & 0 & -\hat{\omega}/C^2 \end{pmatrix} \\ &+\begin{pmatrix} -ie_{xx} & -ie_{xy} & -ie_{xz} & i\tilde{g}_x & -i\Gamma_x \\ -ie_{yx} & -ie_{yy} & -ie_{yz} & i\tilde{g}_y & -i\Gamma_y \\ -ie_{zx} & -ie_{zy} & -ie_{zz} & i\tilde{g}_z & -i\Gamma_z \\ -\frac{i}{\rho}\frac{\partial\rho_{pot}}{\partial x} & -\frac{i}{\rho}\frac{\partial\rho_{pot}}{\partial y} & -\frac{i}{\rho}\frac{\partial\rho_{pot}}{\partial z} & 0 & 0 \\ i\Gamma_x & i\Gamma_y & i\Gamma_z & 0 & 0 \end{pmatrix} \\ &+\frac{1}{\rho}\frac{D\rho}{Dt}\begin{pmatrix} \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & i & \frac{i\rho^{1/2}}{C^4}\left(\frac{\partial\rho^{1/2}C^2}{\partial\rho}\right)_{sS} \end{pmatrix}\ ,\end{aligned} \tag{98.122}$$

where I have changed the order of the matrices in (98.122) to make the comparison with (98.118) easier.

There are some similarities and some differences in comparing (98.118) with (98.121). As can be seen, the first two matrices in (98.118) can be compared with the first two matrices in (98.122), and the sum of the last two matrices in (98.118) are comparable with the last matrix in (98.122).

Notice first that (98.118) has six equations (rows), whereas (98.122) has only five equations (rows). That is because the former needed to include the variable s (composition), whereas the latter did not need to include composition. We shall discuss the reason for this later. Because of that difference, we shall at first neglect the fifth equation (row) in (98.118) when making comparisons.

Next, we notice that the shear tensor in the upper three rows and the left three columns of the second matrix of (98.118) has been split into its symmetric part in the second matrix in (98.122) and its antisymmetric part, which has been included in $\tilde{\mathbf{\Omega}}$ in the first matrix in (98.122). Otherwise, the first two matrices in (98.118) are the same as the first two matrices in (98.122) when we replace all of the variables in the matrices of the former with background values. When replacing by background values in the former, we see that the variable μ in the fourth column becomes equal to 1, and the variable ν in the fifth column becomes 0.

In comparing the sum of the last two matrices in (98.118) with the last matrix in (98.122), we see that the comparison shows equality (when taking all variables to be background variables) except for the terms containing α and β. Notice that if β were zero in addition to ν, the composition variable s would be uncoupled from the rest of the equations. That β (and, in general, ν) are not zero requires six equation in (98.118), whereas the absence of those terms in (98.122) allows the latter system to have only five equations.

Now, we investigate the interesting reason for this difference. The short answer is that for nonlinear differential equations, taking the difference of the linearization of two equations may not be the same as the linearization of the difference of those same two equations. Specifically, for the present case, the two equations involved are the continuity equation (98.66) and the adiabatic equation of state (98.53) or (98.58). The difference of (98.66) and (98.53) is (98.67). A linearization of (98.67) leads to the sixth equation (row) in (98.118) in the limit as the elements of the matrix approach background values. On the other hand, the fifth equation (row) in (98.122) is the difference in the linearizations of (98.66) and (98.58). We have already seen that those final results are different. Now, I shall demonstrate explicitly how that happens.

A perturbation (linearization) of (98.67) is

$$\frac{1}{C^2}\frac{D\delta p}{Dt}+\frac{1}{C^2}\delta\mathbf{U}\cdot\nabla p+\rho\nabla\cdot\delta\mathbf{U}+\delta\rho\nabla\cdot\mathbf{U}-\frac{D\rho}{Dt}\frac{\delta C^2}{C^2}=0\ . \tag{98.123}$$

A perturbation (linearization) of the adiabatic equation of state (98.58) is

$$\frac{D\delta\tilde{\rho}_{\text{pot}}}{Dt}+\delta\mathbf{U}\cdot\nabla\tilde{\rho}_{\text{pot}}=0\ . \tag{98.124}$$

A perturbation (linearization) of the continuity equation (98.66) is

$$\frac{D\delta\rho}{Dt}+\delta\mathbf{U}\cdot\nabla\rho+\rho\nabla\cdot\delta\mathbf{U}-\frac{1}{\rho}\frac{D\rho}{Dt}\delta\rho=0\ . \tag{98.125}$$

A perturbation (linearization) of (98.47) gives

$$\delta p=C^2\delta\rho+\left(\frac{\partial p}{\partial S}\right)_{\rho,s}\delta S+\left(\frac{\partial p}{\partial s}\right)_{\rho,S}\delta s\ , \tag{98.126}$$

where I have used (98.51). A perturbation (linearization) of (98.55) is

$$\delta p_a=0=C^2\delta\tilde{\rho}_{\text{pot}}+\left(\frac{\partial p}{\partial S}\right)_{\rho,s}\delta S+\left(\frac{\partial p}{\partial s}\right)_{\rho,S}\delta s\ , \tag{98.127}$$

again using (98.51). Taking the difference of (98.126) and (98.127) gives

$$\delta p=C^2\left(\delta\rho-\delta\tilde{\rho}_{\text{pot}}\right)\ , \tag{98.128}$$

A similar calculation gives

$$\nabla p=C^2\left(\nabla\rho-\nabla\tilde{\rho}_{\text{pot}}\right)\ . \tag{98.129}$$

Taking the difference of (98.125) and (98.124) is

$$\frac{D\left(\delta\rho-\delta\tilde{\rho}_{\text{pot}}\right)}{Dt}+\delta\mathbf{U}\cdot\left(\nabla\rho-\nabla\tilde{\rho}_{\text{pot}}\right)+\rho\nabla\cdot\delta\mathbf{U}-\frac{1}{\rho}\frac{D\rho}{Dt}\delta\rho=0\ . \tag{98.130}$$

Using (98.128) and (98.129) in (98.130) gives

$$\frac{D\left(\delta p/C^2\right)}{Dt}+\delta\mathbf{U}\cdot\nabla p/C^2+\rho\nabla\cdot\delta\mathbf{U}-\frac{1}{\rho}\frac{D\rho}{Dt}\delta\rho=0\ . \tag{98.131}$$

Taking the derivative indicated in (98.131) and using the continuity equation (98.66) gives

$$\frac{1}{C^2}\frac{D\delta p}{Dt}+\frac{1}{C^2}\delta\mathbf{U}\cdot\nabla p+\rho\nabla\cdot\delta\mathbf{U}+\nabla\cdot\mathbf{U}\delta\rho-\frac{\delta p}{C^4}\frac{DC^2}{Dt}=0\ . \tag{98.132}$$

Equations (98.123) and (98.132) are identical except for the last term in each. The last term in (98.123) is

$$-\frac{D\rho}{Dt}\frac{\delta C^2}{C^2} = -\frac{1}{C^2}\frac{D\rho}{Dt}\left(\alpha\delta\tilde{\rho}_{\text{pot}} + \beta\delta s + \gamma\delta p\right) = -\frac{1}{C^2}\frac{D\rho}{Dt}\left(\alpha\delta\tilde{\rho}_{\text{pot}} + \beta\delta s + \frac{1}{C^2}\left(\frac{\partial C^2}{\partial \rho}\right)_{S,s}\delta p\right) . \tag{98.133}$$

The last term in (98.132) is

$$-\frac{\delta p}{C^4}\frac{DC^2}{Dt} = -\frac{1}{C^4}\left(\frac{\partial C^2}{\partial \rho}\right)_{S,s}\frac{D\rho}{Dt}\delta p . \tag{98.134}$$

As can be seen, (98.133) and (98.134) agree except for the α and β terms, which appear in (98.133), but not in (98.134). These are exactly the terms that were in the sixth row of (98.118), but not in the fifth row of (98.122).

Thus, we have an explanation for why linearizing the nonlinear equation (98.118) did not lead to the previously linearized form (98.121) and (98.122). All linearization methods are not equivalent, although both of these two methods are valid. The linearization in (98.121) and (98.122) has the advantage that the composition variable s is absent, and the resulting 5×5 matrix leads to a simpler dispersion relation. As far as I know, there is no way to avoid bringing in the composition variable s for the full nonlinear case.

It is not clear what the significance of the extra α and β terms are. That they appear in the matrix multiplied by $D\rho/Dt$, and that that matrix is usually ignored in calculating a dispersion relation is interesting. It might be that the effects of the terms in that matrix on the dispersion relation are small because the fluid can be considered nearly incompressible for wave motion.

Before linearization, however, there are many terms in the first two matrices in (98.118) that have significant nonlinear effects, even without the last two matrices in (98.118).

98.7 Generalization to cosmology

In cosmology, the universe is treated as a perfect gas with regard to an equation of state. Therefore, the analysis here should apply to the universe and General Relativity with a few changes. The momentum equation will have to be replaced by a relativistic 4-dimensional version, and proper time will have to be introduced. There may be some other changes as well.

Chapter 99

King on the mountain[1]

abstract

General Relativity and quantum theory are the two most successful theories we have. In fact, each of these theories has universal application. General Relativity provides the background geometry upon which all of physics, including the other three fundamental interactions and quantum theory reside. Quantum theory, in turn, has general principles that apply to all interactions, including gravitation, and therefore to General Relativity. The other three fundamental interactions have been successfully quantized, but not gravitation. A quantum field has at least three properties. First, a classical particle becomes a wave. Second, a classical particle becomes a superposition of particles. Third, a classical field becomes quantized as lumps, or quanta. In the case of gravitation, this means that a classical geometry is replaced by a quantum superposition of 3-geometries, and the gravitational interaction has a smallest quantum, called a graviton. The difficulty here, is that, as far as I know, we do not know how to make a quantum theory that resides on a geometry that is itself quantized.

Thus, we are in a bind that is caused by two competing theories that both claim universality. We cannot have two kings on the mountain. If we have two kings, we have to enlarge the mountain. In this case, we do this by replacing the complex domain by either quaternions or octonions.

99.1 Introduction

General Relativity and quantum theory are the two most successful theories we have. In fact, each of these theories has universal application. General Relativity provides the background geometry upon which all of physics, including the other three fundamental interactions and quantum theory reside. Quantum theory, in turn, has general principles that apply to all interactions, including gravitation, and therefore to General Relativity. The other three fundamental interactions have been successfully quantized, but not gravitation. A quantum field has at least three properties. First, a classical particle becomes a wave. Second, a classical particle becomes a superposition of particles. Third, a classical field becomes quantized as lumps, or quanta. In the case of gravitation, this means that a classical geometry is replaced by a quantum superposition of 3-geometries, and the gravitational interaction has a smallest quantum, called a graviton. The difficulty here, is that, as far as I know, we do not know how to make a quantum theory that resides on a geometry that is itself quantized. Thus, we are in a bind that is caused by two competing theories that both claim universality. We cannot have two kings on the mountain. If we have two kings, we have to enlarge the mountain.

[1] 13 Jan 2012 - 21 September 2017, 13:10 - 29 January 2020, 14:00

This is similar to the situation with Fermat's principle, which states that the path that contributes most to a path integral has a phase that is stationary with respect to variation of the path. However, in an absorbing medium, we can equally require that the path that contributes the most has a minimum attenuation. A compromise would not satisfy either, and paths in ordinary space do not have enough parameters to satisfy both requirements. The solution is to enlarge the available number of parameters by allowing each of the coordinates to take on complex values. The resulting "ray tracing in complex space" solves the problem by a mathematical trick. A plane wave pointed in a complex direction represents an inhomogeneous plane wave.

Here, the solution is similar. We must enlarge the domain so that we have enough parameters to satisfy both requirements. That is, we must find a way that both General Relativity and quantum theory can claim universality without getting in each other's way. In this case, we do this by replacing the complex domain by either quaternions or octonions. More specifically, can we make General Relativity a theory involving quaternions or octonions?

99.2 Addition, January 2020

Now I am thinking of a different solution. Part of the problem is that we represent gravitation as geometry. Einstein's field equations for General Relativity are somewhat analogous to Maxwell's equations for electricity and magnetism. It is not absolutely necessary that we interpret gravitation as geometry. When we have a solution of the field equations, the coordinates represent coordinates in a geometry, but we do not have to interpret $g_{\mu\nu}$ as a metric, nor $R_{\mu\nu\epsilon\delta}$ as curvature. They are just fields on a manifold.

More specifically, the analogy between gravitation and electromagnetism is not complete. If we restrict our coordinate system to be fixed to the body in question, then the connection Γ is analogous to the electric field. We don't worry if something is a tensor unless we are making coordinate transformations. Also, there are no magnetic forces in a frame fixed with the body. We could say that $\Gamma_{\alpha\beta\gamma}$ is analogous to $F_{\mu\nu}$, even though in the frame of the body, only part of each will have any effect.

It is incorrect to say that gravitation is represented by geometry. Geometry exists without any gravitational sources. Let us now introduce sources of electromagnetism, that is, charges and currents. If we now consider a body with a charge and a mass, that body will accelerate. If we look in the frame of the body, the total force will be zero. That is, the electric force represented by $F_{\mu\nu}$ times the electric charge of the body will balance the gravitational force represented by $\Gamma_{\alpha\beta\gamma}$ times the mass of the body. So, we now have a gravitational field $\Gamma_{\alpha\beta\gamma}$ even though we have no sources of that field. So, it looks like geometry really does represent gravitation.

It seems now that we have the possibility of a gravitational field without a source for that field, since the stress-energy tensor is usually considered to be a source of a gravitational field. However, we are looking at the situation from an accelerated frame. In that case, we have to consider the Unruh effect. That will give an Unruh temperature. Will that give a stress-energy tensor to be the source of the gravitational field? If so, then it must be through the generation of curvature. Is there curvature in flat spacetime when observed by an accelerated observer? By the equivalence principle, acceleration is equivalent to a gravitational field, so there really is a gravitational field, so there must be a source for that field.

Continuing on, Consider we have sources of both electromagnetism and gravitation. Consider Maxwell's equations written in terms of $F_{\mu\nu}$ and Einstein's field equations written in terms of $\Gamma_{\alpha\beta\gamma}$. In both cases, these are first-order differential equations, with the latter having some additional nonlinear terms. Both forms are quasi-linear. In both cases, when we solve the equations, we determine $F_{\mu\nu}$ and $\Gamma_{\alpha\beta\gamma}$ as functions of position and time. It is only when we connect $\Gamma_{\alpha\beta\gamma}$ with the metric that we think of gravitation as being represented by geometry.

Chapter 100

Myths in physics[1]

abstract

There are several myths in physics that lead to misconceptions.

100.1 Introduction

There are several myths in physics that lead to misconceptions.

1. Gravitation is not important in quantum mechanics.
2. Laws of physics should involve only observables.

100.2 Gravitation is not important in quantum mechanics.

What people really mean when they say that gravitation is not important in quantum mechanics is that the scalar gravitational field is not important in atomic physics because the gravitational attraction of a proton and an electron is 40 orders of magnitude below the electric attraction between them. However, we have known since Einstein's General Relativity that inertia is a gravitational force,[2] and it is inertia that exactly balances the electrical attraction between an electron and a proton.

The problem is that inertia does not enter directly in quantum theory as a force as the electromagnetic force does. Instead, it is hidden in the wave properties of particles. De Broglie wavelength is put in as a gradient term that hides the inertia behind that term.

100.3 Laws of physics should involve only observables.

Saying that the Laws of physics should involve only observables sounds good, but there are good reasons to ignore that advice. Often including non-observables simplifies the theory. EM potentials give one example. Another example is Kepler's laws. In assuming elliptical orbits for planetary orbits, he assumed that the planets moved in orbits that involved changing their distance from the

[1] 21 Feb 2012

[2] Inertia is contained in the geodesic equation, which follows from Einstein's field equations for General Relativity. In General Relativity, at least, inertia is a gravitational force, since General Relativity is a theory of gravitation. There are theories, such as Newton's theory, for which inertia is not considered a gravitational force, but such theories are no longer considered seriously.

Earth, even though there were (at that time) no observations that could validate that.[3] The only observables were two-dimensional position in the sky. A theory based on only those two observables (and time) would have been very complicated.

[3]Although Kepler knew of Copernicus's view of a sun-centered solar system, that was also based on the assumption that the sun and planets moved in a 3-dimensional space. There were no measurements (at that time) to verify the assumed changes in distance between the Earth and the various planets.

Chapter 101

Renormalization[1]

abstract

Renormalization gives accurate and correct calculations by re-normalizing the mass and charge of particles such as the electron. However, the corrections are infinite, which means that the bare mass and bare charge of the electron must be infinite. It is extremely unlikely that that is the actual case, so even though renormalization is extremely successful, it is unlikely to be a correct theory. What might be wrong? Most likely is the assumption that inertia and inertial frames are intrinsic rather than being determined by an interaction with the bulk of matter in the universe. Hopefully, taking that into account may lead to an improved method for doing renormalization. Propagators for particles in inertial frames have to be replaced by primitive propagators coupled by graviton propagators with every other mass in the universe.

101.1 Introduction

Renormalization gives accurate and correct calculations by re-normalizing the mass and charge of particles such as the electron. However, the corrections are infinite, which means that the bare mass and bare charge of the electron must be infinite. It is extremely unlikely that that is the actual case, so even though renormalization is extremely successful, it is unlikely to be a correct theory. What might be wrong? Most likely is the assumption that inertia and inertial frames are intrinsic rather than being determined by an interaction with the bulk of matter in the universe. Hopefully, taking that into account may lead to an improved method for doing renormalization.

101.2 "Dressed," "bare," and "primitive" propagators

Particle propagators, including graviton propagators, are altered during renormalization. In renormalization, "dressed" mass, charge, and propagator refer to the mass, charge, and propagator that we observe, and that include the effects of vacuum fluctuations. "Bare" mass, charge, and propagator refer to those quantities before taking into account vacuum fluctuations. However, both "dressed" and "bare" include interaction with all of the matter in the universe that produces inertia and inertial frames. We need a terminology for mass, charge, and propagator in the absence of the interaction with all of the matter in the universe that produces inertia and inertial frames. So as not to change the terminology already in use, I will call these "primitive" mass, charge, and propagator.

[1] 13 - 19 Mar 2012

Because we do not know what the primitive mass, charge, and propagator are, it seems that we should simply write out the correct Feynman diagrams, add them up, and determine what the primitive elements should be to give the known dressed elements.

Chapter 102

Phase interference can solve the rotation problem[1]

abstract

A semi-classical approximation to quantum gravity is used to explain why inertial frames do not seem to rotate relative to the average matter distribution in the universe by showing that wave interference would cancel out cosmologies with significant relative rotation of the average inertial frame and matter. Specific calculations, using various Bianchi models[2] show that only cosmologies with a present relative rotation of the average inertial frame and matter smaller than $(2\hbar\sqrt{3(\Lambda+8\pi\rho)}/r^3)^{1/2} \approx 2\times10^{-71}$ radians per year[3] would contribute significantly to a measurement of relative rotation rate, where Λ is the cosmological constant, ρ is the present density of matter (including dark matter) in the universe, and r is the present radius of the visible universe, which we take to be the Hubble radius. It may be possible to extend this estimate to more general cases in the future.

102.1 Introduction

The "Rotation Problem" is to explain "If the universe can rotate, why does it rotate so slowly?"[298]. It has long been known that inertial frames seem not to rotate with respect to the visible stars. It can easily be seen by comparing the rotation of the plane of a Foucault pendulum with the movement of the stars relative to the Earth. However, investigation of the possible rotation of inertial frames with respect to matter has a long history, and more accurate estimates of this effect are now available, mostly because of the availability of isotropy measurements on the cosmic microwave background radiation.

One example was given by Hawking[164], who used measurements on the cosmic microwave background radiation to put limits on the rotation of the universe. Specifically, he showed that if the universe contains a large-scale homogeneous vorticity, then the rotation rate corresponding to that vorticity cannot be larger than somewhere between 7×10^{-17} rad yr^{-1} and 10^{-14} rad yr^{-1} if the universe is closed and about 2×10^{-46}/(present density in g cm^{-3}) if it is open.

[1] This is the version I was revising to re-submit somewhere. I rewrote it according to the footnotes in the following pages. Chapter 106 has the revised published version.

[2] Bianchi $\mathrm{VI_h}$ model

[3] Using a simple formula for the volume in the action that ignores regions that could not possibly contribute to local physics based on causality. Taking the volume to be infinite would give exactly zero for the relative rotation of matter and inertial frames.

Wolfe[299] uses a global analogy with the Compton-Getting effect to place limiting values on the shear and vorticity of the same order as those set by the isotropy of the microwave background (over a small solid angle) if intergalactic hydrogen now exists at densities $> 10^{-6}\mathrm{cm}^{-3}$ for the shear model, and $> 10^{-7}\mathrm{cm}^{-3}$ for the vorticity model.

Ellis[300, 301] uses limits on the anisotropy of the microwave background radiation to set limits on the anisotropy and inhomogeneity of the universe (specifically shear and vorticity, based on references to previous work) of $\sigma/H < 3 \times 10^{-3}$ and $\omega/H < 3 \times 10^{-3}$, where H is the Hubble parameter.

Collins and Hawking[302] show that "the set of spatially homogeneous cosmological models which approach isotropy at infinite times is of measure zero in the space of all spatially homogeneous models" and "It therefore seems that there is only a small set of initial conditions that would give rise to universe models which would be isotropic to within the observed limits at the present time." Their solution is to invoke the "anthropic principle," namely that our existence requires the observed isotropy.

Collins and Hawking[303] also base their analysis on vorticity, showing "If the universe is closed, ... it is rotating at a rate of less than 3×10^{-11} second of arc/century if the microwave background was last scattered at a redshift of about 7 and less than 2×10^{-14} second of arc/century if the last scattering was at a redshift of 1000."

Fennelly [304] shows that number-count relations can be affected by vorticity.

Bayin and Cooperstock[305] find perturbations to Friedmann universes to include rotation.[4]

Raine and Thomas [306] show that the inhomgeneous source free shear cannot contribute significantly to the observed microwave anisotropy.

Barrow et al.[307] find that current (in 1985) observations of the microwave background permit $(\omega/H)_0$ to be no larger than 3.9×10^{-13} if the universe is closed with $\Omega_0 < 2$, no larger than 1.9×10^{-5} if it is flat ($\Omega_0 = 1$) and no larger than 10^{-4} if it is open ($\Omega_0 \geq 0.05$).

Ellis and Wainwright[308] give limits on the shear of $\sigma/H < 10^{-9}$ or $\sigma/H < 3 \times 10^{-8}$, where H is the Hubble parameter, depending on the model.

Jaffe et al. 2005[309] give limits on the vorticity of $\omega/H = 6.1 \times 10^{-10}$ by fitting WMAP data to a Bianchi VII$_h$ model (where H is the Hubble parameter). This corresponds to a rotation rate of about 4.5×10^{-20} radians per year. The same team[310] in a later paper, point out "that the shear and vorticity values in our original paper (Jaffe et al. 2005) contain an error in amplitude", although their corrected value is not much different.

Ellis[311] gives limits on the shear and vorticity of $\sigma/H < 10^{-9}$ and $\omega/H < 10^{-6}$, respectively, based on isotropy of the cosmic microwave background (CMB) combined with measurements of chemical element abundances.

Su and Chu [312] compare the 2nd-order Sachs-Wolfe effect due to rotation with the CMBA data to constrain the angular speed of the rotation to be less than 10^{-9} rad yr^{-1} at the last scattering surface.

There have been various possible explanations of the Rotation problem.

Ellis and Olive[298] argue that inflation would lead to very small shear and rotation rates for our present universe.

Braccesi[313] argues that inflation would lead to even smaller rotation rates, with a minimum rotational period as long as $10^{135\sim155}$ years for the present universe.

Although not an explanation of the Rotation Problem, a related consideration is about the origin of inertia in a hypothetical empty universe.

Donald Lynden-Bell [314] points out that "There are many exact solutions of Einstein's equations that do not obey Mach's Principle, the simplest being Minkowski space in which nongravita-

[4] I need to be more specific about what is in this reference.

tional dynamics is replete with inertia despite there being no gravitational-inertial interaction to cause it."

Unless we have absolute space (as proposed by Newton), it is difficult to explain the absence of relative rotation classically without assuming very finely tuned initial conditions for the universe because there are many solutions of Einstein's field equations for General Relativity that have large-scale rotation of matter and inertial frames. Although the relative rotation of an expanding universe decreases with time, so the relative rotation in a cosmology would have been much greater in the distant past than it would be today, allowing for arbitrary initial conditions for the relative rotation would still allow a nearly arbitrary relative rotation today. In General Relativity, gravitation (including inertia) (as expressed by the metric tensor) is determined not only by the distribution of matter (in terms of the stress-energy tensor), but also by initial and boundary conditions. Although gravitation is sometimes equated to curvature of the geometry, inertia (which is a gravitational force) exists in flat Minkowski space.

Ernst Mach suggested[120, 102, 1, 121, 122, 15] that inertia might be determined by distant matter. Various versions of that proposal have come to be known as Mach's principle. Since we now know (from General Relativity) that inertia is a gravitational force, such an implementation of Mach's principle would require that the gravitational field (or at least part of it) be determined only by its sources (matter) rather than having independent degrees of freedom (in terms of initial and boundary conditions). If the many proposals to implement exactly that for General Relativity[11, 16, 156, 109, 315, 159] were correct, then gravitation would behave very differently from the electromagnetic interaction, in which electric and magnetic fields are determined not only from sources (charges and currents), but also from initial and boundary conditions.

To explain the degree of non-rotation of the universe, without accepting absolute space, without requiring very finely tuned initial or boundary conditions, and without giving up on independent degrees of freedom for the gravitational field, we are forced to give up on a classical explanation and turn to quantum cosmology. (Although the above motivation can be criticized, the calculation to follow stands on its own.)

It is well known that we do not have a theory of quantum gravity, and therefore, no theory of quantum cosmology. However, we have some ideas on how such a theory should look. Specifically, we have a rough idea for what the action should be for a theory of quantum gravity. If the action dominates such a calculation (as we expect), it might be possible to calculate the upper bounds on the rotation of the universe in terms of a calculation using a semi-classical approximation to quantum cosmology, such as a sum-over-histories [316, 165, 317, 318, 319], decoherent histories[320, 321, 322, 323, 324, 325, 326, 327], or consistent histories[328, 329, 330, 331] approach.

Section 102.2 sets up a calculation of the amplitude. Section 102.3 calculates the action for a perfect fluid. Section 102.5 considers the special case of a Bianchi VI_h model, which has relative rotation of matter and inertial frames. Section 102.6 discusses the radiation, matter, and dark-energy eras. Section 102.4 gives the generalized Friedmann equation and approximations for small and large relative rotation. Section 102.7 makes the saddlepoint approximation to the integral. Section 102.8 discusses the validity of the saddlepoint approximation. Section 102.9 discusses possible problems. Appendix 102.10 estimates the action for the Bianchi VI_h model in the various eras.

102.2 Amplitude for measuring a rotation of the universe

Hawking[164] used a Bianchi IX model to represent a closed universe and a Bianchi V model to represent an open universe to estimate vorticity, so I shall use those plus some others.[5]

[5]So far, I have used only a Bianchi $\mathrm{VI_h}$ model.

Let $\psi_{\omega_0}(\Omega)$ be the amplitude for measuring relative rotation[6] Ω given that the universe is characterized by the parameter ω_0, in which relative rotation (which may be a function of time) is proportional to ω_0. Also, let $\Psi(\omega_0)$ be the amplitude that our universe is characterized by the parameter ω_0.

The amplitude for measuring the relative rotation to be Ω is[7]

$$\psi(\Omega) = \int_{-\infty}^{\infty} \psi_{\omega_0}(\Omega)\Psi(\omega_0)\,\mathrm{d}\omega_0 \ . \tag{102.1}$$

In this calculation, we take the universe that is characterized by the parameter ω_0 to be a classical cosmology because a cosmology that is not a solution of Einstein's field equations would not contribute significantly to the integral [220, 221]. The parameter ω_0, classically determined by initial conditions, represents an independent degree-of-freedom of the gravitational field. Although in general, we should integrate over all parameters that specify initial conditions for the model in (102.1), here, we are interested only in relative rotation. Halliwell[332] considers more general path integrals.

Actually, taking the ω_0 integration from $-\infty$ to ∞ in (102.1) is not physically realistic. The largest relative rotation that could possibly be considered without having a theory of quantum gravity would be one rotation of the universe in the Planck time. This would correspond to taking the maximum value of ω_0 to be the reciprocal of the Planck length, L^*, or $\omega_{\mathrm{max}} = 0.62 \times 10^{33}\mathrm{cm}^{-1}$, which is close enough to infinity for our purposes. Thus, we can write (102.1) as[8]

$$\psi(\Omega) = \int_{-\omega_{\mathrm{max}}}^{\omega_{\mathrm{max}}} \psi_{\omega_0}(\Omega)\Psi(\omega_0)\,\mathrm{d}\omega_0 \ . \tag{102.2}$$

We anticipate that the properties of $\Psi(\omega_0)$ will dominate the integral in (102.2), so we shall start with that. We can express that as

$$\Psi(\omega_0) \approx e^{iI/\hbar} \ , \tag{102.3}$$

where I is the action.

Section 102.9.3 considers the effect of using a Euclidean path integral in (102.3) instead of a Lorentzian path integral. The path of integration is moved into the complex plane to pass through the appropriate saddlepoints in either a stationary-phase path or a steepest-descent path [134, 333, 219]. Although there should be a factor multiplying the exponential in (102.3) that is a slowly varying function ω_0, including it would not have a significant effect on the final result, and a correct theory of quantum gravity would be necessary to provide that factor.

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[6] Specifically, "the local angular velocity, in the restframe of an observer of a set of Fermi-propagated axes with respect to a triad of orthonormal spacelike vectors spanning the tangent plane to the surfaces of transitivity."

[7] I need to consider the initial wave function of the universe.

[8] I can make the initial wave function of the universe broad, but still nearly zero at $\pm\omega_{\mathrm{max}}$. We can represent a quantum cosmology as either the amplitude for a given 3-geometry that changes with time or as the amplitude for a given 4-geometry. In either case, we must construct the amplitude for a given 4-geometry to make calculations. To do either, we start with an initial wave function that gives the amplitude for initial conditions.

In general, we allow arbitrary 4-geometries, not just classical 4-geometries. However, to get the main contribution, we make a saddlepoint approximation to the g^{ij} integral by setting $\delta g^{ij} = 0$. In the case of General Relativity, that leads to Einstein's field equations. In the case of other gravitational theories, based on different Lagrangians, that leads to different field equations. In either case, we can restrict our calculations to classical 4-geometries. The parameters for the 4-geometries then become initial conditions for those 4-geometries, and we assume an initial wave function (amplitude) for each of those parameters. We then integrate over those parameters to get the amplitude for a given measurement.

102.3 Calculating the action for a perfect fluid

MacCallum and Taub [334] and Matzner[335] give formulas for the Lagrangian.

We can take the action in (102.3) to be

$$I = \int (-g^{(4)})^{1/2}(L_{geom} + L_{matter})d^4x + \frac{1}{8\pi}\int (g^{(3)})^{1/2}Kd^3x \ , \tag{102.4}$$

where [183, 123] show the importance of the surface term. Hawking[123] also points out a potential problem in that the action can be changed by conformal transformations, but suggests a solution. The quantity

$$K = g^{(3)ij}K_{ij} \tag{102.5}$$

is the trace of the extrinsic curvature. The extrinsic curvature is given by

$$K_{ij} = -\frac{1}{2}\frac{\partial g^{(3)}_{ij}}{\partial t} \ , \tag{102.6}$$

where $g^{(3)}_{ij}$ is the 3-metric. In this example, we take the Lagrangian for the geometry as

$$L_{geom} = \frac{R^{(4)} - 2\Lambda}{16\pi} \ , \tag{102.7}$$

where $R^{(4)}$ is the four-dimensional scalar curvature and Λ is the cosmological constant.

For a perfect fluid, the energy momentum tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \ , \tag{102.8}$$

where p is the pressure, ρ is the density, and u is the 4-velocity. For solutions to Einstein's field equations for a perfect fluid, (102.7) becomes

$$L_{geom} = \frac{1}{2}\rho - \frac{3}{2}p + \frac{\Lambda}{8\pi} \ , \tag{102.9}$$

and we can take the Lagrangian for the matter as

$$L_{matter} = \rho + \alpha(p - \rho) \ , \tag{102.10}$$

where α is a constant, and we can take $\alpha = 0$ [161] or $\alpha = 1$ [334, 162] or $\alpha = \frac{3}{2}$ [20, combining (102.8) with (21.33a) in MTW]. However, as we shall see, the exact formula for the matter Lagrangian does not significantly alter the final result.

102.4 Generalized Friedmann equation

I am using [300, 301, 336, 337, 338, 339, 340, 341, 342] to help me derive a useful version of a generalized Friedmann equation from the Raychaudhuri equation.

To derive a generalization of the Friedmann equation that includes relative rotation of matter and inertial frames (specifically, shear and vorticity), we start with the Raychaudhuri equation[343, 344] [345, eq. (1.3.4)][346, eq. (36)][300, 301, eq. (4.12)][336, 347, 348, 349, 338, 339] or Raychaudhuri-Ehlers equation[342, eq. (6.4)].

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a} + 4\pi(\rho + 3p) - \Lambda = 0 \ , \tag{102.11}$$

where[300, 301]

$$\theta = u^a{}_{;a} \tag{102.12}$$

is the volume expansion,

$$\sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab} \tag{102.13}$$

is the square of the shear,

$$\omega^2 = \frac{1}{2}\omega^{ab}\omega_{ab} \tag{102.14}$$

is the square of the vorticity,

$$\dot{u}_a = u_{a;c}u^c \tag{102.15}$$

is the acceleration,

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta h_{ab} \tag{102.16}$$

is the shear tensor,

$$\omega_{ab} = h_a{}^b h_b{}^d u_{[c;d]} \tag{102.17}$$

is the vorticity tensor, where [ ] indicates anti-symmetrization,

$$\theta_{ab} = h_a{}^b h_b{}^d u_{(c;d)} \ , \tag{102.18}$$

where ( ) indicates symmetrization,

$$h_a{}^b = \delta_a{}^b + u_a u^b \tag{102.19}$$

is a projection tensor into the rest space of an observer moving with 4-velocity u^a, ρ is density, p is pressure, and Λ is the cosmological constant. If we define a representative length scale by

$$\frac{\dot{\ell}}{\ell} = \frac{1}{3}\theta \ , \tag{102.20}$$

then we can write the Raychaudhuri equation (102.11) as [345, eq. (2.2.9)][346, eq. (81)][300, 301, eq. (4.13)][342, eq. (6.5)]

$$3\frac{\ddot{\ell}}{\ell} + 2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a} + 4\pi(\rho + 3p) - \Lambda = 0 \ . \tag{102.21}$$

If we multiply (102.21) by $2\ell\dot{\ell}$, then we find that the Raychaudhuri equation can be written as

$$(3\dot{\ell}^2 - \Lambda\ell^2)^{\cdot} + 2\ell\dot{\ell}[2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a}] + 8\pi\ell\dot{\ell}(\rho + 3p) = 0 \ . \tag{102.22}$$

For a perfect fluid, the energy conservation equation can be written as [342, equation (5.38)]

$$\dot{\rho} + (\rho + p)\theta = 0 \ . \tag{102.23}$$

Using (102.20) and multiplying (102.23) by ℓ^2 gives

$$(\ell^2\rho)^{\cdot} + \ell\dot{\ell}(\rho + 3p) = 0 \ . \tag{102.24}$$

Substituting (102.24) into (102.22) gives

$$(3\dot{\ell}^2 - \Lambda\ell^2 - 8\pi\rho\ell^2)^{\cdot} + 2\ell\dot{\ell}[2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a}] = 0 \ . \tag{102.25}$$

The second of the two major terms represents extensions to the isotropic FLRW case. We can neglect the acceleration term to give[9]

$$(3\dot{\ell}^2 - \Lambda\ell^2 - 8\pi\rho\ell^2)^{\cdot} + 4\ell\dot{\ell}(\sigma^2 - \omega^2) = 0 \ . \tag{102.26}$$

Integrating (102.26) with respect to time gives

$$3\dot{\ell}^2 - \Lambda\ell^2 - 8\pi\rho\ell^2 + 4\int \ell(\sigma^2 - \omega^2)\,\mathrm{d}\ell = \text{ constant.} \tag{102.27}$$

Rearranging gives a generalized Friedmann equation

$$\frac{\mathrm{d}\,\ell}{\mathrm{d}\,t} = \dot{\ell} = \ell\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4}{3\ell^2}\left(\int \ell(\omega^2 - \sigma^2)\,\mathrm{d}\ell + \text{ constant }\right)} \ . \tag{102.28}$$

In the radiation era ($\ell < \ell_2$), we can take [300, 301][342, Table 6.1] $\omega \propto \ell^{-1}$. That is, we can let[10]

$$\omega = \left(\frac{\ell}{\ell_2}\right)^{-1} \omega_2 \ . \tag{102.29}$$

Substituting (102.29) into (102.28) and choosing constants of integration in a useful way gives

$$\frac{\mathrm{d}\,\ell}{\mathrm{d}\,t} = \dot{\ell} = \ell\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{\ell^2} - \left(\frac{2}{3} + 4\ln\frac{\ell_2}{\ell}\right)\left(\frac{\ell_2}{\ell}\right)^2 \omega_2^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell} \tag{102.30}$$

for the radiation era, where $k = +1$ for a closed universe, $k = -1$ for an open universe, and $k = 0$ for a flat universe.

In the matter-dominant era ($\ell > \ell_2$), we can take [300, 301][342, Table 6.1] $\omega \propto \ell^{-2}$. That is, we can let[11]

$$\omega = \left(\frac{\ell}{\ell_2}\right)^{-2} \omega_2 \ . \tag{102.31}$$

Substituting (102.31) into (102.28) and choosing constants of integration in a useful way gives

$$\frac{\mathrm{d}\,\ell}{\mathrm{d}\,t} = \dot{\ell} = \ell\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{\ell^2} - \frac{2}{3}\left(\frac{\ell_2}{\ell}\right)^4 \omega_2^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell} \tag{102.32}$$

for the matter-dominant era.

At this point, we change variables from ℓ to r.

$$\omega = \left(\frac{r_2}{r}\right)\omega_2 \text{ for } r \le r_2 \ . \tag{102.33}$$

$$\frac{\mathrm{d}\,r}{\mathrm{d}\,t} = \dot{r} = r\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{r^2} - \left(\frac{2}{3} + 4\ln\frac{r_2}{r}\right)\left(\frac{r_2}{r}\right)^2 \omega_2^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell} \text{ for } r \le r_2 \ . \tag{102.34}$$

$$\omega = \left(\frac{r_2}{r}\right)^2 \omega_2 \text{ for } r > r_2 \ . \tag{102.35}$$

$$\frac{\mathrm{d}\,r}{\mathrm{d}\,t} = \dot{r} = r\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{r^2} - \frac{2}{3}\left(\frac{r_2}{r}\right)^4 \omega_2^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell} \text{ for } r > r_2 \ . \tag{102.36}$$

[9] Actually, I should consider models for which the acceleration is zero. I do not want to make any approximations here. However, I may not want to be so restrictive in the models I consider.

[10] This is an approximation. I probably do not want to use this approximation here.

[11] This is an approximation. I probably do not want to use this approximation here.

The smallest size of the visible part of the universe we consider is r_0. Therefore, from (102.33), we have

$$\omega_0 = \left(\frac{r_2}{r_0}\right)\omega_2 \ , \tag{102.37}$$

so that

$$\omega_2 = \left(\frac{r_0}{r_2}\right)\omega_0 \ . \tag{102.38}$$

Substituting (102.38) into (102.34) gives

$$\frac{\mathrm{d}\,r}{\mathrm{d}\,t} = \dot{r} = r\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{r^2} - \left(\frac{2}{3} + 4\ln\frac{r_2}{r}\right)\left(\frac{r_0}{r}\right)^2\omega_0^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell} \text{ for } r \le r_2 \ . \tag{102.39}$$

Substituting (102.38) into (102.36) gives

$$\frac{\mathrm{d}\,r}{\mathrm{d}\,t} = \dot{r} = r\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{r^2} - \frac{2}{3}\left(\frac{r_0^2 r_2^2}{r^4}\right)\omega_0^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell} \text{ for } r > r_2 \ . \tag{102.40}$$

ω_0 is the initial rotation corresponding to the initial vorticity when the universe has the size r_0. It is an initial condition for the universe. To treat (102.39) and (102.40) in a uniform way, we write (102.39) and (102.40) as

$$\frac{\mathrm{d}\,r}{\mathrm{d}\,t} = \dot{r} = r\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{r^2} - f(r)\omega_0^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell} \ , \tag{102.41}$$

where

$$f(r) = \left(\frac{2}{3} + 4\ln\frac{r_2}{r}\right)\left(\frac{r_0}{r}\right)^2 \text{ for } r \le r_2 \ , \tag{102.42}$$

and

$$f(r) = \frac{2}{3}\left(\frac{r_0^2 r_2^2}{r^4}\right) \text{ for } r > r_2 \ . \tag{102.43}$$

The radical in (102.41) will appear in the denominator of an integrand. There are two cases to consider for that radical , depending on whether the ω_0^2 (relative rotation) term is small or large relative to the rest of the radicand.

When the relative rotation term is small, we can expand the radical for small ω_0^2.[12]

$$\begin{aligned}\frac{1}{\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{r^2} - f(r)\omega_0^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell}} &\approx \pm\sqrt{\frac{3}{\Lambda + 8\pi\rho - \frac{3k}{r^2}}}\left(1 + \frac{3f(r)\omega_0^2}{2(\Lambda + 8\pi\rho - \frac{3k}{r^2})}\right) \\ &= \pm\sqrt{\frac{3}{\Lambda + 8\pi\rho - \frac{3k}{r^2}}} \pm \frac{3\sqrt{3}f(r)\omega_0^2}{2(\Lambda + 8\pi\rho - \frac{3k}{r^2})^{3/2}} \ ,\end{aligned} \tag{102.44}$$

where the $\pm$ refers to the two Riemann sheets. We shall take the upper sign for the physical Riemann sheet.

When we cannot make the approximation that ω_0^2 is small, we take the ω_0^2 term to be the dominant term. Then, we take[13]

$$\frac{1}{\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{r^2} - f(r)\omega_0^2 - \frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell}} \approx \pm\frac{i}{\sqrt{f(r)}|\omega_0|} \ , \tag{102.45}$$

where the $\pm$ refers to the two Riemann sheets. We shall take the upper sign for the physical Riemann sheet.

[12] I do not need to make these approximations. I can find the saddlepoint and calculate the WKB approximation without these approximations.

[13] I do not need to make these approximations. I can find the saddlepoint and calculate the WKB approximation without these approximations.

102.5 Bianchi models

Ellis and MacCallum[163] describe several Bianchi models, some of which include relative rotation. Those models that have no relative rotation also have no shear. Those that have relative rotation also have shear such that $\sigma^2 - \omega^2 = 0$. Therefore, for those models, there is to contribution of vorticity or shear to the generalized Friedmann equation (102.28). This will also mean no contribution of vorticity or shear to the action. Thus, we must consider different models.

Barrow and others [307] consider the Bianchi VII model, for which $\sigma^2 - \omega^2 \neq 0$.

The metric used by Ellis and MacCallum[163] is given by their equation (6.6) and the field equations for that model by their equations (6.7). MacCallum[191] gives a correction to the last term (the relative rotation term) in equation (6.7b) in the paper by Ellis and MacCallum[163]. Using this model, we can calculate the surface term in (102.4).

MacCallum, Stewart, and Schmidt[350] give some corrections to [163].

From (102.6) and [163, eq. 6.6], we have

$$K \equiv K^i_i = -\frac{1}{2} g^{(3)ij} \frac{\partial g^{(3)}_{ij}}{\partial t} = -\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z} = -3\frac{\dot{r}}{r} \ , \tag{102.46}$$

where r in (102.46) is defined by

$$\left[\frac{r(t)}{r_3}\right]^3 \equiv X(t)Y(t)Z(t) \ , \tag{102.47}$$

r_3 is an arbitrary constant, which we take to be the present radius of the visible universe, and $X(t)$, $Y(t)$, and $Z(t)$ are variables in the model. This gives

$$r^3 K = -3r^2\dot{r} \ , \tag{102.48}$$

so that

$$K = -\frac{3}{r^3} \int (r^2\dot{r})^{\cdot} dt \ . \tag{102.49}$$

Using

$$\Lambda + 4\pi(\rho - p) = r^{-3}(r^2\dot{r})^{\cdot} - \frac{2}{3}\frac{3a_0^2 + q_0^2}{X^2} \ , \tag{102.50}$$

(which comes from adding together the last three equations in [163, eq. 6.7a], where a_0 and q_0 are parameters of the model) gives

$$K = -\frac{3}{r^3} \int r^3 \left[\Lambda + 4\pi(\rho - p) + \frac{2}{3}\frac{3a_0^2 + q_0^2}{X^2}\right] dt \ . \tag{102.51}$$

Equation (102.51) allows us to convert the surface integral in (102.4) to a volume integral so that it can be combined with the volume integral in the first term.

So, using $-g^{(4)} = g^{(3)}$ gives

$$I = \int (-g^{(4)})^{1/2} \left[L_{geom} + L_{matter} - \frac{3\Lambda}{8\pi} - \frac{3}{2}(\rho - p) - \frac{3a_0^2 + q_0^2}{4\pi X^2}\right] d^4x \ . \tag{102.52}$$

Using (102.9) and (102.10) gives

$$I = \int (-g^{(4)})^{1/2} \left[\alpha(p - \rho) - \frac{\Lambda}{4\pi} - \frac{3a_0^2 + q_0^2}{4\pi X^2}\right] d^4x \ . \tag{102.53}$$

From (102.6) and [163, eq. 6.6], we have

$$-g^{(4)} = g^{(3)} = X^2Y^2Z^2e^{-4a_0x^1} = r_3^{-6}r^6e^{-4a_0x^1} \ . \tag{102.54}$$

Thus,

$$I = \int\int e^{-2a_0x^1}d^3xr_3^{-3}r^3\left[\alpha(p-\rho) - \frac{\Lambda}{4\pi} - \frac{3a_0^2+q_0^2}{4\pi X^2}\right]dt \ . \tag{102.55}$$

Or,

$$I = \int V(t)r_3^{-3}r^3\left[\alpha(p-\rho) - \frac{\Lambda}{4\pi} - \frac{3a_0^2+q_0^2}{4\pi X^2}\right]dt \ , \tag{102.56}$$

where $V(t)$ is the spatial volume. The question whether V should be infinite because this is an open cosmology [351] is considered in section 102.9.2.

The Friedmann equation for this model, given by equation (3.5) in Appendix II of [163], is

$$\Lambda+8\pi\rho+3a^2+q^2+\sigma^2 = \theta_1\theta_2+\theta_2\theta_3+\theta_1\theta_3 = \frac{1}{3}(\theta_1+\theta_2+\theta_3)^2-\frac{1}{6}(\theta_1-\theta_2)^2-\frac{1}{6}(\theta_1-\theta_3)^2-\frac{1}{6}(\theta_3-\theta_2)^2 \ , \tag{102.57}$$

where σ is the magnitude of the relative rotation of matter and inertial frames, $\theta_1 = \dot{X}/X$, $\theta_2 = \dot{Y}/Y$, and $\theta_3 = \dot{Z}/Z$. Using (102.46) gives

$$\Lambda + 8\pi\rho + 3a^2 + q^2 + \sigma^2 = 3\left(\frac{\dot{r}}{r}\right)^2 - \frac{1}{6}(\theta_1-\theta_2)^2 - \frac{1}{6}(\theta_1-\theta_3)^2 - \frac{1}{6}(\theta_3-\theta_2)^2 \ . \tag{102.58}$$

Thus, we have[14]

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{r} = r\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{3a_0^2+q_0^2}{3r^2}r_3^2 + \frac{\sigma^2}{3} + \frac{1}{18}(\theta_1-\theta_2)^2 + \frac{1}{18}(\theta_1-\theta_3)^2 + \frac{1}{18}(\theta_3-\theta_2)^2} \ . \tag{102.59}$$

The final three terms in the radical in (102.59) represent anisotropies of the model that are unrelated to relative rotation. Since we are not interested in those anisotropies, we set them to zero. This allows us to take $X(t) = Y(t) = Z(t) = r(t)/r_3$. In this model, we must have $q_0 = -3a_0$ whenever the relative rotation σ is not zero. In addition, we have that

$$\sigma = \sigma_{12} = \frac{b}{Y^2Z} = \frac{br_3^3}{r^3} \ , \tag{102.60}$$

where b is a parameter of the model. Thus, (102.59) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{r} = r\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4a_0^2r_3^2}{r^2} + \frac{b^2r_3^6}{3r^6}} \ , \tag{102.61}$$

giving the usual Friedmann equation for an open universe with a relative rotation term added. The third term under the radical (proportional to a_0^2) is the curvature term. Present measurements indicate that the curvature of our universe is zero as far as we can tell. The calculations to follow neglect that term when evaluating formulas.

We can change integration variables in (102.56) to give

$$I = \int V(r)r_3^{-3}r^2\frac{\alpha(p-\rho) - \frac{\Lambda}{4\pi} - \frac{3a_0^2r_3^2}{\pi r^2}}{\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4a_0^2r_3^2}{r^2} + \frac{b^2r_3^6}{3r^6}}}\,\mathrm{d}r \ . \tag{102.62}$$

[14]I should be able to get this formula directly from (102.28) with the appropriate formulas for ω and σ.

Substituting (102.62) into (102.3) gives

$$\Psi(b) \approx \exp\left[\frac{i}{\hbar}\int V(r) r_3^{-3} r^2 \frac{\alpha(p-\rho) - \frac{\Lambda}{4\pi} - \frac{3a_0^2 r_3^2}{\pi r^2}}{\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4a_0^2 r_3^2}{r^2} + \frac{b^2 r_3^6}{3r^6}}}\,\mathrm{d}r\right] . \tag{102.63}$$

Substituting (102.63) into (102.2) gives

$$\psi(\Omega) = \int_{-b_{\max}}^{b_{\max}} \psi_b(\Omega) \exp\left[\frac{i}{\hbar}\int_{r_0}^{r_3} V(r) r_3^{-3} r^2 \frac{\alpha(p-\rho) - \frac{\Lambda}{4\pi} - \frac{3a_0^2 r_3^2}{\pi r^2}}{\sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4a_0^2 r_3^2}{r^2} + \frac{b^2 r_3^6}{3r^6}}}\,\mathrm{d}r\right] db . \tag{102.64}$$

The r integration in (102.64) goes from r_0 to r_3, where r_0 is very small, but large enough that quantum effects can be neglected, and r_3 is the present radius of the visible universe. The results do not depend critically on the value of r_0, as long as it is small.

We also have, from the first equation in equation (6.7a) of [163],

$$\frac{\dot\rho}{\rho + p} = -\frac{\dot X}{X} - \frac{\dot Y}{Y} - \frac{\dot Z}{Z} = -3\frac{\dot r}{r} . \tag{102.65}$$

For an equation of state, we take $p = w\rho$, where $w = 1/3$ in the radiation-dominated era, and $w = 0$ in the matter-dominated era. This gives

$$\rho = \rho_{\mathrm{ref}}(r/r_{\mathrm{ref}})^{-3(1+w)} , \tag{102.66}$$

where ρ_{ref} and r_{ref} are constants of integration.

102.6 Radiation, matter, and dark-energy eras

It is not possible to perform the r and b integrations in (102.64) in closed form. It is necessary to make approximations to the radical in (102.64) (to be discussed in Section 102.4), as appropriate. The appropriate approximation to make depends on the values of r and b. Although the b dependence is explicit, the r dependence is partly implicit through the dependence of p and ρ on r.

For the r dependence, there are three eras to consider. In the early universe, radiation dominates over matter to determine the density ρ in the radical in (102.64). When the radius of the visible universe r reaches a certain size (which we define as r_1), matter begins to dominate over radiation to determine the density ρ. When the radius r of the visible universe gets even larger (to a size we define as r_2), the density of matter has fallen low enough that the cosmological constant Λ begins to dominate over the density term in (102.64). We define the present radius of the visible universe to be r_3, which we take to be the Hubble radius, approximately [352, p. 252] 1.3×10^{28} cm.

We can take [352, p. 206]

$$\frac{r_1}{r_3} = \frac{\Omega_{\mathrm{rad}}}{\Omega_{\mathrm{mat}}} \approx \frac{0.000084}{.27} \approx 0.00031 = 3.1 \times 10^{-4} , \tag{102.67}$$

and

$$\frac{r_2}{r_3} = \left(\frac{\Omega_{\mathrm{mat}}}{\Omega_\Lambda}\right)^{1/3} \approx \left(\frac{.27}{.73}\right)^{1/3} \approx 0.72 , \tag{102.68}$$

where Ω_{rad} is the present fraction of radiation density, Ω_{mat} is the present fraction of matter density (including dark matter), and Ω_Λ is the present fraction of dark energy, not to be confused with Ω, which we use to represent relative rotation. We also have [352, p. 235] that 9 billion years after the big bang, the density of matter and dark energy were equal and were

$$\rho_2 = \frac{\Lambda}{8\pi} \approx 5.2 \times 10^{-58}\ \mathrm{cm}^{-2} , \tag{102.69}$$

which gives

$$\Lambda \approx 1.3 \times 10^{-56}\ \mathrm{cm}^{-2}\ . \tag{102.70}$$

We also have

$$\hbar \to \frac{\hbar G}{c^2} = L^{*2} \approx 2.616 \times 10^{-66}\ \mathrm{cm}^2\ . \tag{102.71}$$

In each of these three eras, we consider at what value of b the relative rotation term becomes larger than the otherwise dominant term. Let b_0 be the value of b where the relative rotation term is equal to the dominant term at $r = r_0$. Let b_1 be the value of b where the relative rotation term is equal to the dominant term at $r = r_1$. Let b_2 be the value of b where the relative rotation term is equal to the dominant term at $r = r_2$. Let b_3 be the value of b where the relative rotation term is equal to the dominant term at $r = r_3$. Define r_b to be the value of r at which the relative rotation term equals the otherwise dominant term for the given value of b.

We can break the r integration into three parts: First, from r_0 to r_1 where the density term dominates in the radiation-dominant era. Second, from r_1 to r_2 where the density term dominates in the matter-dominant era. Third, from r_2 to r_3 where the Λ term dominates in the dark-energy dominant era.

Section 102.4 gives the approximations we need for small and large relative rotation.

102.7 Saddlepoint approximation

[15]

Using the above two approximations in the appropriate cases allows us to approximate (102.64) as

$$\psi(\Omega) \approx \int_{-b_{\mathrm{max}}}^{b_{\mathrm{max}}} \psi_b(\Omega) \exp\left\{-iAf_1(b)\right\} \exp\left\{i\left(\frac{b}{\sigma_{\mathrm{m}}}\right)^2 f_2(b)\right\} \exp\left\{-iAf_3(b)\right\} \mathrm{d}b\ , \tag{102.72}$$

where

$$A \equiv \frac{r_3^3\sqrt{\Lambda + 8\pi\rho_3}}{6\hbar\sqrt{3}} = \frac{r_3^3\sqrt{\Lambda + 8\pi\rho_3}}{6(L^*)^2\sqrt{3}} \approx 10^{121}\ , \tag{102.73}$$

$$\sigma_{\mathrm{m}} \equiv \left(\frac{6\hbar\sqrt{3(\Lambda+8\pi\rho_3)}}{r_3^3}\right)^{1/2} \approx 4\times 10^{-89}\mathrm{cm}^{-1} \approx 10^{-78}\mathrm{s}^{-1} \approx 4\times 10^{-71}\mathrm{rad\ per\ year}\ , \tag{102.74}$$

$$f_1 \equiv -6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}} \int_{r_t}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}\right] \mathrm{d}r\ , \tag{102.75}$$

$$f_2 \equiv -6\sqrt{3(\Lambda+8\pi\rho_3)} \int_{r_t}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right]\left[\frac{\sqrt{3}r_3^6}{2r^6(\Lambda+8\pi\rho)^{3/2}}\right] \mathrm{d}r\ , \tag{102.76}$$

and

$$f_3 \equiv -6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}} \int_{r_0}^{r_t} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right]\left[\frac{\sqrt{3}}{|b|}\left(\frac{r}{r_3}\right)^3\right] \mathrm{d}r\ , \tag{102.77}$$

where we have neglected the curvature terms proportional to a_0^2.

If we define r_b as the value of r where the relative rotation term in the radical equals the rest of the radicand, then we define $r_t = r_b$ if $r_0 \le r_b \le r_3$, $r_t = r_0$ if $r_b < r_0$, and $r_t = r_3$ if $r_b > r_3$. $f_1(b)$ and $f_2(b)$ are slowly varying positive functions of b of order unity for b less than b_3. $f_3(b)$ is negligible for b much less than b_3 but is the dominant term for b larger than b_3. b_3 is the value of b for which $r_b = r_3$.

[15] I should be able to make the saddlepoint approximation directly on (102.28).

The maximum value of b, $b_{\rm max} \approx 0.62 \times 10^{33}{\rm cm}^{-1}$ is the reciprocal of the Planck length, L^*, and corresponds to a rotation rate of one rotation per Planck time. Considering rotation rates faster than that would require a correct theory of quantum gravity, and is beyond the scope of this paper.

Appendix 102.10 calculates f_1, f_2, and f_3 for the various values of b and r to show that the integral in (102.72) can be calculated by integrating along the complete stationary-phase path along the real b axis. Alternatively, the integral in (102.72) can be calculated by integrating along the steepest-descent path through the saddlepoint at $b = 0$, and then completing the integration at each end by bringing the path back to the stationary-phase path on the real b axis for large b where the $f_3(b)$ term dominates the integration.

The saddlepoint at $b = 0$ in (102.72) is isolated from the branch points given by (102.87). Along the complete stationary-phase path, the exponent in the path integral continues to go through many cycles (except at the saddlepoint at the origin) because of the large value of A in (102.73), leading to the conclusion that the only significant contribution to the integral is at the saddlepoint at $b = 0$.

From (102.60), we see that b equals the relative rotation rate at the present time. The amplitude for measuring the present rotation rate to be Ω given that the cosmology is described by the parameter b is $\psi_b(\Omega)$. To estimate that formula, we should consider the details of the experiment designed to make that measurement. Instead, let us simply assume that it is an accurate measurement with an error of ϵ. Then we can take

$$\psi_b(\Omega) = \left(\frac{2}{\pi\epsilon^2}\right)^{\frac{1}{4}} \exp\left[-\left(\frac{\Omega - b}{\epsilon}\right)^2\right] , \tag{102.78}$$

where the formula includes correct normalization. Substituting (102.78) into (102.72) (and neglecting the f_3 term, which is negligible at the saddlepoint) gives

$$\psi(\Omega) = \left(\frac{2}{\pi\epsilon^2}\right)^{\frac{1}{4}} \int_{-b_{\rm max}}^{b_{\rm max}} \exp\left[-\left(\frac{\Omega - b}{\epsilon}\right)^2\right] \exp\left\{-iAf_1(b)\right\} \exp\left[i\left(\frac{b}{\sigma_{\rm m}}\right)^2 f_2(b)\right] {\rm d}b \, . \tag{102.79}$$

Actually, the saddlepoint is shifted away from zero when including $\psi_b(\Omega)$. It is shifted to

$$b_{\rm sp} = \frac{\sigma_{\rm m}^2}{\epsilon^2} \frac{\Omega}{-if_2(b_{\rm sp}) + \sigma_{\rm m}^2/\epsilon^2} , \tag{102.80}$$

which is still very nearly zero as long as $\epsilon^2 \gg \sigma_{\rm m}^2$. Then (102.79) can be written

$$\psi(\Omega) = \left(\frac{2}{\pi\epsilon^2}\right)^{\frac{1}{4}} \exp\left[-\frac{\Omega^2}{\epsilon^2}\left(1 - \frac{b_{\rm sp}}{\Omega}\right)\right] \int_{-b_{\rm max}}^{b_{\rm max}} \exp\left\{-iAf_1(b)\right\} \exp\left[-\frac{\Omega}{b_{\rm sp}}\left(\frac{b - b_{\rm sp}}{\epsilon}\right)^2\right] {\rm d}b \, . \tag{102.81}$$

Making a saddlepoint approximation gives

$$\psi(\Omega) = \left(\frac{2}{\pi\epsilon^2}\right)^{\frac{1}{4}} \epsilon\sqrt{\frac{\pi b_{\rm sp}}{\Omega}} \exp\left[-\frac{\Omega^2}{\epsilon^2}\left(1 - \frac{b_{\rm sp}}{\Omega}\right)\right] \exp\left\{-iAf_1(b_{\rm sp})\right\} . \tag{102.82}$$

This gives

$$\psi^*(\Omega)\psi(\Omega) = \frac{\epsilon}{\Omega}\sqrt{2\pi}\, |b_{\rm sp}| \exp\left[-\frac{\Omega^2}{\epsilon^2}\left(2 - \frac{b_{\rm sp} + b_{\rm sp}^*}{\Omega}\right)\right] . \tag{102.83}$$

Using (102.80) gives

$$\psi^*(\Omega)\psi(\Omega) = \frac{\sigma_{\rm m}^2}{\epsilon}\sqrt{\frac{2\pi}{f_2^2(b_{\rm sp}) + \sigma_{\rm m}^4/\epsilon^4}} \exp\left[-2\frac{\Omega^2}{\epsilon^2}\left(\frac{f_2^2(b_{\rm sp})}{f_2^2(b_{\rm sp}) + \sigma_{\rm m}^4/\epsilon^4}\right)\right] . \tag{102.84}$$

Equation (102.84) is not normalized because (102.3) was not normalized. We could normalize it now, but that is not necessary.

Equation (102.84) shows several things. First, it shows that the most expected measurement of relative rotation rate Ω is zero, as expected.

Second, from appendix 102.10 , (102.122) gives

$$f_2(b_{\text{sp}}) = \left[\frac{3}{2} - \frac{1}{2}\left(\frac{r_2}{r_3}\right)^3\right]\alpha + \frac{2}{3} + 3\ln\frac{r_3}{r_2} = 1.3\alpha + 1.7\ . \tag{102.85}$$

We can take $\alpha = 0$ [161] or $\alpha = 1$ [162]. In either case, f_2 is a number of order unity, so that when the measurement error ϵ is larger than σ_{m} given by (102.74) (which would be the usual case), then the spread in measurement values for Ω would be ϵ, as expected.

However, as the measurement error is decreased by improving the measurements until ϵ equals σ_{m} in (102.74), we reach a point where the spread in measurement values for Ω no longer decreases as ϵ continues to decrease, but actually increases, indicating that we have reached the true spread in values for the rotation of inertial frames relative to matter, σ_{m}. Of course, we could have seen this from the saddlepoint approximation by noticing that only values of $|b|$ less than σ_{m} contributed significantly to the integration.

It is not surprising that we do not measure a relative rotation of inertial frames and matter. The calculations here show that only those cosmologies with a present relative rotation of inertial frames and matter is less than $\sigma_{\text{m}} = \left(\frac{6\hbar\sqrt{3(\Lambda+8\pi\rho_3)}}{r_3^3}\right)^{1/2} \approx 10^{-71}$ radians per year contribute significantly to a calculation of relative rotation rate. Cosmologies having rotation rates larger than that cancel each other out in pairs by phase interference. However, see the discussion of the generality of this result in section 102.9.

102.8 Validity of the saddlepoint approximation

There are several conditions that must apply (in addition to those already discussed) for the saddlepoint approximation to (102.72) to be valid. Basically, the functions $f_1(b)$, $f_2(b)$, and $f_3(b)$ must be such that there are no significant contributions to the integral other than at the saddlepoint at $b = 0$.

To make certain that is the case, appendix 102.10 calculates the values of $f_1(b)$, $f_2(b)$, and $f_3(b)$ using appropriate approximations. In making those calculations, we notice that there are five regimes for the value of b to be considered. If we define r_b as the value of r where the appropriate approximation switches from large b using the approximation (102.45) to that for small b using the approximation (102.44), then the five possibilities are:

1. r_b can be less than r_0,
2. r_b can be in the radiation era between r_0 and r_1,
3. r_b can be in the matter era between r_1 and r_2,
4. r_b can be in the dark-energy era between r_2 and r_3, or
5. r_b can be larger than r_3.

There are three considerations: First, the values of $f_1(b)$ and $f_2(b)$ should be of order unity, and the value of $f_3(b)$ should be negligible in the first four of the above five regimes where $b < b_3$ (i.e., $r_b < r_3$). Second, in the fifth of the above five regimes, where r_b can be larger than r_3 and the values of $f_1(b)$ and $f_2(b)$ are negligible, the value of $f_3(b)$ should be such that there are no significant contributions to the integral in that regime. Third, the saddlepoint must be isolated from the branch points.

102.8.1 Behavior of $f_1(b)$, $f_2(b)$, and $f_3(b)$ for $b < b_3$ (i.e., $r_b < r_3$)

In calculating the values of $f_1(b)$, $f_2(b)$, and $f_3(b)$ in these five regimes, appendix 102.10 shows that the values of $f_1(b)$ and $f_2(b)$ are of order unity in the first four regimes and $f_3(b)$ has insignificant effect on the integral in the first four regimes.

102.8.2 Behavior of $f_1(b)$, $f_2(b)$, and $f_3(b)$ for $b > b_3$ (i.e., $r_b > r_3$)

In the fifth regime, where $f_3(b)$ is the dominant term, appendix 102.10 shows that because the factor A in (102.72) is so large, there is no significant contribution to the integral from $f_3(b)$ for the stationary-phase path as long as the integral stops at $b_{\max} = 1/L^*$ rather than continuing to infinity.

Appendix 102.10 shows that f_3 varies from $0.67 + 0.18\alpha$ to $1.8 \times 10^{-61} \times (0.67 + 0.18\alpha)$ as b varies from $b_3 = \sqrt{\Lambda}$ to $b_{\max} = 1/L^*$. The latter value of f_3 may seem like a small number, but since it is multiplied by $A \approx 10^{121}$, the exponent in (102.72) will still be changing very rapidly along the stationary-phase path.

Actually, the condition on how large $b_{\max}$ can be should be more strict. It is the relative rotation σ that should be less than $1/L^*$ at any time during the evolution of the universe. We see from (102.60) that σ will be largest when the radius of the visible universe equals r_0. Thus, from (102.60), we have

$$b_{\max} = \left(\frac{r_0}{r_3}\right)^3 \sigma_{\max} = \left(\frac{r_0}{r_3}\right)^3 \frac{1}{L^*} . \tag{102.86}$$

So, for example, if we take $r_0 = (\Lambda L^{*2})^{1/6} r_3 \approx 5.7 \times 10^{-21} r_3 \approx 7 \times 10^7$ cm, then we have $b_{\max} = b_3 = \sqrt{\Lambda} \approx 10^{-28}$ cm^{-1}, which is quite small. On the other hand, if we take $r_0 = r_1 \approx 3.1 \times 10^{-4} r_3$, then we would have $b_{\max} \approx (3.1 \times 10^{-4})^3 / L^* \approx 1.8 \times 10^{22}$ cm^{-1}, which is quite large. In either case, the contribution to this part of the integral will still be negligible, as pointed out above.

102.8.3 Branch points

Here, we show that the saddlepoint at $b = 0$ is isolated from the branch points in (102.64) as long as r_0 is much greater than $250L^*$, where L^* is the Planck length.

The branch points in the radical in the denominator of (102.64) are at

$$b_{\mathrm{br}} = \pm i\sqrt{3} r_3^{-3} r^3 \sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} + \frac{4a_0^2 r_3^2}{r^2}} \rightarrow \pm i r_3^{-3} r^3 \sqrt{\Lambda + 8\pi\rho} , \tag{102.87}$$

where the curvature term is neglected. We can take the branch cuts to run along the imaginary b axis from each branch point away from the real axis to $+i\infty$ or $-i\infty$. Each of the two Riemann sheets has a saddlepoint at $b = 0$. The stationary-phase path runs along the real b axis in each case. For the Riemann sheet corresponding to the physical problem (the one for which we take the positive square root at the saddlepoint), the steepest descent path runs from the third quadrant to the first quadrant.

Actually, the calculation of the branch points is more complicated. The formula for the branch points in (102.87) depends on r. To get the correct formula for the branch points, it is first necessary to perform the r integration. It is not possible to do that in closed form without making approximations. However, the correct formula involves evaluating (102.87) at the endpoints of the integral. The smallest value of (102.87) will occur at $r = r_0$. So, the smallest the branch point could be (the closest to the saddlepoint) is

$$b_{\mathrm{br}} \approx \pm i \left(\frac{r_0}{r_3}\right)\left(\frac{r_1}{r_3}\right)^{1/2}\left(\frac{r_2}{r_3}\right)^{3/2}\sqrt{\Lambda}\ . \tag{102.88}$$

For the branch point to be far enough from the saddlepoint to have no significant effect on the saddlepoint approximation requires that $|b_{\mathrm{br}}| \gg \sigma_{\mathrm{m}}$, where σ_{m} is given by (102.74). This requires $r_0 \gg 250L^*$. This can easily be satisfied while still keeping r_0 small enough.

102.9 Discussion

There are several things to consider in this calculation of the rotation of the universe: First, how general is the result, considering that only one cosmological model, the Bianchi VI_h model, was used? Second, since the Bianchi VI_h model is an open model, shouldn't the action be infinite because it should be integrated over an infinite volume, as pointed out to me by James Hartle [351]? Third, would the result change if we were to use a Euclidean instead of a Lorentzian path integral? Fourth, how much does the result depend on the choice of Lagrangian? Fifth, how much does the result depend on the implementation of the semi-classical approximation to quantum gravity?

102.9.1 Generality

The Bianchi VI_h model is so far the only closed-form solution of Einstein's field equations I have found that has relative rotation of matter and inertial frames and that approaches the Robertson-Walker metric continuously as the relative rotation approaches zero. We will not know for sure how general the result that "only those models that have small relative rotation contribute significantly to any measurement of relative rotation" is until we have found other models containing relative rotation to make the calculation with, so we should try to find such models and make the calculation.

It may be possible to make a general calculation that is applicable to a wide range of models, and we should try to do that as well. In addition, the approximation of our actual universe by a homogeneous model may introduce errors that need to be investigated. The effect of this approximation has been looked at with regard to the existence of dark energy.

From symmetry, we would expect there to be a saddlepoint at zero relative rotation for homogeneous models. However, we cannot be certain that the size of the visible universe relative to the Planck length will always lead to having significant contributions to measurements of relative rotation coming only from cosmologies that have very small relative rotation.

102.9.2 Infinite action for an open universe?

Formally, $V(r)$ should be infinite for an open universe to give an infinite action, as pointed out to me by James Hartle [351], however it is not physically realistic to consider contributions to the action outside of the visible universe.

It seems to be a stretch to think that matter outside of the visible universe should have a significant effect on observations we make here. When we finally get a correct theory of quantum gravity, we may find that the effect of matter outside of the visible universe is multiplied by a factor that decays exponentially as a function of the distance outside of the Hubble distance. This would be similar to the exponential decay of effects outside of the light cone[353].

However, if we actually did take the action to be infinite, then the parameter A in (102.73) would also be infinite and the parameter σ_{m} in (102.74) would be zero, with the result that the relative rotation of inertial frames and matter would be even more sharply peaked than the calculation given here. The relative rotation rate would be exactly zero, with zero width. That is, if the action were really infinite, the explanation for why we do not observe a relative rotation of the average inertial frame and the average matter distribution would be even stronger.

102.9.3 Euclidean versus Lorentzian path integral

Replacing the Lorentzian path integral in (102.3) by a Euclidean path integral[124] would give

$$\Psi(b) \approx e^{-I/\hbar} \ , \tag{102.89}$$

where the saddlepoint would still be at $b = 0$, the steepest descent path would be along the real b axis, and it would be necessary to take the opposite root in (102.77) to keep the steepest-descent path along the complete real b axis. As pointed out by Hajicek[217], the value of the path integral is independent of the regime, Lorentzian or Euclidean, that one is using to calculate it.

102.9.4 Dependence on the choice of Lagrangian

Using a standard form for the Lagrangian for General Relativity with two choices [161, 162] for the matter Lagrangian shows no significant difference on the main result for how only a narrow range of possible values of relative rotation near zero contribute significantly to a measurement of global relative rotation. It is difficult to imagine a reasonable Lagrangian that would give a significantly different result.

102.9.5 Dependence on the implementation of the semi-classical approximation to quantum gravitly

Since we do not yet have a theory of quantum gravity, we are not quite sure what a correct semi-classical approximation would be. However, it is difficult to imagine a formulation that would give results that differ significantly from those obtained here.

102.10 Appendix: Estimate of the action

It is not possible to perform the r and b integrations in (102.64) in closed form. It is necessary to make approximations to the radical in (102.64), discussed in Section 102.4, as appropriate. The appropriate approximation to make depends on the values of r and b. Although the b dependence is explicit, the r dependence is partly implicit through the dependence of p and ρ on r. For the r dependence, there are three eras to consider. In the early universe, radiation dominates over matter to determine the density ρ in the radical in (102.64). When the radius of the visible universe r reaches a certain size (which we define as r_1), matter begins to dominate over radiation to determine the density ρ. When the radius r of the visible universe gets even larger (to a size we define as r_2), the density of matter has fallen low enough that the cosmological constant Λ begins to dominate over the density term in (102.64). We define the present radius of the visible universe to be r_3, which we take to be the Hubble radius, which we take to be [352, p. 252] 1.3×10^{28} cm.

In each of these eras, we consider at what value of b the relative rotation term becomes larger than the otherwise dominant term. Let b_0 be the value of b where the relative rotation term is equal to the dominant term at $r = r_0$. Let b_1 be the value of b where the relative rotation term is equal to the dominant term at $r = r_1$. Let b_2 be the value of b where the relative rotation term is equal to the dominant term at $r = r_2$. Let b_3 be the value of b where the relative rotation term is equal to the dominant term at $r = r_3$. I define r_b to be the value of r at which the relative rotation term equals the otherwise dominant term for the given value of b.

We can break the r integration into three parts: First, from r_0 to r_1 where the density term dominates in the radiation-dominant era. Second, from r_1 to r_2 where the density term dominates in the matter-dominant era. Third, from r_2 to r_3 where the Λ term dominates in the dark-energy dominant era.

Tables 102.1 through 102.5 summarize the five possible cases, depending on the five regimes of the parameter $|b|$, which is proportional to the relative rotation. Table 102.1 shows the case in which $|b|$ is small enough that the relative rotation term in the Friedmann equation is smaller than the dominant term throughout the domain of the r integration.

Table 102.1: The case where $r_{b0} < r_0$ (that is, where $b^2 < b_0^2 = \Lambda r_0^2 r_1 r_2^3 r_3^{-6}$)

r	w	ρ	comment
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation large
$r_{b0} = \frac{br_3^3}{\sqrt{\Lambda r_1 r_2 r_2}}$	1/3	$\rho_{b0} = \frac{\Lambda^3}{8\pi} r_1^3 r_2^9 r_3^{-12} b^{-4}$	$8\pi\rho_{b0} = b^2 \frac{r_3^6}{r_{b0}^6}$
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation small
r_0	1/3	$\rho_0 = \frac{\Lambda}{8\pi} r_0^{-4} r_1 r_2^3$	smallest universe considered
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation small
r_1		$\rho_1 = \rho_0 \left(\frac{r_1}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1^{-3} r_2^3$	$\rho_{\text{radiation}} = \rho_{\text{matter}}$
r	0	$\rho = \rho_1 \left(\frac{r}{r_1}\right)^{-3} = \frac{\Lambda}{8\pi} r_2^3 r^{-3}$	matter era, relative rotation small
r_2	0	$\rho_2 = \frac{\Lambda}{8\pi}$	$8\pi\rho_2 = \Lambda$
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small
r_3	0	$\rho_3 = \frac{\Lambda}{8\pi} \frac{\Omega_{\text{mat}}}{\Omega_\Lambda} = \frac{\Lambda}{8\pi} \frac{.27}{.73} = \frac{\Lambda}{21.6\pi}$	now
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small

Table 102.2 shows the case where the relative rotation term is dominant at $r = r_0$, but becomes smaller than the density term during the radiation era. (The density term includes both radiation density and matter density, including both normal matter and dark matter.)

Table 102.3 shows the case where the relative rotation term is dominant at the beginning of the matter-dominant era, but becomes smaller than the density term during the matter-dominant era. It is necessary to consider large values of relative rotation because the parameter b must be integrated from $-b_{\text{max}}$ to b_{max}.

Table 102.4 shows the case where the relative rotation term is dominant at the beginning of the dark-energy era, but becomes smaller than the Λ term in the dark-energy era, but before the present time.

Table 102.5 shows the case where the relative rotation term is dominant throughout the domain of the r integration.

It is possible to break the b integration in (102.64) into several parts as follows:

$$\int_{-b_{\text{max}}}^{b_{\text{max}}} = \int_{-b_{\text{max}}}^{-b_3} + \int_{-b_3}^{-b_2} + \int_{-b_2}^{-b_1} + \int_{-b_1}^{-b_0} + \int_{-b_0}^{b_0} + \int_{b_0}^{b_1} + \int_{b_1}^{b_2} + \int_{b_2}^{b_3} + \int_{b_3}^{b_{\text{max}}} , \tag{102.90}$$

where the shorthand notation should be obvious.

Table 102.2: The case where $r_0 < r_{b0} < r_1$ (that is, where $b_0^2 = \Lambda r_0^2 r_1 r_2^3 r_3^{-6} < b^2 < b_1^2 = \Lambda r_1^3 r_2^3 r_3^{-6}$)

r	w	ρ	comment
r_0	1/3	$\rho_0 = \frac{\Lambda}{8\pi} r_0^{-4} r_1 r_2^3$	smallest universe considered
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation large
$r_{b0} = \frac{b r_3^3}{\sqrt{\Lambda r_1 r_2 r_2}}$	1/3	$\rho_{b0} = \frac{\Lambda^3}{8\pi} r_1^3 r_2^9 r_3^{-12} b^{-4}$	$8\pi\rho_{b0} = b^2 \frac{r_3^6}{r_{b0}^6}$
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation small
r_1		$\rho_1 = \rho_0 \left(\frac{r_1}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1^{-3} r_2^3$	$\rho_{\text{radiation}} = \rho_{\text{matter}}$
r	0	$\rho = \rho_1 \left(\frac{r}{r_1}\right)^{-3} = \frac{\Lambda}{8\pi} r_2^3 r^{-3}$	matter era, relative rotation small
r_2	0	$\rho_2 = \frac{\Lambda}{8\pi}$	$8\pi\rho_2 = \Lambda$
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small
r_3	0	$\rho_3 = \frac{\Lambda}{8\pi} \frac{\Omega_{\text{mat}}}{\Omega_\Lambda} = \frac{\Lambda}{8\pi} \frac{.27}{.73} = \frac{\Lambda}{21.6\pi}$	now
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small

Table 102.3: The case where $r_1 < r_{b1} < r_2$ (that is, where $b_1^2 = \Lambda r_1^3 r_2^3 r_3^{-6} < b^2 < b_2^2 = \Lambda r_2^6 r_3^{-6}$)

r	w	ρ	comment
r_0	1/3	$\rho_0 = \frac{\Lambda}{8\pi} r_0^{-4} r_1 r_2^3$	smallest universe considered
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation large
r_1		$\rho_1 = \rho_0 \left(\frac{r_1}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1^{-3} r_2^3$	$\rho_{\text{radiation}} = \rho_{\text{matter}}$
r	0	$\rho = \rho_1 \left(\frac{r}{r_1}\right)^{-3} = \frac{\Lambda}{8\pi} r_2^3 r^{-3}$	matter era, relative rotation large
$r_{b1} = \frac{b^{2/3} r_3^2}{\Lambda^{1/3} r_2}$	0	$\rho_{b1} = \frac{\Lambda^2 r_2^6 r_3^{-6}}{8\pi b^2}$	$8\pi\rho_{b1} = b^2 \frac{r_3^6}{r_{b1}^6}$
r	0	$\rho = \rho_1 \left(\frac{r}{r_1}\right)^{-3} = \frac{\Lambda}{8\pi} r_2^3 r^{-3}$	matter era, relative rotation small
r_2	0	$\rho_2 = \frac{\Lambda}{8\pi}$	$8\pi\rho_2 = \Lambda$
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small
r_3	0	$\rho_3 = \frac{\Lambda}{8\pi} \frac{\Omega_{\text{mat}}}{\Omega_\Lambda} = \frac{\Lambda}{8\pi} \frac{.27}{.73} = \frac{\Lambda}{21.6\pi}$	now
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small

Table 102.4: The case where $r_2 < r_{b2} < r_3$ (that is, where $b_2^2 = \Lambda r_2^6 r_3^{-6} < b^2 < b_3^2 = \Lambda$)

r	w	ρ	comment
r_0	1/3	$\rho_0 = \frac{\Lambda}{8\pi} r_0^{-4} r_1 r_2^3$	smallest universe considered
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation large
r_1		$\rho_1 = \rho_0 \left(\frac{r_1}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1^{-3} r_2^3$	$\rho_{\text{radiation}} = \rho_{\text{matter}}$
r	0	$\rho = \rho_1 \left(\frac{r}{r_1}\right)^{-3} = \frac{\Lambda}{8\pi} r_2^3 r^{-3}$	matter era, relative rotation large
r_2	0	$\rho_2 = \frac{\Lambda}{8\pi}$	$8\pi\rho_2 = \Lambda$
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation large
$r_{b2} = \frac{b^{1/3}}{\Lambda^{1/6}} r_3$	0	$\rho_{b2} = \frac{\Lambda^{3/2} r_2^3 r_3^{-3}}{8\pi b}$	$\Lambda = b^2 \frac{r_3^6}{r_{b2}^6}$
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small
r_3	0	$\rho_3 = \frac{\Lambda}{8\pi} \frac{\Omega_{\text{mat}}}{\Omega_\Lambda} = \frac{\Lambda}{8\pi} \frac{.27}{.73} = \frac{\Lambda}{21.6\pi}$	now
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small

Table 102.5: The case where $r_{b3} > r_3$ (that is, where $b^2 > b_3^2 = \Lambda$)

r	w	ρ	comment
r_0	1/3	$\rho_0 = \frac{\Lambda}{8\pi} r_0^{-4} r_1 r_2^3$	smallest universe considered
r	1/3	$\rho = \rho_0 \left(\frac{r}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1 r_2^3 r^{-4}$	radiation era, relative rotation large
r_1		$\rho_1 = \rho_0 \left(\frac{r_1}{r_0}\right)^{-4} = \frac{\Lambda}{8\pi} r_1^{-3} r_2^3$	$\rho_{\text{radiation}} = \rho_{\text{matter}}$
r	0	$\rho = \rho_1 \left(\frac{r}{r_1}\right)^{-3} = \frac{\Lambda}{8\pi} r_2^3 r^{-3}$	matter era, relative rotation large
r_2	0	$\rho_2 = \frac{\Lambda}{8\pi}$	$8\pi\rho_2 = \Lambda$
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation large
r_3	0	$\rho_3 = \frac{\Lambda}{8\pi} \frac{\Omega_{\text{mat}}}{\Omega_\Lambda} = \frac{\Lambda}{8\pi} \frac{.27}{.73} = \frac{\Lambda}{21.6\pi}$	now
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation large
$r_{b3} = \frac{b^{1/3}}{\Lambda^{1/6}} r_3$	0	$\rho_{b3} = \frac{\Lambda^{3/2} r_2^3 r_3^{-3}}{8\pi b}$	$\Lambda = b^2 \frac{r_3^6}{r_{b3}^6}$
r	0	$\rho = \rho_2 \left(\frac{r}{r_2}\right)^{-3} = \frac{\Lambda}{8\pi} \left(\frac{r}{r_2}\right)^{-3}$	dark energy era, relative rotation small

Each of these integrals can be calculated as the first example shows.

$$\int_{-b_{\max}}^{-b_3} = \int_{-b_{\max}}^{-b_3} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\overbrace{\underbrace{\overbrace{\int_{r_0}^{r_1}\mathrm{d}r}^{w=1/3} + \overbrace{\int_{r_1}^{r_2}\mathrm{d}r}^{w=0}}_{\rho\text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_3}\mathrm{d}r}^{w=0}}_{\Lambda\text{ dominant}}}^{b^2\text{ large}}\right)\right]. \tag{102.91}$$

Similarly,

$$\int_{-b_3}^{-b_2} = \int_{-b_3}^{-b_2} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\underbrace{\overbrace{\int_{r_0}^{r_1}\mathrm{d}r + \int_{r_1}^{r_2}\mathrm{d}r}^{b^2\text{ large}}}_{\rho\text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_{b2}}\mathrm{d}r}^{b^2\text{ large}} + \overbrace{\int_{r_{b2}}^{r_3}\mathrm{d}r}^{b^2\text{ small}}}_{\Lambda\text{ dominant}}\right)\right], \tag{102.92}$$

$$\int_{-b_2}^{-b_1} = \int_{-b_2}^{-b_1} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\underbrace{\overbrace{\int_{r_0}^{r_1}\mathrm{d}r + \int_{r_1}^{r_{b1}}\mathrm{d}r}^{b^2\text{ large}} + \overbrace{\int_{r_{b1}}^{r_2}\mathrm{d}r}^{b^2\text{ small}}}_{\rho\text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_3}\mathrm{d}r}^{b^2\text{ small}}}_{\Lambda\text{ dominant}}\right)\right], \tag{102.93}$$

$$\int_{-b_1}^{-b_0} = \int_{-b_1}^{-b_0} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\underbrace{\overbrace{\int_{r_0}^{r_{b0}}\mathrm{d}r}^{b^2\text{ large}} + \overbrace{\int_{r_{b0}}^{r_1}\mathrm{d}r + \int_{r_1}^{r_2}\mathrm{d}r}^{b^2\text{ small}}}_{\rho\text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_3}\mathrm{d}r}^{b^2\text{ small}}}_{\Lambda\text{ dominant}}\right)\right], \tag{102.94}$$

$$\int_{-b_0}^{b_0} = \int_{-b_0}^{b_0} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\overbrace{\underbrace{\int_{r_0}^{r_1}\mathrm{d}r + \int_{r_1}^{r_2}\mathrm{d}r}_{\rho\text{ dominant}} + \underbrace{\int_{r_2}^{r_3}\mathrm{d}r}_{\Lambda\text{ dominant}}}^{b^2\text{ small}}\right)\right], \tag{102.95}$$

$$\int_{b_0}^{b_1} = \int_{b_0}^{b_1} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\underbrace{\overbrace{\int_{r_0}^{r_{b0}}\mathrm{d}r}^{b^2\text{ large}} + \overbrace{\int_{r_{b0}}^{r_1}\mathrm{d}r + \int_{r_1}^{r_2}\mathrm{d}r}^{b^2\text{ small}}}_{\rho\text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_3}\mathrm{d}r}^{b^2\text{ small}}}_{\Lambda\text{ dominant}}\right)\right], \tag{102.96}$$

$$\int_{b_1}^{b_2} = \int_{b_1}^{b_2} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\underbrace{\overbrace{\int_{r_0}^{r_1}\mathrm{d}r + \int_{r_1}^{r_{b1}}\mathrm{d}r}^{b^2\text{ large}} + \overbrace{\int_{r_{b1}}^{r_2}\mathrm{d}r}^{b^2\text{ small}}}_{\rho\text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_3}\mathrm{d}r}^{b^2\text{ small}}}_{\Lambda\text{ dominant}}\right)\right], \tag{102.97}$$

$$\int_{b_2}^{b_3} = \int_{b_2}^{b_3} \mathrm{d}b\,\psi_b(\Omega)\exp\left[\frac{i}{\hbar}\left(\underbrace{\overbrace{\int_{r_0}^{r_1}\mathrm{d}r + \int_{r_1}^{r_2}\mathrm{d}r}^{b^2\text{ large}}}_{\rho\text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_{b2}}\mathrm{d}r}^{b^2\text{ large}} + \overbrace{\int_{r_{b2}}^{r_3}\mathrm{d}r}^{b^2\text{ small}}}_{\Lambda\text{ dominant}}\right)\right], \tag{102.98}$$

$$\int_{b_3}^{b\text{max}} = \int_{b_3}^{b\text{max}} \mathrm{d}b\psi_b(\Omega) \exp\left[\frac{i}{\hbar} \left(\overbrace{\underbrace{\overbrace{\int_{r_0}^{r_1} \mathrm{d}r}^{w=1/3} + \overbrace{\int_{r_1}^{r_2} \mathrm{d}r}^{w=0}}_{\rho \text{ dominant}} + \underbrace{\overbrace{\int_{r_2}^{r_3} \mathrm{d}r}^{w=0}}_{\Lambda \text{ dominant}}}^{b^2 \text{ large}} \right) \right] , \tag{102.99}$$

where r_{b0}, r_{b1}, and r_{b2} are the values of r where the relative rotation term equals the dominant term in the various cases.

102.10.1 Appendix: Main contribution, interval from $-b_0 \to b_0$

Equation (102.95) gives the main contribution. Equation (102.44) is a valid approximation for the complete integration in (102.95). Using (102.44) and (102.64), (102.95) becomes

$$\int_{-b_0}^{b_0} = \int_{-b_0}^{b_0} \psi_b(\Omega) \exp\left\{ \frac{i}{\hbar} \int_{r_0}^{r_3} V(r) r_3^{-3} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi} \right] \left[\sqrt{\frac{3}{\Lambda + 8\pi\rho}} - \frac{\sqrt{3} r_3^6 b^2}{2r^6(\Lambda + 8\pi\rho)^{3/2}} \right] \mathrm{d}r \right\} \mathrm{d}b , \tag{102.100}$$

where I have taken the upper sign for the physical Riemann sheet, and I have neglected the curvature term. Equation (102.100) can be written

$$\int_{-b_0}^{b_0} = \int_{-b_0}^{b_0} \psi_b(\Omega) \exp\{-iAf_1\} \exp\left\{ \frac{iAb^2 f_2}{\Lambda + 8\pi\rho_3} \right\} = \int_{-b_0}^{b_0} \psi_b(\Omega) \exp\{-iAf_1\} \exp\left\{ i \left(\frac{b}{\sigma_\text{m}} \right)^2 f_2 \right\} \mathrm{d}b , \tag{102.101}$$

where

$$A = \frac{r_3^3 \sqrt{\Lambda + 8\pi\rho_3}}{6\hbar\sqrt{3}} \approx 10^{121} , \tag{102.102}$$

r_3 is the present radius of the visible universe, which we take to be the Hubble radius,

$$\sigma_\text{m} = \left(\frac{6\hbar\sqrt{3(\Lambda + 8\pi\rho_3)}}{r_3^3} \right)^{1/2} \approx 4 \times 10^{-89} \text{cm}^{-1} \approx 10^{-78} \text{s}^{-1} \approx 4 \times 10^{-71} \text{rad per year} , \tag{102.103}$$

$$f_1 = -6\sqrt{\frac{3}{\Lambda + 8\pi\rho_3}} \int_{r_0}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi} \right] \left[\sqrt{\frac{3}{\Lambda + 8\pi\rho}} \right] \mathrm{d}r , \tag{102.104}$$

and

$$f_2 = -6\sqrt{3(\Lambda + 8\pi\rho_3)} \int_{r_0}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi} \right] \left[\frac{\sqrt{3} r_3^6}{2r^6(\Lambda + 8\pi\rho)^{3/2}} \right] \mathrm{d}r . \tag{102.105}$$

f_1 and f_2 have values that are of order unity. To calculate them, we use the formulas in table 102.1. Because we know from measurements that the universe is nearly spatially flat, we can use

$$V(r) = \frac{4}{3}\pi r^3 . \tag{102.106}$$

We also use $p = w\rho$ for the equation of state and split the integral for f_1 into three parts corresponding to the three eras. This gives

$$f_1 = -6\sqrt{\frac{3}{\Lambda + 8\pi\rho_3}} \int_{r_0}^{r_1} \frac{4}{3}\pi r^3 r_3^{-6} r^2 \left[\alpha(w-1)\rho - \frac{\Lambda}{4\pi} \right] \left[\sqrt{\frac{3}{\Lambda + 8\pi\rho}} \right] \mathrm{d}r$$

$$-6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}}\int_{r_1}^{r_2}\frac{4}{3}\pi r^3 r_3^{-6} r^2\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}\right]\,\mathrm{d}r$$
$$-6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}}\int_{r_2}^{r_3}\frac{4}{3}\pi r^3 r_3^{-6} r^2\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}\right]\,\mathrm{d}r\ . \tag{102.107}$$

We do not have to be very precise in our calculation of f_1. Within a factor of two or so is sufficient. At r_1, we switch from the radiation-dominant era to the matter dominant error. At r_2, we switch from the matter-dominant era to the dark-energy era. Although the actual transitions are smooth, it is sufficient for our purposes to make the transitions abrupt. In that way, we can do the integrations in closed form. We also neglect $8\pi\rho_3$ with respect to Λ. Thus, we can write (102.107) as

$$f_1=-8\pi\sqrt{\frac{3}{\Lambda}}\int_{r_0}^{r_1} r^3 r_3^{-6} r^2\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{8\pi\rho}}\right]\,\mathrm{d}r$$
$$-8\pi\sqrt{\frac{3}{\Lambda}}\int_{r_1}^{r_2} r^3 r_3^{-6} r^2\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{8\pi\rho}}\right]\,\mathrm{d}r$$
$$-8\pi\sqrt{\frac{3}{\Lambda}}\int_{r_2}^{r_3} r^3 r_3^{-6} r^2\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda}}\right]\,\mathrm{d}r\ . \tag{102.108}$$

We now use the values in table 102.1 for w and ρ to give

$$f_1=-8\pi\sqrt{\frac{3}{\Lambda}}r_3^{-6}\int_{r_0}^{r_1} r^3 r^2\left[\alpha(-\frac{2}{3})\frac{\Lambda}{8\pi}r_1 r_2^3 r^{-4}-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda r_1 r_2^3 r^{-4}}}\right]\,\mathrm{d}r$$
$$-8\pi\sqrt{\frac{3}{\Lambda}}r_3^{-6}\int_{r_1}^{r_2} r^3 r^2\left[\alpha(-1)\frac{\Lambda}{8\pi}r_2^3 r^{-3}-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda r_2^3 r^{-3}}}\right]\,\mathrm{d}r$$
$$-8\pi\sqrt{\frac{3}{\Lambda}}r_3^{-6}\int_{r_2}^{r_3} r^3 r^2\left[\alpha(-1)\frac{\Lambda}{8\pi}r_2^3 r^{-3}-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda}}\right]\,\mathrm{d}r\ . \tag{102.109}$$

Evaluating the integrals gives

$$f_1=-6r_1^{-1/2}r_2^{-3/2}r_3^{-6}\left[-\frac{\alpha}{3}r_1 r_2^3\frac{r_1^4-r_0^4}{4}-\frac{r_1^8-r_0^8}{8}\right]$$
$$-6r_2^{-3/2}r_3^{-6}\left[-\frac{\alpha}{2}r_2^3\frac{r_2^{9/2}-r_1^{9/2}}{9/2}-\frac{r_2^{15/2}-r_1^{15/2}}{15/2}\right]$$
$$-6r_3^{-6}\left[\frac{-\alpha}{2}r_2^3\frac{r_3^3-r_2^3}{3}-\frac{r_3^6-r_2^6}{6}\right]\ . \tag{102.110}$$

Or,

$$f_1=\frac{3}{2}\left(\frac{r_1}{r_3}\right)^{9/2}\left(\frac{r_2}{r_3}\right)^{3/2}\left[\frac{\alpha}{3}\left(1-\left(\frac{r_0}{r_1}\right)^4\right)+\frac{1}{2}\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{r_0}{r_1}\right)^8\right)\right]$$
$$+2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right]$$
$$+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right]\ . \tag{102.111}$$

We can neglect r_1 because of (102.67). Also, r_0 is smaller than r_1, so we can neglect it also. We cannot take r_0 equal to zero because of possible quantum effects that cannot be taken into account by this semi-classical approximation, but we can take it to be small enough. Neglecting these terms gives

$$f_1 = 2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}+\frac{2}{5}\right] + \left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right] . \quad (102.112)$$

Or,

$$f_1 = \frac{2\alpha}{3}\left(\frac{r_2}{r_3}\right)^6+\frac{4}{5}\left(\frac{r_2}{r_3}\right)^6+\alpha\left[\left(\frac{r_2}{r_3}\right)^3-\left(\frac{r_2}{r_3}\right)^6\right]+\left[1-\left(\frac{r_2}{r_3}\right)^6\right] . \quad (102.113)$$

Using (102.68) gives

$$f_1 = \left[1-\frac{1}{3}\left(\frac{r_2}{r_3}\right)^3\right]\left(\frac{r_2}{r_3}\right)^3\alpha+\left[1-\frac{1}{5}\left(\frac{r_2}{r_3}\right)^6\right] = .32\alpha+.97 . \quad (102.114)$$

We can take $\alpha = 0$[161] or $\alpha = 1$[162]. In either case, f_1 is of order unity.

We also use (102.106) for the volume, $p = w\rho$ for the equation of state, and split the integral in(102.105) for f_2 into three parts corresponding to the three eras. This gives

$$\begin{aligned} f_2 = &-12\pi\sqrt{\Lambda+8\pi\rho_3}\int_{r_0}^{r_1} r^{-1}\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(\Lambda+8\pi\rho)^{3/2}}\right]\mathrm{d}r \\ &-12\pi\sqrt{\Lambda+8\pi\rho_3}\int_{r_1}^{r_2} r^{-1}\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(\Lambda+8\pi\rho)^{3/2}}\right]\mathrm{d}r \\ &-12\pi\sqrt{\Lambda+8\pi\rho_3}\int_{r_2}^{r_3} r^{-1}\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(\Lambda+8\pi\rho)^{3/2}}\right]\mathrm{d}r . \end{aligned} \quad (102.115)$$

As with calculating f_1, we can abruptly change the formulas between eras without significant error. We also neglect $8\pi\rho_3$ with respect to Λ.

$$\begin{aligned} f_2 = &-12\pi\sqrt{\Lambda}\int_{r_0}^{r_1} r^{-1}\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(8\pi\rho)^{3/2}}\right]\mathrm{d}r \\ &-12\pi\sqrt{\Lambda}\int_{r_1}^{r_2} r^{-1}\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(8\pi\rho)^{3/2}}\right]\mathrm{d}r \\ &-12\pi\sqrt{\Lambda}\int_{r_2}^{r_3} r^{-1}\left[\alpha(w-1)\rho-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(\Lambda)^{3/2}}\right]\mathrm{d}r . \end{aligned} \quad (102.116)$$

We now use the values in table 102.1 for w and ρ to give

$$\begin{aligned} f_2 = &-12\pi\sqrt{\Lambda}\int_{r_0}^{r_1} r^{-1}\left[\alpha(-\frac{2}{3})\frac{\Lambda}{8\pi}r_1r_2^3r^{-4}-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(\Lambda r_1r_2^3r^{-4})^{3/2}}\right]\mathrm{d}r \\ &-12\pi\sqrt{\Lambda}\int_{r_1}^{r_2} r^{-1}\left[\alpha(-1)\frac{\Lambda}{8\pi}r_2^3r^{-3}-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(\Lambda r_2^3r^{-3})^{3/2}}\right]\mathrm{d}r \\ &-12\pi\sqrt{\Lambda}\int_{r_2}^{r_3} r^{-1}\left[\alpha(-1)\frac{\Lambda}{8\pi}r_2^3r^{-3}-\frac{\Lambda}{4\pi}\right]\left[\frac{1}{(\Lambda)^{3/2}}\right]\mathrm{d}r . \end{aligned} \quad (102.117)$$

Evaluating the integrals gives

$$f_2 = -3r_1^{-3/2}r_2^{-9/2}\left[-\frac{\alpha}{3}r_1r_2^3\frac{r_1^2-r_0^2}{2}-\frac{r_1^6-r_0^6}{6}\right]$$

$$-3r_2^{-9/2}\left[-\frac{\alpha}{2}r_2^3\frac{r_2^{3/2}-r_1^{3/2}}{3/2}-\frac{r_2^{9/2}-r_1^{9/2}}{9/2}\right]$$
$$-3\left[-\frac{\alpha}{2}r_2^3\frac{r_3^{-3}-r_2^{-3}}{-3}-\ln\frac{r_3}{r_2}\right] \ . \tag{102.118}$$

Or,

$$f_2=\frac{1}{2}r_1^{3/2}r_2^{-3/2}\left[\alpha\left(1-\left(\frac{r_0}{r_1}\right)^2\right)+\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{r_0}{r_1}\right)^6\right)\right]$$
$$+\left[\alpha\left(1-\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right]$$
$$+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right] \ . \tag{102.119}$$

As with f_1, we can neglect r_0 and r_1 to give

$$f_2=\left[\alpha+\frac{2}{3}\right]+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right] \ . \tag{102.120}$$

Or,

$$f_2=\alpha+\frac{2}{3}-\frac{\alpha}{2}\left[\left(\frac{r_2}{r_3}\right)^3-1\right]+3\ln\frac{r_3}{r_2}=\left[\frac{3}{2}-\frac{1}{2}\left(\frac{r_2}{r_3}\right)^3\right]\alpha+\frac{2}{3}+3\ln\frac{r_3}{r_2} \ . \tag{102.121}$$

Using (102.68) gives

$$f_2=\left[\frac{3}{2}-\frac{1}{2}\left(\frac{r_2}{r_3}\right)^3\right]\alpha+\frac{2}{3}+3\ln\frac{r_3}{r_2}=1.3\alpha+1.7 \ . \tag{102.122}$$

We can take $\alpha=0$[161] or $\alpha=1$[162]. In either case, f_2 is of order unity.

102.10.2 Appendix: Interval $b_0\to b_1$

Thus, using (102.45), (102.44), and (102.64), (102.96) becomes

$$\int_{b_0}^{b_1}=\int_{b_0}^{b_1}\psi_b(\Omega)\exp\left\{\frac{i}{\hbar}\int_{r_0}^{r_{b0}}V(r)r_3^{-3}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\sqrt{3}\left(\frac{r}{r_3}\right)^3\frac{1}{b}\right]\mathrm{d}r\right.$$
$$\left.+\frac{i}{\hbar}\int_{r_{b0}}^{r_3}V(r)r_3^{-3}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}-\frac{\sqrt{3}r_3^6b^2}{2r^6(\Lambda+8\pi\rho)^{3/2}}\right]\mathrm{d}r\right\}\mathrm{d}b \ , \tag{102.123}$$

Equation (102.123) can be written

$$\int_{b_0}^{b_1}=\int_{b_0}^{b_1}\psi_b(\Omega)\exp\{-iAf_1\}\exp\left\{\frac{iAb^2f_2}{\Lambda+8\pi\rho_3}\right\}\exp\{-iAf_3\}\,\mathrm{d}b \ , \tag{102.124}$$

where A is given by (102.102),

$$f_1=-6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}}\int_{r_{b0}}^{r_3}V(r)r_3^{-6}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}\right]\mathrm{d}r \ , \tag{102.125}$$

$$f_2 = -6\sqrt{3(\Lambda+8\pi\rho_3)}\int_{r_{b0}}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\frac{\sqrt{3}r_3^6}{2r^6(\Lambda+8\pi\rho)^{3/2}}\right] \mathrm{d}r \ , \tag{102.126}$$

and

$$f_3 = -6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}}\int_{r_0}^{r_{b0}} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\frac{\sqrt{3}}{b}\left(\frac{r}{r_3}\right)^3\right] \mathrm{d}r \ . \tag{102.127}$$

As before, we can separate the integrals in (102.125) and (102.126) into three parts. Evaluating gives

$$\begin{aligned} f_1 = \frac{3}{2}\left(\frac{r_1}{r_3}\right)^{9/2}\left(\frac{r_2}{r_3}\right)^{3/2}&\left[\frac{\alpha}{3}\left(1-\left(\frac{r_{b0}}{r_1}\right)^4\right)+\frac{1}{2}\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{r_{b0}}{r_1}\right)^8\right)\right] \\ &+2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right] \\ &+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right] \end{aligned} \tag{102.128}$$

and

$$\begin{aligned} f_2 = \frac{1}{2}\left(\frac{r_1}{r_2}\right)^{3/2}&\left[\alpha\left(1-\left(\frac{r_{b0}}{r_1}\right)^2\right)+\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{r_{b0}}{r_1}\right)^6\right)\right] \\ &+\left[\alpha\left(1-\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right] \\ &+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right] \ . \end{aligned} \tag{102.129}$$

Evaluating (102.127) gives

$$f_3 = 2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\frac{b_1}{b}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{r_{b0}}{r_1}\right)^5-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{r_{b0}}{r_1}\right)^9-\left(\frac{r_0}{r_1}\right)^9\right)\right] \ . \tag{102.130}$$

Because r_1 could be neglected in evaluating f_1 and f_2 previously, we should be able to neglect r_{b0}, since it is smaller than r_1. That would give (102.114) and (102.122), as before. On the other hand, we have from Table 102.2 that

$$\frac{r_{b0}}{r_1} = \frac{b}{b_1} = \frac{b}{b_0}\frac{r_0}{r_1} \ . \tag{102.131}$$

Therefore, (102.128), (102.129), and (102.130) become

$$\begin{aligned} f_1 = \frac{3}{2}\left(\frac{r_1}{r_3}\right)^{9/2}\left(\frac{r_2}{r_3}\right)^{3/2}&\left[\frac{\alpha}{3}\left(1-\left(\frac{b}{b_1}\right)^4\right)+\frac{1}{2}\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{b}{b_1}\right)^8\right)\right] \\ &+2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right] \\ &+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right] \ , \end{aligned} \tag{102.132}$$

$$f_2 = \frac{1}{2}\left(\frac{r_1}{r_2}\right)^{3/2}\left[\alpha\left(1-\left(\frac{b}{b_1}\right)^2\right)+\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{b}{b_1}\right)^6\right)\right]$$

$$+\left[\alpha\left(1-\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right]$$
$$+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right], \tag{102.133}$$

and

$$f_3=2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\frac{b_1}{b}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{b}{b_1}\right)^5-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{b}{b_1}\right)^9-\left(\frac{r_0}{r_1}\right)^9\right)\right]. \tag{102.134}$$

Alternatively, we have

$$f_1=\frac{3}{2}\left(\frac{r_1}{r_3}\right)^{9/2}\left(\frac{r_2}{r_3}\right)^{3/2}\left[\frac{\alpha}{3}\left(1-\left(\frac{b}{b_0}\frac{r_0}{r_1}\right)^4\right)+\frac{1}{2}\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{b}{b_0}\frac{r_0}{r_1}\right)^8\right)\right]$$
$$+2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right]$$
$$+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right], \tag{102.135}$$

$$f_2=\frac{1}{2}\left(\frac{r_1}{r_2}\right)^{3/2}\left[\alpha\left(1-\left(\frac{b}{b_0}\frac{r_0}{r_1}\right)^2\right)+\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{b}{b_0}\frac{r_0}{r_1}\right)^6\right)\right]$$
$$+\left[\alpha\left(1-\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right]$$
$$+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right], \tag{102.136}$$

and

$$f_3=2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\frac{b_0}{b}\frac{r_1}{r_0}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{b}{b_0}\frac{r_0}{r_1}\right)^5-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{b}{b_0}\frac{r_0}{r_1}\right)^9-\left(\frac{r_0}{r_1}\right)^9\right)\right]. \tag{102.137}$$

We need to know the values of these quantities at the endpoints of the interval. Thus, we have

$$f_1(b_0)=\frac{3}{2}\left(\frac{r_1}{r_3}\right)^{9/2}\left(\frac{r_2}{r_3}\right)^{3/2}\left[\frac{\alpha}{3}\left(1-\left(\frac{r_0}{r_1}\right)^4\right)+\frac{1}{2}\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{r_0}{r_1}\right)^8\right)\right]$$
$$+2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right]$$
$$+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right], \tag{102.138}$$

$$f_2(b_0)=\frac{1}{2}\left(\frac{r_1}{r_2}\right)^{3/2}\left[\alpha\left(1-\left(\frac{r_0}{r_1}\right)^2\right)+\left(\frac{r_1}{r_2}\right)^3\left(1-\left(\frac{r_0}{r_1}\right)^6\right)\right]$$
$$+\left[\alpha\left(1-\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right]$$
$$+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right], \tag{102.139}$$

and

$$f_3(b_0) = 2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\frac{r_1}{r_0}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{r_0}{r_1}\right)^5-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{r_0}{r_1}\right)^9-\left(\frac{r_0}{r_1}\right)^9\right)\right]=0\ . \tag{102.140}$$

and

$$\begin{aligned} f_1(b_1) = \frac{3}{2}\left(\frac{r_1}{r_3}\right)^{9/2}\left(\frac{r_2}{r_3}\right)^{3/2}&\left[\frac{\alpha}{3}\left(1-(1)^4\right)+\frac{1}{2}\left(\frac{r_1}{r_2}\right)^3\left(1-(1)^8\right)\right] \\ +2\left(\frac{r_2}{r_3}\right)^6&\left[\frac{\alpha}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right] \\ &+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right]\ , \end{aligned} \tag{102.141}$$

$$\begin{aligned} f_2(b_1) = \frac{1}{2}\left(\frac{r_1}{r_2}\right)^{3/2}&\left[\alpha\left(1-(1)^2\right)+\left(\frac{r_1}{r_2}\right)^3\left(1-(1)^6\right)\right] \\ &+\left[\alpha\left(1-\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right] \\ &+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right]\ , \end{aligned} \tag{102.142}$$

and

$$f_3(b_1) = 2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left((1)^5-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left((1)^9-\left(\frac{r_0}{r_1}\right)^9\right)\right]\ . \tag{102.143}$$

Equation (102.124) can be written

$$\int_{b_0}^{b_1} = \int_{b_0}^{b_1}\psi_b(\Omega)\exp\left\{-iAf_1\right\}\exp\left\{iA\left[\frac{b^2 f_2}{\Lambda+8\pi\rho_3}-f_3\right]\right\}\mathrm{d}b\ , \tag{102.144}$$

We want to compare the f_3 contribution with the f_2 contribution to (102.144) within the interval from b_0 to b_1. Equation (102.140) shows that $f_3(b_0)=0$, so that the contribution of f_3 at that end of the interval can be neglected. On the other hand, using (102.142) and (102.143) shows that the contribution of f_3 is down by about a factor of $(r_1/r_3)^{3/2}\approx 5\times10^{-6}$ compared with the f_2 contribution. Thus, we can neglect the contribution of f_3 in the interval from b_0 to b_1 This will also apply to the interval from $-b_1$ to $-b_0$.

102.10.3 Appendix: Interval $b_1 \to b_2$

Thus, using (102.45), (102.44), and (102.64), (102.97) becomes

$$\begin{aligned} \int_{b_1}^{b_2} = \int_{b_1}^{b_2}\psi_b(\Omega)\exp&\left\{\frac{i}{\hbar}\int_{r_0}^{r_{b1}}V(r)r_3^{-3}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\sqrt{3}\left(\frac{r}{r_3}\right)^3\frac{1}{b}\right]\mathrm{d}r\right. \\ &\left.+\frac{i}{\hbar}\int_{r_{b1}}^{r_3}V(r)r_3^{-3}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}-\frac{\sqrt{3}r_3^6b^2}{2r^6(\Lambda+8\pi\rho)^{3/2}}\right]\mathrm{d}r\right\}\mathrm{d}b\ , \end{aligned} \tag{102.145}$$

Equation (102.145) can be written

$$\int_{b_1}^{b_2} = \int_{b_1}^{b_2}\psi_b(\Omega)\exp\left\{-iAf_1\right\}\exp\left\{\frac{iAb^2 f_2}{\Lambda+8\pi\rho_3}\right\}\exp\left\{-iAf_3\right\}\mathrm{d}b\ , \tag{102.146}$$

where A is given by (102.102),

$$f_1 = -6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}}\int_{r_{b1}}^{r_3} V(r)r_3^{-6}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}\right]\mathrm{d}r\ , \tag{102.147}$$

$$f_2 = -6\sqrt{3(\Lambda+8\pi\rho_3)}\int_{r_{b1}}^{r_3} V(r)r_3^{-6}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\frac{\sqrt{3}r_3^6}{2r^6(\Lambda+8\pi\rho)^{3/2}}\right]\mathrm{d}r\ , \tag{102.148}$$

and

$$f_3 = -6\sqrt{\frac{3}{\Lambda+8\pi\rho_3}}\int_{r_0}^{r_{b1}} V(r)r_3^{-6}r^2\left[\alpha(p-\rho)-\frac{\Lambda}{4\pi}\right]\left[\frac{\sqrt{3}}{b}\left(\frac{r}{r_3}\right)^3\right]\mathrm{d}r\ . \tag{102.149}$$

As before, we can separate the integrals in (102.147), (102.148), and (102.149) into parts. Evaluating gives

$$\begin{aligned} f_1 = 2\left(\frac{r_2}{r_3}\right)^6&\left[\frac{\alpha}{3}\left(1-\left(\frac{r_{b1}}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_{b1}}{r_2}\right)^{15/2}\right)\right]\\ &+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right]\ , \end{aligned} \tag{102.150}$$

$$\begin{aligned} f_2 = &\left[\alpha\left(1-\left(\frac{r_{b1}}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_{b1}}{r_2}\right)^{9/2}\right)\right]\\ &+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right]\ , \end{aligned} \tag{102.151}$$

and

$$\begin{aligned} f_3 = 2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\frac{b_1}{b}&\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^9\right)\right.\\ &\left.+\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{r_{b1}}{r_1}\right)^6-1\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{r_{b1}}{r_1}\right)^9-1\right)\right]\ . \end{aligned} \tag{102.152}$$

We have from Table 102.3 that

$$\frac{r_{b1}}{r_2} = \left(\frac{b}{b_2}\right)^{2/3} = \left(\frac{b}{b_1}\right)^{2/3}\frac{r_1}{r_2}\ . \tag{102.153}$$

Therefore, (102.150), (102.151), and (102.152) become

$$\begin{aligned} f_1 = 2\left(\frac{r_2}{r_3}\right)^6&\left[\frac{\alpha}{3}\left(1-\left(\frac{b}{b_1}\right)^3\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{b}{b_1}\right)^5\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right]\\ &+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right]\ , \end{aligned} \tag{102.154}$$

$$\begin{aligned} f_2 = &\left[\alpha\left(1-\left(\frac{b}{b_1}\right)\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{b}{b_1}\right)^3\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right]\\ &+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right]\ , \end{aligned} \tag{102.155}$$

and

$$f_3 = 2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\frac{b_1}{b}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^9\right)\right.$$
$$\left.+\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{b}{b_1}\right)^4-1\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{b}{b_1}\right)^6-1\right)\right]. \quad (102.156)$$

Alternatively, we have

$$f_1 = 2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}\left(1-\left(\frac{b}{b_2}\right)^3\right)+\frac{2}{5}\left(1-\left(\frac{b}{b_2}\right)^5\right)\right]$$
$$+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right], \quad (102.157)$$

$$f_2 = \left[\alpha\left(1-\left(\frac{b}{b_2}\right)\right)+\frac{2}{3}\left(1-\left(\frac{b}{b_2}\right)^3\right)\right]$$
$$+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right], \quad (102.158)$$

and

$$f_3 = 2\left(\frac{r_1}{r_3}\right)^6\left(\frac{r_3}{r_2}\right)^{3/2}\frac{b_2}{b}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^9\right)\right.$$
$$\left.+\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{b}{b_2}\right)^4\left(\frac{r_2}{r_1}\right)^6-1\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{b}{b_2}\right)^6\left(\frac{r_2}{r_1}\right)^9-1\right)\right]. \quad (102.159)$$

We need to know the values of these quantities at the endpoints of the interval. Thus, we have

$$f_1(b_1) = 2\left(\frac{r_2}{r_3}\right)^6\left[\frac{\alpha}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)+\frac{2}{5}\left(1-\left(\frac{r_1}{r_2}\right)^{15/2}\right)\right]$$
$$+\left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right], \quad (102.160)$$

$$f_2(b_1) = \left[\alpha\left(1-\left(\frac{r_1}{r_2}\right)^{3/2}\right)+\frac{2}{3}\left(1-\left(\frac{r_1}{r_2}\right)^{9/2}\right)\right]$$
$$+\left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right], \quad (102.161)$$

and

$$f_3(b_1) = 2\left(\frac{r_1}{r_3}\right)^{\frac{9}{2}}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^9\right)\right.$$
$$\left.+\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}(1-1)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}(1-1)\right]. \quad (102.162)$$

and

$$f_1(b_2) = 2\left(\frac{r_2}{r_3}\right)^6 \left[\frac{\alpha}{3}\left(1-(1)^{9/2}\right) + \frac{2}{5}\left(1-(1)^{15/2}\right)\right] + \left[\alpha\left(\frac{r_2}{r_3}\right)^3 \left(1-\left(\frac{r_2}{r_3}\right)^3\right) + \left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right] , \tag{102.163}$$

$$f_2(b_2) = \left[\alpha\left(1-(1)^{3/2}\right) + \frac{2}{3}\left(1-(1)^{9/2}\right)\right] + \left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3 \left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right) + 3\ln\frac{r_3}{r_2}\right] , \tag{102.164}$$

and

$$f_3(b_2) = 2\left(\frac{r_1}{r_3}\right)^6 \left(\frac{r_3}{r_2}\right)^{3/2} \left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(1-\left(\frac{r_0}{r_1}\right)^5\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(1-\left(\frac{r_0}{r_1}\right)^9\right) + \frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(\left(\frac{r_2}{r_1}\right)^6 - 1\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(\left(\frac{r_2}{r_1}\right)^9 - 1\right)\right] . \tag{102.165}$$

Equation (102.146) can be written

$$\int_{b_1}^{b_2} = \int_{b_1}^{b_2} \psi_b(\Omega) \exp\{-iAf_1\} \exp\left\{iA\left[\frac{b^2 f_2}{\Lambda + 8\pi\rho_3} - f_3\right]\right\} \mathrm{d}b , \tag{102.166}$$

We want to compare the f_3 contribution with the f_2 contribution to (102.166) within the interval from b_1 to b_2. Using (102.161) and (102.162) shows that the contribution of f_3 is down by about a factor of $(r_1/r_2)^{3/2} \approx 8 \times 10^{-6}$ compared with the f_2 contribution at $b = b_1$. Thus, we can neglect the contribution of f_3 at the beginning of the interval from b_1 to b_2. On the other hand, using (102.164) and (102.165) shows that the contribution of f_3 is comparable with the f_2 contribution at $b = b_2$. This analysis will also apply to the interval from $-b_2$ to $-b_1$.

102.10.4 Appendix: Interval $b_2 \to b_3$

Thus, using (102.45), (102.44), and (102.64), (102.98) becomes

$$\int_{b_2}^{b_3} = \int_{b_2}^{b_3} \psi_b(\Omega) \exp\left\{\frac{i}{\hbar}\int_{r_0}^{r_{b2}} V(r) r_3^{-3} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right]\left[\sqrt{3}\left(\frac{r}{r_3}\right)^3 \frac{1}{b}\right] \mathrm{d}r + \frac{i}{\hbar}\int_{r_{b2}}^{r_3} V(r) r_3^{-3} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}} - \frac{\sqrt{3} r_3^6 b^2}{2r^6(\Lambda+8\pi\rho)^{3/2}}\right] \mathrm{d}r\right\} \mathrm{d}b , \tag{102.167}$$

Equation (102.167) can be written

$$\int_{b_2}^{b_3} = \int_{b_2}^{b_3} \psi_b(\Omega) \exp\{-iAf_1\} \exp\left\{\frac{iAb^2 f_2}{\Lambda + 8\pi\rho_3}\right\} \exp\{-iAf_3\} \,\mathrm{d}b , \tag{102.168}$$

where A is given by (102.102),

$$f_1 = -6\sqrt{\frac{3}{\Lambda + 8\pi\rho_3}} \int_{r_{b2}}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right]\left[\sqrt{\frac{3}{\Lambda+8\pi\rho}}\right] \mathrm{d}r , \tag{102.169}$$

$$f_2 = -6\sqrt{3(\Lambda + 8\pi\rho_3)} \int_{r_{b2}}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right] \left[\frac{\sqrt{3} r_3^6}{2r^6(\Lambda + 8\pi\rho)^{3/2}}\right] \mathrm{d}r \ , \tag{102.170}$$

and

$$f_3 = -6\sqrt{\frac{3}{\Lambda + 8\pi\rho_3}} \int_{r_0}^{r_{b2}} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right] \left[\frac{\sqrt{3}}{b}\left(\frac{r}{r_3}\right)^3\right] \mathrm{d}r \ . \tag{102.171}$$

Evaluating (102.169) and (102.170) gives

$$f_1 = \left[\alpha\left(\frac{r_2}{r_3}\right)^3 \left(1 - \left(\frac{r_{b2}}{r_3}\right)^3\right) + \left(1 - \left(\frac{r_{b2}}{r_3}\right)^6\right)\right] \tag{102.172}$$

and

$$f_2 = \left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3 \left(1 - \left(\frac{r_{b2}}{r_3}\right)^{-3}\right) + 3\ln\frac{r_3}{r_{b2}}\right] \ . \tag{102.173}$$

As before, we can separate the integral in (102.171) into parts. Evaluating gives

$$\begin{aligned} f_3 = 2\left(\frac{r_1}{r_3}\right)^6 \left(\frac{r_3}{r_2}\right)^{3/2} \frac{b_2}{b} \Bigg[&\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(1 - \left(\frac{r_0}{r_1}\right)^5\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(1 - \left(\frac{r_0}{r_1}\right)^9\right) \\ &+ \frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(\left(\frac{r_{b2}}{r_1}\right)^6 - 1\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(\left(\frac{r_{b2}}{r_1}\right)^9 - 1\right)\Bigg] \ . \end{aligned} \tag{102.174}$$

We have from Table 102.4 that

$$\frac{r_{b2}}{r_3} = \left(\frac{b}{b_3}\right)^{1/3} = \left(\frac{b}{b_2}\right)^{1/3} \frac{r_2}{r_3} \ . \tag{102.175}$$

Therefore, (102.172), (102.173), and (102.174) become

$$f_1 = \left[\alpha\left(\frac{r_2}{r_3}\right)^3 \left(1 - \left(\frac{b}{b_2}\right)\left(\frac{r_2}{r_3}\right)^3\right) + \left(1 - \left(\frac{b}{b_2}\right)^2 \left(\frac{r_2}{r_3}\right)^6\right)\right] \ , \tag{102.176}$$

$$f_2 = \left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3 \left(1 - \left(\frac{b}{b_2}\right)^{-1} \left(\frac{r_2}{r_3}\right)^{-3}\right) + 3\ln\frac{r_3}{r_2} - \ln\left(\frac{b}{b_2}\right)\right] \ , \tag{102.177}$$

and

$$\begin{aligned} f_3 = 2\left(\frac{r_1}{r_3}\right)^6 \left(\frac{r_3}{r_2}\right)^{3/2} \frac{b_2}{b} \Bigg[&\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(1 - \left(\frac{r_0}{r_1}\right)^5\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(1 - \left(\frac{r_0}{r_1}\right)^9\right) \\ &+ \frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(\left(\frac{b}{b_2}\right)^2 \left(\frac{r_2}{r_1}\right)^6 - 1\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(\left(\frac{b}{b_2}\right)^3 \left(\frac{r_2}{r_1}\right)^9 - 1\right)\Bigg] \ . \end{aligned} \tag{102.178}$$

Alternatively, we have

$$f_1 = \left[\alpha\left(\frac{r_2}{r_3}\right)^3 \left(1 - \left(\frac{b}{b_3}\right)\right) + \left(1 - \left(\frac{b}{b_3}\right)^2\right)\right] \ , \tag{102.179}$$

$$f_2 = \left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3 \left(1 - \left(\frac{b}{b_3}\right)^{-1}\right) - \ln\left(\frac{b}{b_3}\right)\right] \ , \tag{102.180}$$

and

$$f_3 = 2\left(\frac{r_1}{r_3}\right)^6\left(\frac{r_2}{r_3}\right)^{3/2}\frac{b_3}{b}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^9\right)\right.$$
$$\left.+\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{b}{b_3}\right)^2\left(\frac{r_3}{r_1}\right)^6-1\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{b}{b_3}\right)^3\left(\frac{r_3}{r_1}\right)^9-1\right)\right]. \quad (102.181)$$

We need to know the values of these quantities at the endpoints of the interval. Thus, we have

$$f_1(b_2) = \left[\alpha\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^3\right)+\left(1-\left(\frac{r_2}{r_3}\right)^6\right)\right], \quad (102.182)$$

$$f_2(b_2) = \left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3\left(1-\left(\frac{r_2}{r_3}\right)^{-3}\right)+3\ln\frac{r_3}{r_2}\right], \quad (102.183)$$

and

$$f_3(b_2) = 2\left(\frac{r_1}{r_3}\right)^6\left(\frac{r_3}{r_2}\right)^{3/2}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^9\right)\right.$$
$$\left.+\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{r_2}{r_1}\right)^6-1\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{r_2}{r_1}\right)^9-1\right)\right]. \quad (102.184)$$

and

$$f_1(b_3) = \left[\alpha\left(\frac{r_2}{r_3}\right)^3(1-1)+(1-1)\right] = 0, \quad (102.185)$$

$$f_2(b_3) = \left[-\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3(1-1)\right] = 0, \quad (102.186)$$

and

$$f_3(b_3) = 2\left(\frac{r_1}{r_3}\right)^6\left(\frac{r_2}{r_3}\right)^{3/2}\left[\frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^5\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(1-\left(\frac{r_0}{r_1}\right)^9\right)\right.$$
$$\left.+\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}}\left(\left(\frac{r_3}{r_1}\right)^6-1\right)+\frac{1}{3}\left(\frac{r_1}{r_3}\right)^3\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\left(\left(\frac{r_3}{r_1}\right)^9-1\right)\right]. \quad (102.187)$$

Equation (102.168) can be written

$$\int_{b_2}^{b_3} = \int_{b_2}^{b_3}\psi_b(\Omega)\exp\left\{iA\left[-f_1+\frac{b^2 f_2}{\Lambda+8\pi\rho_3}-f_3\right]\right\}\mathrm{d}b, \quad (102.188)$$

We want to compare the f_3 contribution with the f_2 contribution to (102.188) within the interval from b_2 to b_3. Using (102.183) and (102.184) shows that the contribution of f_3 is comparable with the f_2 contribution at $b = b_2$. On the other hand, (102.186) shows that $f_2 = 0$ at $b = b_3$. Thus the contribution of f_3 in (102.187) dominates that of the f_2 contribution at $b = b_3$. This analysis will also apply to the interval from $-b_3$ to $-b_2$.

In summary, $-f_1 + \frac{b^2 f_2}{\Lambda+8\pi\rho_3} - f_3$ varies from $-.82 - .26\alpha$ to $-.67 - .18\alpha$ as b varies from b_2 to b_3. That is, the quantity does not vary much, indicating that when multiplied by $A \approx 10^{121}$, the exponent in (102.188) will be changing rapidly along the stationary-phase path, or the exponential in (102.188) will remain tiny along the steepest descent path.

102.10.5 Appendix: Contributions for large b, interval $b_3 \to b_{\mathrm{max}}$

Thus, using (102.45), (102.44), and (102.64), (102.99) becomes

$$\int_{b_3}^{b_{\mathrm{max}}} = \int_{b_3}^{b_{\mathrm{max}}} \psi_b(\Omega) \exp\left\{ \frac{i}{\hbar} \int_{r_0}^{r_{b2}} V(r) r_3^{-3} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right] \left[\sqrt{3}\left(\frac{r}{r_3}\right)^3 \frac{1}{b}\right] \mathrm{d}r \right\} \mathrm{d}b \ , \tag{102.189}$$

Equation (102.189) can be written

$$\int_{b_3}^{b_{\mathrm{max}}} = \int_{b_3}^{b_{\mathrm{max}}} \psi_b(\Omega) \exp\{-iAf_3\} \,\mathrm{d}b \ , \tag{102.190}$$

where A is given by (102.102) and

$$f_3 = -6\sqrt{\frac{3}{\Lambda + 8\pi\rho_3}} \int_{r_0}^{r_3} V(r) r_3^{-6} r^2 \left[\alpha(p-\rho) - \frac{\Lambda}{4\pi}\right] \left[\frac{\sqrt{3}}{b}\left(\frac{r}{r_3}\right)^3\right] \mathrm{d}r \ . \tag{102.191}$$

Evaluating gives

$$\begin{aligned} f_3 = 2\left(\frac{r_1}{r_3}\right)^6 \left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \frac{b_3}{b} \Bigg[& \frac{\alpha}{5}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(1 - \left(\frac{r_0}{r_1}\right)^5\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(1 - \left(\frac{r_0}{r_1}\right)^9\right) \\ & + \frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(\left(\frac{r_3}{r_1}\right)^6 - 1\right) + \frac{1}{3}\left(\frac{r_1}{r_3}\right)^3 \left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(\left(\frac{r_3}{r_1}\right)^9 - 1\right)\Bigg] \ . \end{aligned} \tag{102.192}$$

Taking r_0 and r_1 small gives

$$f_3 = 2\left(\frac{r_1}{r_3}\right)^6 \left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \frac{b_3}{b} \left[\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(\frac{r_3}{r_1}\right)^6 + \frac{1}{3}\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(\frac{r_3}{r_1}\right)^6\right] \ . \tag{102.193}$$

Using Table 102.5 for b_3 gives

$$f_3 = 2\frac{\sqrt{\Lambda}}{b}\left(\frac{r_1}{r_3}\right)^6 \left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left[\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left(\frac{r_3}{r_1}\right)^6 + \frac{1}{3}\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}} \left(\frac{r_3}{r_1}\right)^6\right] \ . \tag{102.194}$$

Or,

$$f_3 = 2\frac{\sqrt{\Lambda}}{b}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} \left[\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^{\frac{3}{2}} + \frac{1}{3}\left(\frac{r_2}{r_3}\right)^{-\frac{3}{2}}\right] \ . \tag{102.195}$$

Or,

$$f_3 = 2\frac{\sqrt{\Lambda}}{b}\left[\frac{\alpha}{4}\left(\frac{r_2}{r_3}\right)^3 + \frac{1}{3}\right] \ . \tag{102.196}$$

Substituting (102.196) into (102.190) gives

$$\int_{b_3}^{b_{\mathrm{max}}} = \int_{b_3}^{b_{\mathrm{max}}} \psi_b(\Omega) \exp\left\{-iA\frac{\sqrt{\Lambda}}{b}\left[\frac{\alpha}{2}\left(\frac{r_2}{r_3}\right)^3 + \frac{2}{3}\right]\right\} \mathrm{d}b \ , \tag{102.197}$$

However, in (102.91) and (102.99), there is no small-b term, so we have only the large-b contribution.

We see that f_3 varies from $0.67 + 0.18\alpha$ to $1.8 \times 10^{-61} \times (0.67 + 0.18\alpha)$ as b varies from $b_3 = \sqrt{\Lambda}$ to $b_{\mathrm{max}} = 1/L^*$. The latter value of f_3 may seem like a small number, but since it is multiplied by $A \approx 10^{121}$, the exponent in (102.197) is still changing very rapidly along the stationary-phase path.

To drive it home even farther, we could change variables from b to x, using $x^2 = 1/b$. The integral then has a saddlepoint at $x = 0$ (same as $b = \infty$). Actually, it will be a saddlepoint with a triple pole, but that should be no problem. Then it can be seen that that saddlepoint is not in the integration interval, and is far enough away from the endpoint that it is possible to neglect any contribution from that saddlepoint.

Chapter 103

Possible limitations for the uncertainty relations[1]

abstract

Limiting the range (or domain?) of validity of uncertainty relations may solve some problems in zero-point energy and other problems in quantum theory.

103.1 Introduction

There was a controversy about whether the uncertainty relations were based on measurement or whether they were more fundamental. Although the view that they were more fundamental won out, that view may have led to problems such as the magnitude of zero-point energy being a factor of about 10^{120} too large. Reverting to limiting the uncertainty relations to being based on measurement may solve that problem.

A modern view of Planck's radiation formula [354, 355, 356, 357, 358] is that electromagnetic energy is radiated in quanta of size $\hbar\omega$.[2] In addition, although we now know that stars are powered by nuclear processes in the core of the star, that radiation takes about 10,000 years to reach the surface of the star because it is not simply scattered by the atoms of the star, but is absorbed and re-radiated by those atoms. Planck's radiation formula also tells us that the radiation is characteristic of the surface temperature of the star, having lost all information about how it was originally generated.

We also now accept that electromagnetic energy is also absorbed in quanta of size $\hbar\omega$.

The usual interpretation of Einstein's 1905 paper[128] explaining the photoelectric effect is that the electromagnetic field is itself intrinsically quantized. However, that interpretation is not necessarily correct. In the same way that Compton scattering can be explained in terms of wave interference, the photoelectric effect can be explained in terms of wave interference as follows. Wave interference implies that

$$\omega_{\text{incident radiation}} = \omega_{\text{ionization energy}} + \omega_{\text{kinetic energy}} \tag{103.1}$$

and

$$\mathbf{k}_{\text{incident radiation}} = \mathbf{k}_{\text{ionization energy}} + \mathbf{k}_{\text{kinetic energy}} \,, \tag{103.2}$$

[1] 2-27 May 2014

[2] It is interesting to notice that Max Planck derived the natural units we now refer to as Planck units and had a value for what is now known as Planck's constant in May of 1899 [359], a year and a half before he first suggested that radiation might be quantized [355, 357]. That 1899 paper was the fifth in a series of five papers [360, 361, 362, 363, 359]. See John C. D. Brand's 1995 book [364] for a history of the introduction of the quantum.

with obvious terminology.

So, although we know that electromagnetic energy is generated and absorbed in quanta, we do not know if the electromagnetic field remains as individual quanta. However, because we now treat photons as bosons, we expect that quanta of electromagnetic radiation lose their individuality, so that all of the electromagnetic fields simply add together.

Still, our modern view shows us that the electromagnetic field is not intrinsically a classical field. This is shown by the fact that E and B fields do not commute, indicating that it is impossible to measure both simultaneously with exact precision. Specifically,

$$\left[E_x(x), B_y(x')\right] = ic\hbar\delta(\mathrm{d}\,s) \;, \tag{103.3}$$

where

$$\mathrm{d}\,s = \sqrt{g_{\mu\nu}(x-x')^{\mu}(x-x')^{\nu})} \;. \tag{103.4}$$

The derivation of (103.3) involves using

$$\delta(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx} \;. \tag{103.5}$$

If we replace the Dirac delta function in (103.3) by the sum in (103.5), then we have the original formula before the substitution was made to give the formula in terms of the Dirac delta function. The sum in that case is over electromagnetic modes of radiation. In the usual modern interpretation, this sum really does go to infinity, but leads to well-known problems.

Because we want to consider the uncertainty principle in terms of measurement limitations, we restrict the sum to only include radiation modes that correspond to radiation or absorption modes for the atoms or molecules that actually exist in the region of space and time where the uncertainty principle is being applied because any measurements of the radiation field must be made with actual atoms or molecules.

In regions of space and time that contain no such atoms or molecules, there would be no terms in the sum. Although virtual particles should probably be considered in such cases, it can probably be shown that they do not contribute significantly.

Chapter 104

Box normalization[1]

abstract

One aspect of quantum theory that needs to be examined is choice of modes that comes from box normalization, along with the corresponding creation and annihilation operators. Although this mathematical procedure works in many situations, it clearly does not represent the real situation, and by trying to use it in difficult situations may not work well, for example in non-inertial frames [8].

104.1 Introduction

I think the way to do the calculation is to first transform the Fourier series to a Fourier transform to get a continuous spectrum. That is a more realistic way to represent the actual fields. The second step is to show that emission and absorption can be represented as an integral over frequency. Spectral lines are not sharp in any case, so that is more realistic. Then, if necessary, we can move the frequency integration into the complex frequency plane. Eventually, we shall see what should replace creation and annihilation operators.

[1]- 21 July 2014

Chapter 105

Discrete General Relativity

abstract

Gary Gibbons and George Ellis [365, 366] have developed a formalism for a discrete form of Newtonian cosmology that overcomes some of the problems with cosmology based on General Relativity, such as infinities and the problems with treating the cosmology as a fluid, while still getting a Friedmann equation for the cosmology. However, because their treatment is Newtonian, inertia is still given by the background, rather than from frame-dragging, as in General Relativity.

To overcome that problem, I develop a discrete representation of General Relativity. To do that, we start with a stress-energy tensor for discrete particles, ignoring fields to begin with. In that case, the action for the matter can be expressed as a sum over all of the particles. Also, in that case, the action for the geometry can also be expressed as a sum. Therefore, the total action can be expressed as a sum. At this point, we can ignore that we have a geometry. Then, setting the variation of the action to zero gives (at least formally) the Einstein field equations. However, at this point, we do not know that $g_{\mu\nu}$ represents a metric, nor do we know how the Ricci tensor is related to the connection.

To go further, we consider that there are various fields (to be defined later) that have values at each particle, but no positions yet. At this point, we make a guess that derivatives with respect to coordinates need to (somehow) be replaced with differences of field variables evaluated at different particles. To get geometry, we somehow find that particles whose field variables are highly correlated must be close to each other. (There may be situations where that is not possible. When that happens, we are stuck with a pure quantum gravity, similar to the physics in a sparse universe.)

Another approach to writing General Relativity in terms of discrete masses, starts with the SWG equations [16] using a discrete energy-momentum tensor.

105.1 Introduction

Gary Gibbons and George Ellis [365, 366] have developed a formalism for a discrete form of Newtonian cosmology that overcomes some of the problems with cosmology based on General Relativity, such as infinities and the problems with treating the cosmology as a fluid, while still getting a Friedmann equation for the cosmology. However, because the treatment is Newtonian, inertia is still given by the background, rather than from frame-dragging, as in General Relativity.

To overcome that problem, I develop a discrete representation of General Relativity.

105.2 Discrete representation of General Relativity without fields

To develop a discrete representation of General Relativity, we start with the usual formula for the action.

$$S = \int \left[\frac{R}{2k} + \mathcal{L}_M \right] \sqrt{-g} \mathrm{d}^4 x \ , \tag{105.1}$$

where R is the Ricci scalar, $\mathcal{L}_M$ is the matter Lagrangian, $k = 8\pi G c^{-4}$, $g = \det(g_{\mu\nu})$, G is Newton's gravitational constant, and c is the free-space speed of light.

We need a stress-energy tensor for discrete particles, ignoring fields to begin with.

$$T^{\mu\nu} = \rho u^\mu u^\nu + \frac{p}{c^2}(u^\mu u^\nu - g^{\mu\nu}) \ . \tag{105.2}$$

To make it discrete, we ignore pressure, p, and let the density be given by

$$\rho = \sum_{i=1}^{N} m_i \delta^3(x - x_i) \ . \tag{105.3}$$

Substituting (105.3) into (105.2) gives

$$T^{\mu\nu} = \sum_{i=1}^{N} m_i \delta^3(x - x_i) u^\mu u^\nu \ . \tag{105.4}$$

When the field equations are satisfied, $R^{\mu\nu}$ will also be discrete. We can write it as

$$R^{\mu\nu} = \sum_{i=1}^{N} R_i^{\mu\nu} \delta^3(x - x_i) \ . \tag{105.5}$$

We can also write $\mathcal{L}_M$ as discrete.

$$\mathcal{L}_M = \sum_{i=1}^{N} \mathcal{L}_{i\text{Matter}} \delta^3(x - x_i) \ . \tag{105.6}$$

In that case, the action for the matter can be expressed as a sum over all of the particles. Also, in that case, the action for the geometry can also be expressed as a sum. Therefore, the total action can be expressed as a sum. Substituting (105.5) and (105.6) into (105.1) gives

$$S = \sum_{i=1}^{N} \int \left[\frac{g_{i\mu\nu} R_i^{\mu\nu}}{2k} + \mathcal{L}_{i\text{Matter}} \right] \sqrt{-g_i} = \sum_{i=1}^{N} \int \left[\frac{g_i^{\mu\nu} R_{i\mu\nu}}{2k} + \mathcal{L}_{i\text{Matter}} \right] \sqrt{-g_i} \mathrm{d}t \ . \tag{105.7}$$

I'm not sure what I do about the t integral. I used only a three-dimensional delta function in (105.3) to make the units correct. Actually, I have to leave it this way unless I want to change physics here. I will change the physics later, but not here. This should be the correct way to do present physics in the discrete case.

At this point, we can ignore that we have a geometry (except for time). Then, setting the variation of the action to zero gives (at least formally) the Einstein field equations. The variation of (105.7) is

$$\delta S = \sum_{i=1}^{N} \int \left[R_{i\mu\nu} + \frac{g_i^{\alpha\beta} \delta R_{i\alpha\beta}}{\delta g_i^{\mu\nu}} + \frac{R_i}{\sqrt{-g_i}} \frac{\delta \sqrt{-g_i}}{\delta g_i^{\mu\nu}} + \frac{2k}{\sqrt{-g_i}} \frac{\delta(\mathcal{L}_{i\text{Matter}} \sqrt{-g_i})}{\delta g_i^{\mu\nu}} \right] \frac{\sqrt{-g_i}}{2k} \delta g_i^{\mu\nu} \mathrm{d}t \ . \tag{105.8}$$

Using

$$\frac{1}{\sqrt{-g_i}}\frac{\delta\sqrt{-g_i}}{\delta g_i^{\mu\nu}} = -\frac{1}{2}g_{i\mu\nu} \tag{105.9}$$

and

$$T_{i\mu\nu} = \frac{-2}{\sqrt{-g_i}}\frac{\delta(\mathcal{L}_i\text{Matter}\sqrt{-g_i})}{\delta g_i^{\mu\nu}} \tag{105.10}$$

gives

$$\delta S = \sum_{i=1}^{N}\int\left[R_{i\mu\nu} + \frac{g_i^{\alpha\beta}\delta R_{i\alpha\beta}}{\delta g_i^{\mu\nu}} - \frac{1}{2}g_{i\mu\nu}R_i - kT_{i\mu\nu}\right]\frac{\sqrt{-g_i}}{2k}\delta g_i^{\mu\nu}\mathrm{d}t\ . \tag{105.11}$$

However, at this point, we do not know that $g_{\mu\nu}$ represents a metric (but we do know that it can be used to raise or lower indexes), nor do we know how the Ricci tensor is related to the connection. At this point, however, we do need to put in how the Ricci tensor is related to the connection. Because that definition involves derivatives, it cannot be expressed without geometry. Thus, at this point, we do need to use geometry. Using the usual definition of the Riemann tensor in terms of the connection, we get

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \left(g^{\alpha\beta}\delta\Gamma^{\sigma}_{\beta\alpha} - g^{\alpha\sigma}\delta\Gamma^{\rho}_{\rho\alpha}\right)_{;\sigma}\ . \tag{105.12}$$

Because this term involves derivatives, it is not really valid to substitute it into (105.11). Instead, it is necessary to substitute for the second term in (105.11) what it would be from taking the variation of (105.1). That gives

$$\delta S = \sum_{i=1}^{N}\int\left[R_{i\mu\nu} - \frac{1}{2}g_{i\mu\nu}R_i - kT_{i\mu\nu}\right]\frac{\sqrt{-g_i}}{2k}\delta g_i^{\mu\nu}\mathrm{d}t + \frac{1}{2k}\int g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}\mathrm{d}^4x\ . \tag{105.13}$$

Substituting (105.12) into (105.13) gives

$$\delta S = \sum_{i=1}^{N}\int\left[R_{i\mu\nu} - \frac{1}{2}g_{i\mu\nu}R_i - kT_{i\mu\nu}\right]\frac{\sqrt{-g_i}}{2k}\delta g_i^{\mu\nu}\mathrm{d}t + \frac{1}{2k}\int\left(g^{\alpha\beta}\delta\Gamma^{\sigma}_{\beta\alpha} - g^{\alpha\sigma}\delta\Gamma^{\rho}_{\rho\alpha}\right)_{;\sigma}\sqrt{-g}\mathrm{d}^4x\ . \tag{105.14}$$

Or,

$$\delta S = \sum_{i=1}^{N}\int\left[R_{i\mu\nu} - \frac{1}{2}g_{i\mu\nu}R_i - kT_{i\mu\nu}\right]\frac{\sqrt{-g_i}}{2k}\delta g_i^{\mu\nu}\mathrm{d}t + \frac{1}{2k}\int\left[\left(g^{\alpha\beta}\delta\Gamma^{\sigma}_{\beta\alpha} - g^{\alpha\sigma}\delta\Gamma^{\rho}_{\rho\alpha}\right)\sqrt{-g}\right]_{,\sigma}\mathrm{d}^4x\ . \tag{105.15}$$

The 4-dimensional integral gives only a boundary term because of Stokes' theorem. If $\delta g^{\mu\nu}$ vanishes at infinity, this term will not contribute to the variation of the action. Thus, we have

$$\delta S = \sum_{i=1}^{N}\int\left[R_{i\mu\nu} - \frac{1}{2}g_{i\mu\nu}R_i - kT_{i\mu\nu}\right]\frac{\sqrt{-g_i}}{2k}\delta g_i^{\mu\nu}\mathrm{d}t\ . \tag{105.16}$$

Setting (105.16) to zero gives

$$R_{i\mu\nu} - \frac{1}{2}g_{i\mu\nu}R_i - kT_{i\mu\nu} = 0\ . \tag{105.17}$$

The derivation of (105.17) requires the use of geometry, so it is valid for the case of geometry. The parameters in (105.17) are not the usual ones, because of the definitions in (105.5), (105.6), and (105.10). $R_{i\mu\nu}$ still depends on derivatives, so we have not yet eliminated geometry even for this discrete representation.

Although we could probably include fields by including them in $\mathcal{L}_{i\text{Matter}}$, their contribution would not be discrete because the fields would have values in regions where there was no matter. Similarly, we could include a cosmological constant by adding it to (105.1), but we would have to consider whether it had a value in regions where there was no matter.

The derivatives in $R_{i\mu\nu}$ probably prevent this approach from representing a geometry-free situation, in addition to the fact that we still have time.

105.3 Adding fields

To go further, we consider that there are various fields (to be defined later) that have values at each particle, but no positions yet. At this point, we make a guess that derivatives with respect to coordinates need to (somehow) be replaced with differences of field variables evaluated at different particles.

105.4 Fields, tensors, derivatives, and frames of reference in the absence of geometry

Imagine that we have a finite, discrete set of particles. At each particle, we have values for some fields. These fields might be scalars, vectors, or tensors. We can define a frame of reference as the frame of reference relative to a particular particle. Thus, the values of the various fields at each of the particles will depend on which particle we have chosen to be our frame of reference. We need to generalize the usual law for tensor transformation to this situation where geometry does not exist. There are no distances, no coordinates. We can number the particles from 1 to N. There is no time coordinate, but there is a discrete time in the sense that the configuration (the values of the fields on all of the particles) can change discretely in a sequence from 1 to M.

To generalize the law for tensor transformation, first consider a rank-1 tensor (vector). The usual transformation law is

$$T^\mu = T'^\nu \frac{\partial x^\mu}{\partial x'^\nu} \ . \tag{105.18}$$

Let us consider very carefully the meaning of (105.18). The value of the μth contravariant component of a tensor T at the point x^μ in the unprimed coordinate system is given by the formula in (105.18), where T'^ν is the value of the νth contravariant component of the tensor in the primed coordinate system, and the formula is summed over the repeated index ν. We assume that we have formulas relating the unprimed coordinates to the primed coordinates so that the various partial derivatives are available.

I have tried to think of a corresponding formula in the absence of geometry, but I cannot do that yet. Still, let us consider the problem. Let us consider the values of all of the components of a tensor T at the ith particle in a reference frame on the jth particle and in a reference frame on the kth particle. Call these $T^\mu(i,j)$ and $T^\mu(i,k)$. Any transformation between the two frames cannot depend on the particular way the particles are numbered.

I just found out that much of this has already been done by Rafael Sorkin and others using causal sets [367, 368].

105.5 Geometry emerges

To get geometry, we somehow find that particles whose field variables are highly correlated must be close to each other. (There may be situations where that is not possible. When that happens, we are stuck with a pure quantum gravity, similar to the physics in a sparse universe.)

Maybe instead of correlation, we need to look for where the action is stationary.

Somehow, we need a physical mechanism for geometry to arise out of a bunch of particles in pregeometry. First, let us imagine how light cones arise. Imagine light in pregeometry having infinite speed. (Actually, since there is no distance, and no time, speed is undefined.) Still, consider that light interacts with all particles simultaneously.

In the interior of a star, it takes tens or hundreds of thousands of years for light to reach the surface from the interior. Relative to that speed, the speed of light in free space could be considered infinite. Is it possible that the mechanism for light to be slowed from actually infinite to c is a similar mechanism? In a star, a photon has a mean free path of about a centimeter. It gets absorbed and then re-radiated by black-body radiation in a random direction. The only reason the energy ever gets to the surface is that there is a temperature gradient, so that radiation nearer the center of the star outward is slightly greater than radiation inward from a place that is slightly farther from the center of the star. That exact mechanism won't work for getting a light-cone structure from pregeometry.

In the pregeometry case, we consider that a photon interacts with a particle by gravitation. Suppose it is absorbed and re-radiated. Suppose that happens in a Planck time. So, each Planck time, the photon moves from one particle to the next. If the particles are separated by a Planck distance, then the speed will be c. That does not help.

The interaction is gravitational rather than electromagnetic. We don't have anything like temperature or the formula for black-body radiation.

This is like a path-integral approach. The photon can take any path. (Path means the order of interacting with the particles.) Then phase gives the path that contributes most to the amplitude for the process. The hard part is getting the formula for the Lagrangian. (Or, in the more general case, the infinitesimal particle propagator.) Phase interference still takes place to give us geometry. We need the formula to give us the correct phase.

In Minkowski space, from the frame of the photon, all of the particles are flattened into a pancake.

Also, for the wave function of an ordinary particle, the wave function corresponds only to an ensemble average. The actual particle (or, at least some of its energy) is jumping all around. So, in a sense, pregeometry is what is actually happening. Geometry is only an ensemble average.

This may have something to do with re-normalization. Those infinities we sweep under the rug may be something after all. The renormalization we do by re-normalizing the mass and charge isn't enough. We need to re-normalize by starting with the pregeometry version (once we have discovered that). The renormalization part can be anything, because it is subtracted.

Present formulations for quantum gravity in terms of quantum geometrodynamics, in which we represent the situation as a quantum superposition of 3-geometries is probably only an approximation, in that, "geometry" is probably not a fundamental concept of nature. Similarly, "spin" may not be a fundamental concept of nature, but a concept that arises from the Poincaré group.

We note in passing, that a Stern-Gerlach apparatus does not measure spin, but magnetic moment. In actuality, what happens is that the wave for a particle with a magnetic moment propagating in an inhomogeneous magnetic field is birefringent, so that the wave is split into two characteristic polarizations that have different effective indexes of refraction.

Let's go back to a path-integral formulation. In the absence of geometry, we have to change "path" to "way," in the sense that "all of the ways an interaction can occur." That means that when calculating the amplitude for a process, we sum over all of the "ways" an interaction can occur. It really is a sum instead of an integral in this case, because, without geometry, it really is a discrete sum. Thus, we can replace "path-integral method" by "way-sum method." In the way-sum method, each "way" that an interaction can occur is that the particle first interacts with particle 1, then interacts with particle 2, then with particle 3, etc., in which the order of the particles can be anything, and repeat interactions are allowed. If there are N total particles and M total interactions, then there will be $(N-1)^M$ total "ways," if we exclude self interactions, N^M, if we do not exclude self interactions.

In the usual path-integral method, we look at all of the paths light can take to get from A to B. In a homogeneous medium, the path that contributes most to the amplitude is a straight line because only that path has a phase which is stationary with respect to variation of the path. In the way-sum method, we first need to ask, "What is the analog of a straight line?" I am guessing that the analog of a straight line is the "way" in which a particle interacts with each other particle exactly once. (That is, no doubling back.) Neglecting self interactions, this would give $(N-1)!$ possible "ways" to make a "straight" line.

Next, we need to consider how such a "straight-line" path is stationary in the way-sum method. The interactions in this case are gravitational, and the interaction is with all particles in the universe. At some point, we need to decide whether all particles should be considered nucleons (protons and neutrons) because they have nearly all of the mass in the universe, or quarks because they are more fundamental. Since we need to include all of the energy (with the possible exception of gravitational energy), maybe we should consider nucleons to start with. In calculating the amplitude for a "way," we need to consider an interaction factor. Somehow, we need to make a distinction between the first encounter with some particular particle and the second and further encounter in the interaction factor if we are to make a distinction between a "straight line" and a "curved line." It seems likely that the interaction factors (propagators?) do not commute with one another.

It seems likely that quantum world is pregeometry, while geometry is emergent with ensemble averages. Wave functions, such as atomic wave functions, are not directly measurable, so are probably only calculated. It is probably only in a measurement, such as in an atomic transition, that geometry emerges with regard to the effect of wave functions.

What is the simplest thing for which I can calculate the amplitude that would test out this process?

Actually, I think that pregeometry must have slightly more structure. There have to be maybe relative distances. That is, distances have to be ordered in the sense that this distance is larger than that distance. We probably have directions ordered also. Somehow we have to have a formula for calculating the amplitude for a process in terms of the contributions from each way that process could occur. In addition, all of the particles are moving, so they do not have fixed positions, especially since each particle may be testing out various paths.

105.6 Another approach: SWG equations

Another approach to writing General Relativity in terms of discrete masses, starts with the SWG equations (52.9) [16] using a discrete energy-momentum tensor. That is, we start with

$$g^{\acute{\alpha}\acute{\beta}} = 16\pi G \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{\mu}(T^{\mu}_{\nu} - \frac{1}{2}T^{\lambda}_{\lambda}\delta^{\mu}_{\nu})[-g(x)]^{\frac{1}{2}}d^4x = 16\pi G \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{\mu}R^{\mu}_{\nu}[-g(x)]^{\frac{1}{2}}d^4x \ . \tag{105.19}$$

Substituting (105.4) into (105.19) gives

$$g^{\acute{\alpha}\acute{\beta}} = \sum_{i=1}^{N} 16\pi G \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{i\mu}m_i(u^{\mu}u_{\nu} + \frac{1}{2}\delta^{\mu}_{\nu})[-g(x)]^{\frac{1}{2}}dt = \sum_{i=1}^{N} 16\pi G \int G^{-\acute{\alpha}\acute{\beta}\nu}{}_{i\mu}R^{\mu}_{i\nu}[-g(x)]^{\frac{1}{2}}dt \ . \tag{105.20}$$

the metric as a discrete sum.

This works after we have a geometry, but first we have to get a geometry. Somehow, emergence of geometry should lead to (105.19).

We still have to generalize the Green's function. That will be a little easier if we have some kind of distance, even if it is only ordered distance. The differential equation for the Green's function is a d'Alembertian combined with the Riemann tensor, which acts as an effective (square of) mass term. In this case, the Riemann tensor gives a 16 by 16 matrix, whose eigenvalues have the effect of the square of an effective mass. How can we generalize that to the case where we have only ordered distances?

Chapter 106

The rotation problem[1]

abstract

Any reasonable form of quantum gravity can explain (by phase interference) why, on a large scale, inertial frames seem not to rotate relative to the average matter distribution in the universe without the need for absolute space, finely tuned initial conditions, or without giving up independent degrees of freedom for the gravitational field. A simple saddlepoint approximation to a path-integral calculation for a perfect fluid cosmology shows that only cosmologies with an average present relative rotation rate smaller than about $T^*H^2 \approx 10^{-71}$ radians per year could contribute significantly to a measurement of relative rotation rate in our universe, where $T^* \approx 10^{-51}$ years is the Planck time and $H \approx 10^{-10}$ yr^{-1} is the present value of the Hubble parameter. A more detailed calculation (taking into account that with vorticity flow lines are not normal to surfaces of constant global time, and approximating the action to second order in the mean square vorticity) shows that the saddlepoint at zero vorticity is isolated and that only cosmologies with an average present relative rotation rate smaller than about $T^*H^2a_1^{1/2} \approx 10^{-73}$ radians per year could contribute significantly to a measurement of relative rotation rate in our universe, where $a_1 \approx 10^{-4}$ is the value of the cosmological scale factor at the time when matter became more significant than radiation in the cosmological expansion. Including inflation with 60 e-foldings in the calculation of the action further restricts relative rotation rate to be smaller than $\approx 10^{-74}$ radians per year. These calculations are consistent with measurements indicating a present relative rotation rate less than about 10^{-20} radians per year. The observed lack of relative rotation may be evidence for the existence of quantum gravity.

106.1 Introduction

Although there are solutions of Einstein's field equations that allow relative rotation of matter and inertial frames, it has long been known that in our universe inertial frames seem not to rotate with respect to the visible stars. The "Rotation Problem" is to explain "If the universe can rotate, why does it rotate so slowly?"[298]. The rotation problem can easily be seen by comparing the rotation of the plane of a Foucault pendulum with the movement of the stars relative to the Earth. More accurate estimates of the lack of relative rotation are now available, mostly because of the availability of isotropy measurements on the cosmic microwave background radiation.

For example, Hawking [164] showed that if the universe contains a large-scale homogeneous

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vorticity, then the rotation rate corresponding to that vorticity cannot be larger than somewhere between 7×10^{-17} rad yr^{-1} and 10^{-14} rad yr^{-1} if the universe is closed and about 2×10^{-46}/(present density in g cm^{-3}) if it is open. Many studies have been done since then resulting in progressively decreasing estimates of the allowed rotation rate. Table 106.1 summarizes some of these measurements.

Table 106.1: Some measurements of the maximum present relative rotation rate of average inertial frame and average matter distribution.

max. rotation rate (radians per year)	References
2×10^{-17} (note[a])	[164, Hawking, 1969]
2×10^{-13} (note[b])	[300, 301, Ellis, 1971, 2009]
1.5×10^{-18} (note[c])	[303, Collins & Hawking, 1973]
10^{-21} (note[d])	[303, Collins & Hawking, 1973]
0.7×10^{-14} (note[e])	[307, Barrow et al., 1985]
4.3×10^{-20} (note[f])	[309, 310, Jaffe et al., 2005, 2006]
0.7×10^{-16} (note[g])	[311, Ellis, 2006]
2×10^{-9} (note[h])	[312, Su & Chu, 2009]
3×10^{-21} (note[i])	[369, Saadeh et al., 2016]

[a] for an open universe, assuming about 10^{-29} g cm^{-3} for the present density of the universe

[b] assuming 67.74 km/sec per Mpc for the present value of the Hubble parameter

[c] if the universe is closed, and if the microwave background was last scattered at a redshift of about 7

[d] if the universe is closed, and if the microwave background was last scattered at a redshift of 1000

[e] if the universe is open, assuming 67.74 km/sec per Mpc for the present value of the Hubble parameter

[f] assuming 67.74 km/sec per Mpc for the present value of the Hubble parameter

[g] assuming 67.74 km/sec per Mpc for the present value of the Hubble parameter

[h] maximum rotation rate at the last scattering surface

[i] assuming 67.74 km/sec per Mpc for the present value of the Hubble parameter

In General Relativity, gravitation (including inertia) (as expressed by the metric tensor) is determined not only by the distribution of matter (in terms of the stress-energy tensor), but also by initial and boundary conditions. There are many solutions of Einstein's field equations for General Relativity that have large-scale relative rotation of matter and inertial frames, e.g. [347, 348, 163, 370, 371]. It is difficult to explain the absence of relative rotation in our universe classically without absolute space (as proposed by Newton) or without assuming very finely tuned initial conditions for the universe.

Ernst Mach [120, 102, 122, 15] suggested that inertia might be determined by distant matter. Various versions of that proposal have come to be known as Mach's principle. Since we now know (from General Relativity) that inertia is a gravitational force, such an implementation of Mach's principle would require that the gravitational field (or at least part of it) be determined only by its sources (matter) rather than having independent degrees of freedom (in terms of initial and boundary conditions).

If the many proposals to implement Mach's principle for General Relativity, e.g. [11, 16, 156, 109, 315, 159] were correct, then gravitation would behave very differently from the electromagnetic interaction, in that electric and magnetic fields are determined not only from sources (charges and currents), but also from initial and boundary conditions.

Although General Relativity provides a partial mechanism for Mach's principle through "frame dragging," it is necessary to explain why the initial vorticity is small enough to give the very small present value of vorticity. A possible explanation for such a small value of initial vorticity may come from some form of quantum gravity. That is, quantum gravity may provide the mechanism for implementing Mach's principle.

References [298] and [313] suggest that inflation might lead to very small shear and rotation rates for our present universe because shear and rotation rate decrease as the universe expands. Although inflation (and the general expansion) would decrease the relative rotation, it would not be enough to allow for arbitrarily large values of the initial rotation rate.

Section 106.2 argues why a quantum gravity mechanism to explain the complete absence of vorticity is valid even though we have no generally accepted theory of quantum gravity. Section 106.3 gives the form of a path integral for a quantum superposition of cosmologies characterized by the rms value $\langle\omega_3\rangle$ of the present vorticity. Section 106.4 shows that there is a saddlepoint at $\langle\omega_3\rangle = 0$, and that the only cosmologies contributing a significant amount to the path integral have an upper limit for the rms value of the present vorticity. Section 106.5 gives an estimate for the dependence of the Action on $\langle\omega_3\rangle$ and estimates the upper limits on $\langle\omega_3\rangle$. Section 106.6 makes a saddlepoint approximation to the path integral. Section 106.7 discusses possible problems.

Appendix 106.8 sets up a calculation of the amplitude for measuring a rotation of the universe in terms of a path-integral calculation. Appendix 106.9 calculates the action for a perfect fluid. Appendix 106.10 discusses the inflation, radiation, matter, and dark-energy eras. Appendix 106.11 estimates the total relative rotation of the universe for the maximum relative rotation rate that contributes significantly to the amplitude for measuring vorticity. Appendix 106.12 calculates the effect that flow lines are not normal to surfaces of constant global time if vorticity is present. Specifically, it calculates the scale factor along flow lines as a function of cosmological scale factor to second order in the mean square of the present vorticity, $\langle\omega_3\rangle^2 \equiv \overline{\omega_3^2}$. Appendix 106.13 calculates a generalized Friedmann equation to second order in the mean square of the present vorticity when the vorticity is very small. A more accurate calculation that takes into account the coupling between vorticity and shear in terms of a vector perturbation [342, Chapter 10] (not shown) would not change the total action significantly. Appendix 106.14 calculates the approximate action for small vorticity to second order in the mean square of the present vorticity, $\langle\omega_3\rangle^2$.

We take the speed of light c and Newton's gravitational constant G to be 1 throughout.

106.2 Quantum gravity

There are strong reasons why a theory of quantum gravity should exist, e.g.[372], and it is generally believed that such a theory exists. There are many difficulties with formulating a theory of quantum gravity, some of which are discussed in [373, 374]. Although we do not have a final theory of quantum gravity, and therefore, no universally accepted theory of quantum cosmology, we have some speculations for theories of quantum gravity, e.g. [62, 63, 64, 19, 375, 376, 377].

However, some calculations (including the present one) can be made without having a full theory of quantum gravity by using a path-integral representation because the action is most likely to dominate over the measure (which we do not know), and the action in the case of vorticity depends only weakly on the exact form of the Lagrangian.

106.3 Path integrals in quantum cosmology

It is likely that some form of quantum cosmology can explain through phase interference the lack of vorticity. One of the standard formulations of quantum cosmology is in terms of path integrals [123], in which an initial 3-geometry changes to some final 3-geometry along a "path" that is a

4-geometry (i.e., a spacetime, or cosmology). Thus, each "path" is one cosmology. This is related to a sum-over-histories approach [316, 165, 317, 318, 319].[2]

Although in general, the 4-geometries considered in a path integral do not have to be classical cosmologies (that is, solutions of Einstein's field equations), it is known that classical cosmologies usually dominate the path integral, and therefore, here, we shall consider only classical cosmologies in the path integral.

We further restrict cosmologies in our path integral to those cosmologies that differ from the standard cosmological model only in that they have vorticity. However, because vorticity and shear are strongly coupled [342, Chapters 6 and 10], it is not possible to have vorticity without shear. Therefore, vorticity will always be accompanied by some shear. Including shear in the calculation could change the value of the action by a factor of two or three or so, but would not change the fact that there is a saddlepoint in the path integral at zero vorticity. And even a factor of ten one way or another in the action would not change the final result significantly.

Such cosmologies would have begun with an arbitrary initial vorticity ω_i at the beginning of inflation, that varied as a function of position, that would then decrease in a known way (Tables 106.2 and 106.3) as the universe expanded during the inflation era, and vorticity had dropped to a value ω_0 at the end of inflation, still a function of position.

The universe continues to expand during the radiation era when radiation dominated over matter. Because radiation density decreases faster than matter density as the universe expands, we eventually reach a time when the density of radiation and matter are equal, and the vorticity has dropped to a value ω_1, which is still a function of position. From that point on, the vorticity drops at a faster rate (Tables 106.2 and 106.3) as the universe continues to expand until the matter density decreases to a point where the matter term in the Friedmann equation equals the cosmological constant term, and the vorticity has dropped to a value ω_2, which is still a function of position.

Eventually, we reach the present time, where the vorticity has dropped to a value ω_3, still a function of position. Because the vorticity depends in a known way on the expansion during the various eras, ω_3 and ω_i have a known relationship, that allows us to designate each cosmology by the spatial variation of ω_3.

Table 106.2: Inflation, radiation, matter, and dark-energy eras [342, Table 6.1] ℓ is a scale factor along lines of cosmic flow. The constant $w = 1/3$. Some of the quantities in this table are a function of position.

Era	ℓ	$\rho \propto$	$\rho =$	p	$\omega \propto$	$\omega =$
	ℓ_i					ω_i
Inflation	$\ell_i \to \ell_0$	ℓ^{-4}	$3H_3^2/(8\pi)(\ell_1/\ell_2)(\ell_2/\ell)^4$	$w\rho$	ℓ^{-1}	$\omega_3(\ell_3/\ell_1)(\ell_3/\ell)$
	ℓ_0					ω_0
Radiation	$\ell_0 \to \ell_1$	ℓ^{-4}	$3H_3^2/(8\pi)(\ell_1/\ell_2)(\ell_2/\ell)^4$	$w\rho$	ℓ^{-1}	$\omega_3(\ell_3/\ell_1)(\ell_3/\ell)$
	ℓ_1					ω_1
Matter	$\ell_1 \to \ell_2$	ℓ^{-3}	$3H_3^2/(8\pi)(\ell_2/\ell)^3$	0	ℓ^{-2}	$\omega_3(\ell_3/\ell)^2$
	ℓ_2					ω_2
Dark energy	$\ell_2 \to \ell_3$	ℓ^{-3}	$3H_3^2/(8\pi)(\ell_2/\ell)^3$	0	ℓ^{-2}	$\omega_3(\ell_3/\ell)^2$
	ℓ_3					ω_3

There is a complex amplitude associated with each "path", that is, with each cosmology. The complex amplitude has a magnitude and a phase, whose values can be calculated from a correct theory of quantum gravity, once that theory is known. Usually, we expect the value of the phase to dominate the path integral.

[2] also called decoherent histories [320, 321, 322, 323, 324, 325, 326, 327] or consistent histories [328, 329, 330, 331]

Table 106.3: Inflation, radiation, matter, and dark-energy eras [342, Table 6.1] a is the cosmological scale factor. ℓ is a scale factor along lines of cosmic flow. In the presence of vorticity, a and ℓ differ. The dimensionless parameter δ [defined in (106.41)] is proportional to the square of ω_3. Some of the quantities in this table are a function of position.

Era	a	$\rho \approx [1 + f_L(a)\delta + f_{LL}(a)\delta^2] \times$	$\omega \approx [1 + g_\omega(a)\delta + g_{\omega\omega}(a)\delta^2] \times$
	a_i		
Inflation	$a_i \to a_0$	$3H_3^2/(8\pi)(a_1/a_2)(a_2/a)^4$	$\omega_3(a_3/a_1)(a_3/a)$
	a_0		
Radiation	$a_0 \to a_1$	$3H_3^2/(8\pi)(a_1/a_2)(a_2/a)^4$	$\omega_3(a_3/a_1)(a_3/a)$
	a_1		
Matter	$a_1 \to a_2$	$3H_3^2/(8\pi)(a_2/a)^3$	$\omega_3(a_3/a)^2$
	a_2		
Dark energy	$a_2 \to a_3$	$3H_3^2/(8\pi)(a_2/a)^3$	$\omega_3(a_3/a)^2$
	a_3		ω_3

Appendix 106.8 sets up a calculation of the amplitude for measuring a rotation of the universe in terms of a path-integral calculation. Appendix 106.14 shows that the action for each cosmology depends only on the present rms vorticity $\langle\omega_3\rangle$, where $\langle\omega_3\rangle^2 \equiv \overline{\omega_3^2}$, and the average is a spatial average over the volume within the past light cone. However, because there are an infinite number of vorticity configurations for each value of the present rms vorticity $\langle\omega_3\rangle$, the path-integral calculation represents an infinite number of integrations. To correctly deal with that situation, it is useful to make a comparison with a more standard path integral.

To make a path-integral calculation for the propagation of a light wave or a radio wave between a specified source location and a specified observer location in a specified inhomogeneous medium requires an infinite number of parameters to specify the path connecting those two endpoints. Because of that, a path integral corresponds to an infinite number of integrations. For most values of the action, there will be an infinite number of paths that have the same action. However, there are usually only a finite number of discrete paths for which the action is an extremum, and these paths are saddlepoints for the path integral. This justifies reducing an infinite number of integrations to a small finite number when using saddlepoint approximations to make WKB approximations to path integrals .

The situation in the present case is similar. For most values of the rms vorticity (and therefore the action, which depends only on the rms vorticity), there are an infinite number of configurations of the spatial variation of vorticity that have the same rms vorticity, and therefore the same action. However, there is only one configuration at the saddlepoint for which the rms vorticity is zero, and that corresponds to the configuration in which the vorticity is exactly zero everywhere. Thus, the situation here is exactly analogous to that of the ordinary path integral. Therefore, it is valid to consider only a simple path integral

$$\int A(\langle\omega_3\rangle) \exp[iI(\langle\omega_3\rangle)/\hbar]\mathrm{d}\,\langle\omega_3\rangle \ , \tag{106.1}$$

instead of an infinite number of integrations to represent the amplitude for any process (such as measuring the vorticity), where $A(\langle\omega_3\rangle)$ is a slowly varying function of $\langle\omega_3\rangle$ that gives the magnitude of the contribution of the "path" (cosmology), and the phase is equal to the action $I(\langle\omega_3\rangle)$ divided by the reduced Planck's constant $\hbar$.

106.4 Saddlepoint

We expect that $I(\langle\omega_3\rangle)$ should be a smoothly varying function of $\langle\omega_3\rangle$. Also, we expect that when the rms vorticity $\langle\omega_3\rangle$ is zero, the cosmology will be the standard cosmological model (the standard Robertson-Walker cosmology), and will have the action I_0 associated with that model. Further, because of the symmetry that the action will be unchanged if the vorticity changes sign at each location[3], the action $I(\langle\omega_3\rangle)$ should be an even function of $\langle\omega_3\rangle$. Therefore, the action $I(\langle\omega_3\rangle)$ should look roughly like

$$I(\langle\omega_3\rangle) \approx I_0 + \hbar\left(\frac{\langle\omega_3\rangle}{\omega_m}\right)^2 f_I(\langle\omega_3\rangle) \ , \tag{106.2}$$

where $f_I(\langle\omega_3\rangle)$ is some slowly varying dimensionless even function of $\langle\omega_3\rangle$. Notice that (106.2) reduces to I_0, the action for the standard cosmological model, when the rms vorticity $\langle\omega_3\rangle$ is zero, and that (106.2) is an even function of $\langle\omega_3\rangle$ if $f_I(\langle\omega_3\rangle)$ is an even function of $\langle\omega_3\rangle$, as required. Direct calculation in the appendices shows that (106.2) is correct if $\langle\omega_3\rangle$ is the rms value for ω_3 when taking a spatial average over the past light cone.

Comparing (106.1) and (106.2) shows that (106.1) has a saddlepoint at $\langle\omega_3\rangle = 0$. If that saddlepoint is the only significant saddlepoint (and other criteria are satisfied), then the only significant contributions to the path integral (106.1) comes from values of the present rms vorticity

$$\langle\omega_3\rangle = 0 \pm \omega_m/\sqrt{f_I(0)} \ . \tag{106.3}$$

Thus, there is no doubt that (106.2) gives the correct behavior for the action for small rms vorticity, nor no doubt that (106.3) gives the limits for the present rms vorticity of cosmologies that contribute significantly to the path integral in (106.1). The only question is in the values for ω_m and for the function $f_I(\langle\omega_3\rangle)$.

106.5 Evaluation of the action

Estimating the value of ω_m is straightforward. Estimating the function $f_I(\langle\omega_3\rangle)$ is difficult, as is the task of estimating the dependence of the action on the rms vorticity for all values of the rms vorticity. The appendices make such estimates.

If there is relative rotation of the matter distribution and the inertial frame, then density and pressure will depend on the scale length ℓ along flow lines rather than depend on the cosmological scale factor a. The significance is that the cosmological scale factor a is a function of global time, but scale length ℓ along flow lines is not normal to surfaces of constant global time. This causes the calculation to be much more complicated.

Appendix 106.9 calculates the action in terms of a 4-volume integral of the Lagrangian for a perfect fluid plus a surface term. For some cosmological models, the surface term can be absorbed into the 4-volume integral of an effective Lagrangian. It is argued that a general effective Lagrangian for solutions of Einstein's field equations can be expressed as a linear combination of pressure, density, and the cosmological constant with constant coefficients. Specifically, equation (106.27) in appendix 106.9 gives an effective Lagrangian of the form

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda \ , \tag{106.4}$$

where p is pressure, ρ is density, Λ is the cosmological constant, and α_1, α_2, and α_3 are dimensionless constants of order unity.

[3] A cosmology with an average present vorticity equal to $-\omega_3$ about some given axis should be equivalent to a cosmology with an average present vorticity equal to $+\omega_3$ about that same axis, and should therefore have the same action.

Appendix 106.10 discusses the inflation, radiation, matter, and dark-energy eras because the effective Lagrangian would depend differently on the cosmological scale factor in the four eras. To a first approximation, we can neglect all but the dominant term in each of the four eras.

ω_m is the dominant factor that gives the effect of vorticity on the action in (106.2). The main effect of vorticity on the action can be found by a straightforward calculation of the action integral. The time integral to give the action is converted to an integral over the cosmological scale factor using a generalization of the Friedmann equation that includes vorticity. The spatial integral to give the action is approximated by averaging the integrand times the spatial volume within the past light cone, which is proportional to the cube of present radius of the visible universe. The calculation is expanded to first order in the mean square vorticity.

The result is given by (106.103) in appendix 106.14 as

$$\omega_m = \left(\frac{\hbar H_3}{r_3^3}\right)^{1/2} = \frac{T^*}{r_3}\sqrt{\frac{H_3}{r_3}} \approx T^* H_3^2 \approx 10^{-71}\text{rad yr}^{-1} \;, \tag{106.5}$$

where $H_3 \approx \sqrt{\Lambda/3} \approx 10^{-10}$ yr^{-1} is the present value of the Hubble parameter, r_3 is the present radius of the visible universe (which we approximate by the inverse of the Hubble parameter), $T^* \approx 10^{-51}$ years is the Planck time, and Λ is the cosmological constant.

Including vorticity in the calculation of relative rotation (even approximately) is tedious, even though straightforward. Appendix 106.14 calculates the total action to second order in the mean square of the present vorticity.

The result is given by (106.102) in appendix 106.14, which shows that to lowest order in $\langle\omega_3\rangle/H_3$

$$\begin{aligned} f_I(\langle\omega_3\rangle) &\approx \left(\tfrac{a_3}{a_1}\right)\left[C_I + \tfrac{\langle\omega_3\rangle^2+\sigma_\omega^2/\langle\omega_3\rangle^2}{H_3^2}\left(\tfrac{a_3}{a_1}\right)C_{II}\right] \\ &\approx \left(\tfrac{a_3}{a_1}\right)C_I \text{ for } \langle\omega_3\rangle \ll 10^{59}\omega_m, \end{aligned} \tag{106.6}$$

where σ_ω^2 is the variance of ω_3^2, a_1 is the value of the cosmological scale factor when the density of radiation was equal to the density of matter, $a_3 = 1$ is the present value of the cosmological scale factor, and C_I and C_{II} are dimensionless constants of order unity[4]. Thus, $f_I(\langle\omega_3\rangle)$ is essentially constant in a large region surrounding the saddlepoint at $\langle\omega_3\rangle = 0$. [5]

106.6 Saddlepoint approximation

The saddlepoint at $\langle\omega_3\rangle = 0$ in (106.1) is isolated from other saddlepoints and any possible non-analytic points as shown by (106.6). The integral in (106.1) can be approximated by a saddlepoint integration to give

$$\begin{aligned} &A(0)\omega_m\sqrt{\pi}/\sqrt{f_I(0)}e^{i\pi/4} \\ &\text{for } |\langle\omega_3\rangle| < \tfrac{\omega_m}{\sqrt{|f_I(0)|}} \approx \tfrac{T^*H_3^2}{\sqrt{|f_I(0)|}} \approx \tfrac{T^*H_3^2}{\sqrt{|C_I|}}\left(\tfrac{a_1}{a_3}\right)^{1/2} \approx \tfrac{10^{-73}}{\sqrt{|C_I|}} \text{ rad/year}, \end{aligned} \tag{106.7}$$

≈ 0 otherwise.

[4] within a factor of a hundred, or so. Compared to a factor of 10^{59}, a hundred is of order unity.

[5] A simple calculation that neglects the fact that flow lines are not normal to surfaces of constant global time [378] shows that only cosmologies whose present relative rotation rate is less than about $\omega_m \approx T^*H_3^2 \approx 10^{-71}$ radians per year would contribute significantly to a path integral for a measurement of relative rotation rate. That is, the simple calculation gave $f_I(\langle\omega_3\rangle) \approx 1$. That the resulting maximum total rotation since the initial singularity would be less than about 10^{-61} radians suggests that neglecting that flow lines are not normal to surfaces of constant global time should be a good approximation.

Table 106.4 gives possible approximate values for C_I and C_{II}, depending on reasonable values for α, α_1, α_2, α_3, and α_4[6], and depending on whether the surface term is included in the effective Lagrangian. That C_I can be either positive or negative is not a problem, because the path of integration along the real $\langle\omega_3\rangle$ axis is a stationary-phase path. That the most likely value for α_4 is zero gives $|C_I| \approx 1$, which gives from (106.7), $|\langle\omega_3\rangle| <\approx 10^{-73}$ rad/year. However, larger values of α_4 from Table 106.4 could give larger values for $|C_I|$, which could give limits on $|\langle\omega_3\rangle|$ as small as about 10^{-75} rad/year. Table 106.4 also includes the effect of inflation with 60 e-foldings, and shows an increase of C_I by factors ranging from 52 up to 103. This would restrict allowable rotation rates further by a factor of up to 10. Inflation with 50 e-foldings has negligible effect on the value of the C_I.

Table 106.4: Possible approximate values of C_I from (106.110) and C_{II} from (106.115) in appendix 106.14

References	Include the surface term?		α	α_1	α_2	α_3	α_4	C_I	C_{II}
[161]			0	$-3/2$	$3/2$	$1/8\pi$		$1-686\alpha_4$	$-453-8927\alpha_4+1.04\times10^6\alpha_4^2$
[161]			0	$-3/2$	$3/2$	$1/8\pi$	0	1	-453
[161]			0	$-3/2$	$3/2$	$1/8\pi$	10^{-1}	-68	9054
[161]			0	$-3/2$	$3/2$	$1/8\pi$	1	-685	1.03×10^6
[161]		$\rightarrow$	0	$-3/2$	$3/2$	$1/8\pi$	0	97	
[161]	yes		0	0	0	$-1/4\pi$		$-2+471\alpha_4$	$-9+204\alpha_4-2.08\times10^6\alpha_4^2$
[161]	yes		0	0	0	$-1/4\pi$	0	-2	-9
[161]	yes		0	0	0	$-1/4\pi$	10^{-1}	45	-2.08×10^4
[161]	yes		0	0	0	$-1/4\pi$	1	469	-2.08×10^6
[161]	yes	$\rightarrow$	0	0	0	$-1/4\pi$	0	-58	
[334], [162]			1	$-1/2$	$1/2$	$1/8\pi$		$1-384\alpha_4$	$-148-3044\alpha_4+1.05\times10^6\alpha_4^2$
[334], [162]			1	$-1/2$	$1/2$	$1/8\pi$	0	1	-148
[334], [162]			1	$-1/2$	$1/2$	$1/8\pi$	10^{-1}	-37	10^4
[334], [162]			1	$-1/2$	$1/2$	$1/8\pi$	1	-383	1.05×10^6
[334], [162]		$\rightarrow$	1	$-1/2$	$1/2$	$1/8\pi$	0	52	
[334], [162]	yes		1	1	-1	$-1/4\pi$		$-2+769\alpha_4$	$296+6087\alpha_4-2.06\times10^6\alpha_4^2$
[334], [162]	yes		1	1	-1	$-1/4\pi$	0	-2	296
[334], [162]	yes		1	1	-1	$-1/4\pi$	10^{-1}	75	-1.97×10^4
[334], [162]	yes		1	1	-1	$-1/4\pi$	1	767	-2.05×10^6
[334], [162]	yes	$\rightarrow$	1	1	-1	$-1/4\pi$	0	-103	

The parameter α is defined in (106.22). The parameters α_1, α_2, and α_3 are defined in (106.4). The constant of integration α_4 is defined in (106.78), and its most probable value is zero. The effect of inflation for 60 e-foldings is also shown in the rows with the arrow $\rightarrow$.

106.7 Discussion

The calculation that the only cosmologies that contribute significantly to a measurement of rms vorticity have a present rms vorticity less than about 10^{-73} to 10^{-71} radians per year is fairly robust. Although reasonable different choices for the Lagrangian might possibly change that result

[6] where α_4 is an arbitrary constant of integration in (106.72) in appendix 106.13, whose value is most likely zero.

by a factor of a hundred or so, the result is so much smaller than the upper limit of 10^{-20} radians per year set by measurements, that there seems no doubt that we have found the correct reason for the lack of any significant measurement of rms vorticity.

The significance of inflation on the value of the action can be estimated by the value of the parameter β defined in (106.51). Including inflation in the calculation of the action enhanced the effectiveness of phase interference to restrict allowable rotation rates by a factor of up to 10 for an inflation model with 60 e-foldings. Inflation with only 50 e-foldings had no significant effect on the calculation of the action.

That flow lines are not in general normal to surfaces of constant global time in the presence of vorticity makes the calculation more complicated. Detailed calculation in which the action is calculated through second order in the square of vorticity shows that the saddlepoint at zero vorticity is isolated from any other saddlepoints and from any non-analytic points.

A correct calculation of the action should include the coupling between vorticity and shear in terms of vector perturbations [342, Chapter 10] [379, Chapter 29]. Although neglecting the coupling between vorticity and shear could be justified simply because the coupling would be of second order, a more accurate calculation that takes into account the coupling between vorticity and shear in terms of a vector perturbation [342, Chapter 10] (not shown) would not change the total action significantly.

The observed lack of relative rotation may be evidence for the existence of quantum gravity.

106.8 Amplitude for measuring a rotation of the universe

The amplitude for measuring a particular value for some quantity is equal to the amplitude for measuring that value given a particular 4-geometry times the amplitude for that 4-geometry, and then we sum over all 4-geometries.

For example, following [124], the amplitude for the 3-geometry and matter field to be fixed at specified values on two spacelike hypersurfaces is

$$\langle {}^{(3)}\mathcal{G}_f, \phi_f | {}^{(3)}\mathcal{G}_i, \phi_i \rangle = \int \psi[{}^{(4)}\mathcal{G}, \phi]\, \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi \ , \tag{106.8}$$

where the integral is over all 4-geometries and field configurations that match the given values on the two spacelike hypersurfaces, and

$$\psi[{}^{(4)}\mathcal{G}, \phi] \equiv \exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right) \tag{106.9}$$

is the contribution of the 4-geometry ${}^{(4)}\mathcal{G}$ and matter field ϕ on that 4-geometry to the path integral, where $I[{}^{(4)}\mathcal{G}, \phi]$ is the action. The proper time between the two hypersurfaces is not specified. A correct theory of quantum gravity would be necessary to specify the measures $\mathcal{D}^{(4)}\mathcal{G}$ and $\mathcal{D}\phi$, but that will not be necessary for the purposes here. Hartle and Hawking [124] restricted the integration in (106.8) to compact (closed) 4-geometries, but (106.8) can be applied to open 4-geometries if that is done carefully.

Equation (106.8) is a path integral. In this case, the "path" is the sequence of 3-geometries that form the 4-geometry ${}^{(4)}\mathcal{G}$. Thus, each 4-geometry is one "path." The space in which these paths exist is often referred to as superspace, e.g. [20]. As pointed out by Hajicek [217], there are two kinds of path integrals: those in which the time is specified at the endpoints, and those in which the time is not specified. The path integral in (106.8) is the latter. References [217] and [221] consider refinements to the path integral in (106.8), but such refinements are not necessary here.

Because of diffeomorphisms, a given 4-geometry can be specified by different metrics that are connected by coordinate transformations. This makes it difficult to avoid duplications when making path integral calculations. We avoid that difficulty here by considering only simple models.

Let $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ be the amplitude that the 3-geometry was ${}^{(3)}\mathcal{G}_i$ on some initial space-like hypersurface and that the matter fields on that 3-geometry were ϕ_i. Let $\psi_f({}^{(3)}\mathcal{G}_f, \phi_f)$ be the amplitude that the 3-geometry is ${}^{(3)}\mathcal{G}_f$ on some final space-like hypersurface and that the matter fields on that 3-geometry are ϕ_f. Then, we have

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int \langle {}^{(3)}\mathcal{G}_f, \phi_f | {}^{(3)}\mathcal{G}_i, \phi_i \rangle \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ . \tag{106.10}$$

The condition that there are not finely tuned initial conditions is equivalent to $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ being a broad wave function.

Substituting (106.8) into (106.10) gives

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int \int \psi[{}^{(4)}\mathcal{G}, \phi]\, \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi\, \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ . \tag{106.11}$$

Although in (106.11), the integration is over all possible 4-geometries, not just classical 4-geometries, the main contribution to the integral (in most cases) comes from classical 4-geometries, e.g. [220, 221]. Thus, we shall now restrict (106.11) to be an integration over classical 4-geometries. This is appropriate for our purposes, in any case, since we are trying to explain why we do not measure relative rotation of matter and inertial frames in what appears to be a classical universe.

In principle, the idea is very simple. Any measurement to determine the inertial frame will give a result that depends on the 4-geometry. If several 4-geometries contribute significantly to an amplitude, such as in (106.11), then any measurement to determine an inertial frame might give the inertial frame corresponding to any one of those 4-geometries. However, the probability for the result being a particular inertial frame will depend on the contribution of the corresponding 4-geometry to calculations such as that in (106.11).

In this calculation, we consider 4-geometries characterized by a parameter $\langle\omega_3\rangle$ which we take to be the rms vorticity on the final space-like hypersurface. Thus, we can rewrite (106.11) for our purposes as

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int \int_{-\infty}^{\infty} \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\psi[{}^{(4)}\mathcal{G}, \phi; \langle\omega_3\rangle]\, \mathrm{d}\langle\omega_3\rangle\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ . \tag{106.12}$$

The integral in (106.12) is still a path integral. In this case, each value of $\langle\omega_3\rangle$ specifies one "path", in that it specifies one 4-geometry, and that specifies one sequence of 3-geometries. The space of "paths" in this case is often referred to as a mini-superspace because it is restricted to a much smaller space of 4-geometries. The parameter $\langle\omega_3\rangle$, classically determined by initial conditions on the 4-geometry, represents an independent degree-of-freedom of the gravitational field.

Actually, taking the $\langle\omega_3\rangle$ integration from $-\infty$ to ∞ in (106.12) is not physically realistic, and might lead to problems if the infinite endpoints contribute significantly to the integration. The largest relative rotation that could possibly be considered without having a theory of quantum gravity would be one rotation of the universe in a Planck time. This would correspond to taking the maximum value of $\langle\omega_3\rangle$ to be the reciprocal of the Planck time, T^*, or $\omega_{\mathrm{max}} \approx 10^{44}\mathrm{sec}^{-1}$. Thus, we can rewrite (106.12) as

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) \approx \int \int_{-\omega_{\mathrm{max}}}^{\omega_{\mathrm{max}}} \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\psi[{}^{(4)}\mathcal{G}, \phi; \langle\omega_3\rangle]\, \mathrm{d}\langle\omega_3\rangle\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ . \tag{106.13}$$

We anticipate that the properties of $\psi[{}^{(4)}\mathcal{G}, \phi; \langle\omega_3\rangle]$ will dominate the integral in (106.13), so we shall start with

$$\psi[{}^{(4)}\mathcal{G}, \phi; \langle\omega_3\rangle] \approx e^{iI(\langle\omega_3\rangle)/\hbar} \ , \tag{106.14}$$

where $I(\langle\omega_3\rangle)$ is the action.

Either a stationary-phase path or a steepest-descent path could be used when making the saddlepoint approximation [134, 333, 219], but here, we use a stationary-phase path. Halliwell [332] gives an example of a more detailed path-integral calculation of quantum gravity.

106.9 Action for a perfect fluid

We can take the action in (106.14) to be

$$I = \int (-g^{(4)})^{1/2} L \mathrm{d}^4 x + \frac{1}{8\pi} \int (g^{(3)})^{1/2} K \mathrm{d}^3 x \ , \tag{106.15}$$

where

$$L = L_{\text{geom}} + L_{\text{matter}} \tag{106.16}$$

is the Lagrangian, and the surface term is necessary to insure consistency if the action integral is broken into parts [183, 123]. The quantity

$$K = g^{(3)ij} K_{ij} = -\frac{1}{2} g^{(3)ij} \frac{\partial g_{ij}^{(3)}}{\partial t} \tag{106.17}$$

is the trace of the extrinsic curvature, where $g_{ij}^{(3)}$ is the 3-metric. In this example, we take the Lagrangian for the geometry as

$$L_{\text{geom}} = \frac{R^{(4)} - 2\Lambda}{16\pi} \ , \tag{106.18}$$

where $R^{(4)}$ is the four-dimensional scalar curvature and Λ is the cosmological constant.

For a perfect fluid, the energy momentum tensor is

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \ , \tag{106.19}$$

where p is the pressure, ρ is the density, and u is the 4-velocity. For solutions to Einstein's field equations[7]

$$R^{\mu\nu} = 8\pi (T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T) + \Lambda g^{\mu\nu} \tag{106.20}$$

for a perfect fluid, (106.18) becomes

$$L_{\text{geom}} = \frac{1}{2}\rho - \frac{3}{2} p + \frac{\Lambda}{8\pi} \ , \tag{106.21}$$

and we can take the Lagrangian for the matter as

$$L_{\text{matter}} = \rho + \alpha (p - \rho) \ , \tag{106.22}$$

where α is a constant, and we can take $\alpha = 0$ [161], $\alpha = 1$ [334, 162], or $\alpha = \frac{3}{2}$ (from combining (106.19) with [20, eq. 21.33a]). However, as we shall see, the final result is insensitive to the exact form of the Lagrangian. Substituting (106.21) and (106.22) into (106.16) gives

$$L = (\alpha - \frac{3}{2})(p - \rho) + \frac{\Lambda}{8\pi} \ , \tag{106.23}$$

For some cosmological models, it is possible to represent K as a time-integral of some quantity. For example, for the exact cosmological models considered by [163], the effect of K on the action can be represented as an integral over a 4-volume, allowing us to combine the two terms in (106.15). In that case, the surface term adds the following term to an effective Lagrangian:

$$\frac{3}{2}(p - \rho) - \frac{3\Lambda}{8\pi} - \frac{3a_0^2 + q_0^2}{4\pi X^2} \ , \tag{106.24}$$

[7] using the convention of [20]. The field equations (106.20) take on a slightly different interpretation according to [380]. However, the development here is valid for either interpretation.

where a_0, q_0, and X are parameters of their model, and a_0 in (106.24) has nothing to do with the cosmological scale factor a_0 used here. If so, then we would have

$$I = \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^4 x \ , \tag{106.25}$$

where $\tilde{L}$ can be considered to be an effective Lagrangian, which in the case of (106.23) and (106.24) gives

$$\tilde{L} = \alpha(p - \rho) - \frac{\Lambda}{4\pi} \ . \tag{106.26}$$

More generally, we can assume the effective Lagrangian to take the form

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda \ , \tag{106.27}$$

where α_1, α_2, and α_3 are dimensionless constants of order unity.

Table 106.2 shows that with vorticity, the density ρ depends on the scale factor along a flow line rather than on the cosmological scale factor a. To take this into account, we can write

$$\tilde{L}(\ell) \approx \tilde{L}_0 \left[1 + f_L(a)\delta + f_{LL}(a)\delta^2\right] = H_3^2 F(a) \left[1 + f_L(a)\delta + f_{LL}(a)\delta^2\right] \ , \tag{106.28}$$

where δ (defined in (106.41)) is proportional to the square of the present local vorticity,

$$\begin{array}{lcl} F(a) = & \frac{3}{8\pi}(\alpha_1 w + \alpha_2)\frac{a_1}{a_2}\left(\frac{a_2}{a}\right)^4 & \text{for } a \leq a_1, \\ F(a) = & \frac{3}{8\pi}\alpha_2 \left(\frac{a_2}{a}\right)^3 & \text{for } a_1 \leq a \leq a_2, \\ F(a) = & 3\alpha_3 & \text{for } a_2 \leq a \leq a_3, \end{array} \tag{106.29}$$

$$\begin{array}{lcl} f_L(a) = & f_\ell(a_1) + 3f_\ell(a_2) - 4f_\ell(a) & \text{for } a \leq a_1, \\ f_L(a) = & 3f_\ell(a_2) - 3f_\ell(a) & \text{for } a_1 \leq a \leq a_2, \\ f_L(a) = & 0 & \text{for } a_2 \leq a \leq a_3, \end{array} \tag{106.30}$$

$f_\ell(a)$ is defined in (106.61),

$$\begin{array}{lcl} f_{LL}(a) = & f_{\ell\ell}(a_1) + 3f_\ell(a_2)^2 + 3f_{\ell\ell}(a_2) + 10f_\ell(a)^2 - 4f_{\ell\ell}(a) & \\ & +3f_\ell(a_1)f_\ell(a_2) - 4f_\ell(a_1)f_\ell(a) - 12f_\ell(a_2)f_\ell(a) & \text{for } a \leq a_1, \\ f_{LL}(a) = & -9f_\ell(a_2)f_\ell(a) + 3f_\ell(a_2)^2 + 3f_{\ell\ell}(a_2) + 6f_\ell(a)^2 - 3f_{\ell\ell}(a) & \text{for } a_1 \leq a \leq a_2, \\ f_{LL}(a) = & 0 & \text{for } a_2 \leq a \leq a_3, \end{array} \tag{106.31}$$

and $f_{\ell\ell}(a)$ is defined in (106.62). Using (106.64) in (106.30) gives

$$\begin{array}{ll} f_L(a) = & 1 + 3f_\ell(a_2) - 4\left(\frac{a}{a_1}\right)^2 \text{ for } a \leq a_1, \\ f_L(a) = & -216\sqrt{\frac{a_1}{a_2}} + 138\frac{a_1}{a_2} + 36\frac{a_1}{a_2}\ln\frac{a_1}{a_2} + 216\sqrt{\frac{a_1}{a}} - 138\frac{a_1}{a} - 36\frac{a_1}{a}\ln\frac{a_1}{a} \text{ for } a_1 \leq a \leq a_2, \\ & f_L(a) = 0 \text{ for } a_2 \leq a \leq a_3. \end{array} \tag{106.32}$$

106.10 Inflation, radiation, matter, and dark-energy eras

There are four cosmological eras to consider. In the very early universe is the inflation era, which ran from roughly $t_i = 10^{-36}$ seconds to about $t_0 = 10^{-34}$ seconds [381, p. 109] after the initial singularity. During the inflation era the universe expanded by N e-foldings. We have that $a_0/a_i = e^N$ and $N = H_{0\ \mathrm{inflation}}(t_0 - t_i) \approx H_{0\ \mathrm{inflation}} t_0$. The Planck Collaboration [382] estimates that $50 \le N \le 60$. That the cosmological scale factor a varied as $t^{1/2}$ during the radiation era gives the value at the beginning of the radiation era and the end of inflation as about $a_0 \approx 4\times10^{-27}$. The main effect of inflation on the calculation of the action is through the parameter $\beta = H_3 t_2 (a_0/a_1) e^N/N \approx 1.5\times10^{-23} e^N/N$, where $t_2 \approx 9\times10^9$ years is the time of matter and dark energy equality, and the other parameters are defined below.

After inflation, radiation dominates over matter to determine the density ρ in the radical in (106.83). During the radiation era, a varies as $t^{1/2}$. When the cosmological scale factor a reaches a certain size (which we define as a_1), matter begins to dominate over radiation to determine the density ρ. During the matter era, a varies as $t^{2/3}$. When the cosmological scale factor a gets even larger (to a size we define as a_2), the density of matter has fallen low enough that the cosmological constant Λ begins to dominate over the density term in (106.83).

For an equation of state, we take

$$p = w\rho \ , \tag{106.33}$$

where $w = 1/3$ in the radiation-dominated era, and $w = 0$ in the matter-dominated era. The variation of density ρ with cosmological scale factor a is given by [342, Table 6.1]

$$\rho = \rho_1 (a/a_1)^{-3(1+w)} \ , \tag{106.34}$$

where ρ_1 is the value of ρ at the boundary between the radiation era and the matter era where $a = a_1$. Table 106.2 summarizes some of this.

We can take for the present fraction of radiation density $\Omega_{\mathrm{rad}} = 5.4\times10^{-5}$ [381, p. 78]. Otherwise, we take [382]

$$a_1 = \frac{\Omega_{\mathrm{rad}}}{\Omega_{\mathrm{mat}}} \approx \frac{5.4\times10^{-5}}{.3089} \approx 1.7\times10^{-4} \ , \tag{106.35}$$

and

$$a_2 = \left(\frac{\Omega_{\mathrm{mat}}}{\Omega_\Lambda}\right)^{1/3} \approx \left(\frac{.3089}{.6911}\right)^{1/3} \approx 0.76 \ , \tag{106.36}$$

where Ω_{mat} is the present fraction of matter density (including dark matter), and Ω_Λ is the present fraction of dark energy. From the present value of the Hubble parameter [382]

$$H_3 = 67.74 \text{ km s}^{-1}\text{ Mpc}^{-1} = 2.195\times10^{-18}\text{sec}^{-1} = 7.323\times10^{-29}\text{cm}^{-1} = 6.928\times10^{-11}\text{yr}^{-1} \ , \tag{106.37}$$

we can calculate the critical density, and combining with Ω_Λ, we get the value of the density of matter and dark energy when they were equal, which is

$$\rho_2 = \frac{\Lambda}{8\pi} \approx 4.4\times10^{-58} \text{ cm}^{-2} \ , \tag{106.38}$$

which gives

$$\Lambda \approx 1.1\times10^{-56} \text{ cm}^{-2} \ . \tag{106.39}$$

We also have

$$\hbar \to \frac{\hbar G}{c^2} = L^{*2} \approx 2.616\times10^{-66} \text{ cm}^2 \ . \tag{106.40}$$

106.11 Relative rotation

It is necessary to estimate the effect of the vorticity ω on the total rotation angle θ. Because we expect only very small values of vorticity to contribute significantly to a path integral calculation of vorticity, it is useful to choose a non-dimensional parameter that is small whenever the vorticity is small. In addition, because the action is an even function of the vorticity, and therefore is a function of the square of vorticity, it is useful for that parameter to be proportional to the square of the vorticity. It turns out to be useful to take that parameter as

$$\delta \equiv \frac{1}{6}\left(\frac{\omega_3}{H_3}\right)^2\left(\frac{a_3}{a_1}\right)\left(\frac{a_3}{a_2}\right)^3 , \tag{106.41}$$

where ω_3 is the present value of the local vorticity, H_3 is the present value of the Hubble parameter, a_1 is the value of the cosmological scale factor at the time when the matter density became equal to the radiation density, a_2 is the value of cosmological scale factor at the time when the cosmological constant surpassed the matter density in the Friedmann equation, and $a_3 = 1$ is the present value of the cosmological scale factor.

We start with

$$\theta = \int_0^t \omega \,\mathrm{d}t = \int_{a_i}^a \frac{\omega}{\dot a}\,\mathrm{d}a = \int_{a_i}^a \frac{\omega}{aH_0}\left[1 + f_H(a)\delta + f_{HH}(a)\delta^2\right]\mathrm{d}a , \tag{106.42}$$

where we have used (106.91) for $1/\dot a$. Using table 106.2 for ω gives

$$\begin{aligned}\omega &= H_3\sqrt{6\delta}a_1^{-1/2}a_2^{3/2}(\ell_1/a_1)^{-1}(\ell_3/a_3)^2 a^{-1}(\ell/a)^{-1} \text{ for } a \le a_1, \text{ and}\\ \omega &= H_3\sqrt{6\delta}a_1^{1/2}a_2^{3/2}(\ell_3/a_3)^2 a^{-2}(\ell/a)^{-2} \text{ for } a \ge a_1,\end{aligned} \tag{106.43}$$

where δ is defined in (106.41). Using (106.60) to expand ℓ_1/a_1, ℓ_3/a_3, and ℓ/a to second order in δ allows us to write (106.43) as

$$\begin{aligned}\omega &= \omega_3\frac{a_3}{a_1}\frac{a_3}{a}[1 + g_\omega(a)\delta + g_{\omega\omega}(a)\delta^2] \text{ for } a \le a_1, \text{ and}\\ \omega &= \omega_3\left(\frac{a_3}{a}\right)^2[1 + g_\omega(a)\delta + g_{\omega\omega}(a)\delta^2] \text{ for } a \ge a_1,\end{aligned} \tag{106.44}$$

where

$$\begin{aligned}g_\omega(a) &= 2f_\ell(a_3) - f_\ell(a_1) - f_\ell(a) \text{ for } a \le a_1\\ g_\omega(a) &= 2f_\ell(a_3) - 2f_\ell(a) \text{ for } a \ge a_1 ,\end{aligned} \tag{106.45}$$

$f_\ell(a)$ is defined in (106.61),

$$\begin{aligned}g_{\omega\omega}(a) &= \Big[2f_{\ell\ell}(a_3) - f_{\ell\ell}(a_1) - f_{\ell\ell}(a) + f_\ell(a_3)^2 + f_\ell(a_1)^2 + f_\ell(a)^2\\ &\quad + f_\ell(a_1)f_\ell(a) - 2f_\ell(a_1)f_\ell(a_3) - 2f_\ell(a)f_\ell(a_3)] \text{ for } a \le a_1\\ g_{\omega\omega}(a) &= \Big[2f_{\ell\ell}(a_3) - 2f_{\ell\ell}(a) + f_\ell(a_3)^2 + 3f_\ell(a)^2 - 4f_\ell(a)f_\ell(a_3)\Big] \text{ for } a \ge a_1 ,\end{aligned} \tag{106.46}$$

and $f_{\ell\ell}(a)$ is defined in (106.62).

Using (106.43) for ω, (106.92) for $f_H(a)$, and (106.60) to expand ℓ_1/a_1, ℓ_3/a_3, and ℓ/a to second order in δ allows us to write (106.42) as

$$\theta \approx \sqrt{2\delta}\left[\sqrt{f_\theta(a)} - \frac{1}{2}f_{\theta\theta}(a)\delta\right] , \tag{106.47}$$

where

$$\sqrt{f_\theta(a)} = H_3\sqrt{3}a_1^{-1/2}a_2^{3/2}\int_{a_i}^{a}\frac{a^{-2}}{H_0}\,\mathrm{d}a \text{ for } a \le a_1, \text{ and}$$
$$\sqrt{f_\theta(a)} = \sqrt{f_\theta(a_1)} + H_3\sqrt{3}a_1^{1/2}a_2^{3/2}\int_{a_1}^{a}\frac{a^{-3}}{H_0}\,\mathrm{d}a \text{ for } a \ge a_1, \tag{106.48}$$

and

$$f_{\theta\theta}(a) = -2\left(2f_\ell(a_3) - f_\ell(a_1)\right)\sqrt{f_\theta(a)}$$
$$+2H_3\sqrt{3}a_1^{-1/2}a_2^{3/2}\int_{a_0}^{a}\frac{a^{-2}}{H_0}(f_\ell(a) - f_H(a))\,\mathrm{d}a \text{ for } a \le a_1, \text{ and}$$
$$f_{\theta\theta}(a) = f_{\theta\theta}(a_1) - 4f_\ell(a_3)\left(\sqrt{f_\theta(a)} - \sqrt{f_\theta(a_1)}\right)$$
$$+2H_3\sqrt{3}a_1^{1/2}a_2^{3/2}\int_{a_1}^{a}\frac{a^{-3}}{H_0}(2f_\ell(a) - f_H(a))\,\mathrm{d}a \text{ for } a \ge a_1. \tag{106.49}$$

Evaluating (106.48) gives

$$f_\theta(a) = 3\beta^2\left(1 - \frac{a_0}{a}e^{-N}\right)^2 \text{ for } a_i \le a \le a_0,$$
$$f_\theta(a) = 3\left[\beta + \left(1 - \frac{a_0}{a}\right)\left(\frac{a}{a_1}\right)\right]^2 \text{ for } a_0 \le a \le a_1,$$
$$f_\theta(a) = 3\left[\beta + 3 - 2\left(\frac{a_1}{a}\right)^{1/2} - \frac{a_0}{a_1}\right]^2 \text{ for } a_1 \le a \le a_2, \text{ and}$$
$$f_\theta(a) = 3\left[\beta + 3 - \frac{a_0}{a_1} - \left(\frac{a_1}{a_2}\right)^{1/2}\left(\frac{3}{2} + \frac{1}{2}\left(\frac{a_2}{a}\right)^2\right)\right]^2 \text{ for } a_2 \le a \le a_3, \tag{106.50}$$

where

$$\beta = \frac{H_3}{H_0}(\frac{a_1}{a_i})\left(\frac{a_2}{a_1}\right)^{3/2} = H_3\frac{t_0 - t_i}{N}a_1\frac{e^N}{a_0}\left(\frac{a_2}{a_1}\right)^{3/2} \approx H_3t_0\left(\frac{a_2}{a_1}\right)^{3/2}(\frac{a_1}{a_0})\frac{e^N}{N}$$
$$= H_3t_1\left(\frac{a_2}{a_1}\right)^{3/2}(\frac{a_0}{a_1})\frac{e^N}{N} = H_3t_2(\frac{a_0}{a_1})\frac{e^N}{N} \approx 1.5\times 10^{-23}\frac{e^N}{N} \tag{106.51}$$

gives the effect of inflation. We can neglect a_0 in some places in (106.50) to give

$$f_\theta(a) = 3\beta^2\left(1 - \frac{a_0}{a}e^{-N}\right)^2 \text{ for } a_i \le a \le a_0,$$
$$f_\theta(a) = 3\left(\beta + \frac{a}{a_1}\right)^2 \text{ for } a_0 \le a \le a_1,$$
$$f_\theta(a) = 3\left[\beta + 3 - 2\left(\frac{a_1}{a}\right)^{1/2}\right]^2 \text{ for } a_1 \le a \le a_2, \text{ and}$$
$$f_\theta(a) = 3\left[\beta + 3 - \left(\frac{a_1}{a_2}\right)^{1/2}\left(\frac{3}{2} + \frac{1}{2}\left(\frac{a_2}{a}\right)^2\right)\right]^2 \text{ for } a_2 \le a \le a_3. \tag{106.52}$$

We also have

$$\theta^2 \approx 2\delta\left[f_\theta(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\delta\right], \tag{106.53}$$

and

$$\cos\theta \approx (1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4) \approx \left[1 - f_\theta(a)\delta + \left(\frac{1}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right)\delta^2\right] , \tag{106.54}$$

and

$$\frac{1}{\cos\theta} \approx (1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4) \approx \left[1 + f_\theta(a)\delta + \left(\frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right)\delta^2\right] . \tag{106.55}$$

Taking the approximate mean of (106.53), using (106.52), putting in the appropriate values, including $a_3 = 1$, gives for the total rms rotation

$$\langle\theta\rangle \approx 347\,\langle\omega_3\rangle/H_3 \approx 347\,\langle\omega_3\rangle\sqrt{3/\Lambda} . \tag{106.56}$$

Finally, using the value for Λ from (106.39) and using ω_m from (106.103) for $\langle\omega_3\rangle$ gives

$$\theta \approx 6 \times 10^{-59} \text{ radians } \approx \times 10^{-58} \text{ radians} \tag{106.57}$$

for the total rotation of the universe from the initial singularity to the present, showing that it should be valid to assume flow lines are normal to surfaces of constant time for the calculations.

To find the limits for the validity of the calculation of $f_I(\langle\omega_3\rangle)$, we use (106.56) to find the value of $\langle\omega_3\rangle$ for which $\theta = 1$ to give

$$\langle\omega_3\rangle \approx H_3/347 \approx 0.003 H_3 . \tag{106.58}$$

Thus, the approximate calculation of $f_I(\langle\omega_3\rangle)$ is not valid all the way to $\langle\omega_3\rangle = H_3$, but it is valid for $\langle\omega_3\rangle \gg \omega_m$, which is good enough. Of course this is only the local total rotation. If we allow for an additional few orders of magnitude increase because of spatial variation of the vorticity, we still have a small total rotation.

The point is, that it was a good approximation to consider that surfaces of constant global time are normal to flow lines. In addition, there are a few more orders of magnitude to use to allow the approximation to be valid for $\langle\omega_3\rangle$ to be much larger than ω_m and still be a good approximation.

It also means that the saddlepoint at $\langle\omega_3\rangle = 0$ is isolated from any other saddlepoints or other non-analytic points.

106.12 Approximate global cosmological scale factor

The flow lines are not normal to surfaces of constant global time, so we have to change variables from ℓ to a, where a is the cosmological scale factor. We can use the angle calculations in appendix 106.11 to make that conversion. Because the angles are small, we can make approximations. The formulas below apply to each location in the flow because the vorticity can be a function of location.

We use (106.55) to approximate $1/\cos\theta$ to give

$$\ell \approx \int \frac{\mathrm{d}a}{\cos\theta} \approx \int (1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4)\,\mathrm{d}a \approx \int \left[1 + f_\theta(a)\delta + \left(\frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right)\delta^2\right]\mathrm{d}a , \tag{106.59}$$

where a is the cosmological scale factor, which is normal to surfaces of constant global time, and δ is given by (106.41).

We can write

$$\ell \approx a\left[1 + f_\ell(a)\delta + f_{\ell\ell}(a)\delta^2\right] , \tag{106.60}$$

where

$$f_\ell(a) = \frac{1}{a}\int_{a_i}^{a} f_\theta(a)\,\mathrm{d}a \tag{106.61}$$

and

$$f_{\ell\ell}(a) = \frac{1}{a}\int_{a_0}^{a}\left[\frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right] \mathrm{d}a \ . \tag{106.62}$$

We also have

$$\frac{1}{\ell} \approx \frac{1}{a}\left[1 - f_\ell(a)\delta + \left(f_\ell^2(a) - f_{\ell\ell}(a)\right)\delta^2\right] \ . \tag{106.63}$$

We can use (106.52) in (106.61) and neglect a_0 in some places to give

$$\begin{aligned}
f_\ell(a) &= 3\beta^2\left(1 - 2\frac{a_0}{a}Ne^{-N} - 2\frac{a_0}{a}e^{-N}\ln\frac{a}{a_0} - \frac{a_0^2}{a^2}e^{-2N}\right) \text{ for } a_i \le a \le a_0,\\
f_\ell(a) &= 3\beta^2 + \left(\frac{a}{a_1}\right)^2 \text{ for } a_0 \le a \le a_1,\\
f_\ell(a) &= 3\beta^2 + \left(27 - 72\sqrt{\frac{a_1}{a}} + 46\frac{a_1}{a} - 12\frac{a_1}{a}\ln\frac{a_1}{a}\right) \text{ for } a_1 \le a \le a_2, \text{ and}\\
f_\ell(a) &= f_\ell(a_2)\frac{a_2}{a} + 3\frac{a_1}{a_2}\left[\left(3\sqrt{\frac{a_2}{a_1}} - \frac{3}{2}\right)^2\left(1 - \frac{a_2}{a}\right)\right.\\
&\left. + \left(3\sqrt{\frac{a_2}{a_1}} - \frac{3}{2}\right)\left(\frac{a_2^2}{a^2} - \frac{a_2}{a}\right) - \frac{1}{12}\left(\frac{a_2^4}{a^4} - \frac{a_2}{a}\right)\right] \text{ for } a_2 \le a \le a_3.
\end{aligned} \tag{106.64}$$

From (106.64), we have

$$f_\ell(a_1) = 3\beta^2 + 1 \ . \tag{106.65}$$

Using that from (106.35) a_1 is so small, we can write

$$f_\ell(a_2) \approx 3\beta^2 + 27 \ . \tag{106.66}$$

We can write (106.62) as

$$\begin{aligned}
f_{\ell\ell}(a) &= \frac{1}{a}\int_{a_0}^{a}\left[\frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right] \mathrm{d}a \text{ for } a_0 \le a \le a_1,\\
f_{\ell\ell}(a) &= f_{\ell\ell}(a_1)\frac{a_1}{a} + \frac{1}{a}\int_{a_1}^{a}\left[\frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right] \mathrm{d}a \text{ for } a_1 \le a \le a_2,\\
f_{\ell\ell}(a) &= f_{\ell\ell}(a_2)\frac{a_2}{a} + \frac{1}{a}\int_{a_2}^{a}\left[\frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right] \mathrm{d}a \text{ for } a_2 \le a \le a_3.
\end{aligned} \tag{106.67}$$

Or,

$$\begin{aligned}
f_{\ell\ell}(a) &= \frac{5}{6}\frac{1}{a}\int_{a_0}^{a}(f_\theta(a))^2 \ \mathrm{d}a - \frac{1}{a}\int_{a_0}^{a} f_{\theta\theta}(a)\sqrt{f_\theta(a)}\,\mathrm{d}a \text{ for } a_0 \le a \le a_1,\\
f_{\ell\ell}(a) &= f_{\ell\ell}(a_1)\frac{a_1}{a} + \frac{5}{6}\frac{1}{a}\int_{a_1}^{a}(f_\theta(a))^2 \ \mathrm{d}a - \frac{1}{a}\int_{a_1}^{a} f_{\theta\theta}(a)\sqrt{f_\theta(a)}\,\mathrm{d}a \text{ for } a_1 \le a \le a_2,\\
f_{\ell\ell}(a) &= f_{\ell\ell}(a_2)\frac{a_2}{a} + \frac{5}{6}\frac{1}{a}\int_{a_2}^{a}(f_\theta(a))^2 \ \mathrm{d}a - \frac{1}{a}\int_{a_2}^{a} f_{\theta\theta}(a)\sqrt{f_\theta(a)}\,\mathrm{d}a \text{ for } a_2 \le a \le a_3.
\end{aligned} \tag{106.68}$$

106.13 Approximate Generalized Friedmann equation for small Vorticity

It is possible to find an approximate solution to the field equations that is a perturbation from the standard cosmological model for small vorticity. The difficulty with finding an approximate solution that includes vorticity is that the Raychaudhuri equation is based on a coordinate system that follows flow lines, but surfaces of constant global time cannot be found that are normal to the flow lines. However, in the limit of small vorticity, we can estimate the effect that the scale factor along flow lines ℓ is not quite the same as the global cosmological scale factor a.

Derivation of a generalization of the Friedmann equation that includes relative rotation of matter and inertial frames (specifically, shear and vorticity), starts with the Raychaudhuri equation [343, 344], [345, eq. 1.3.4], [346, eq. (36)], [300, 301, eq. 4.12], [336, 347, 348, 349, 338, 339], or the Raychaudhuri-Ehlers equation [342, eq. 6.4].

We start with the Raychaudhuri-Ehlers equation [342, eq. 6.5]

$$3\frac{\ddot{\ell}}{\ell} = -2\left(\sigma^2 - \omega^2\right) + \nabla_a \dot{u}^a + \dot{u}_a \dot{u}^a - 4\pi G(\rho + 3p) + \Lambda \ , \tag{106.69}$$

where ℓ is a scale factor that follows flow lines, and $\cdot$ is a derivative with respect to a variable τ that increases along the flow line u.

Multiplying (106.69) by $\ell\dot{\ell}$ and integrating gives

$$\dot{\ell}/\ell = \sqrt{H^2 + H_\omega^2 + H_\sigma^2 + H_a^2} \ , \tag{106.70}$$

where

$$H \equiv \sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{\ell^2}} \tag{106.71}$$

is the Hubble parameter without vorticity, shear, or acceleration,

$$H_\omega^2 \equiv \frac{4}{3\ell^2} \int \ell\omega^2 \, \mathrm{d}\ell \tag{106.72}$$

is the vorticity term, ω is vorticity,

$$H_\sigma^2 \equiv -\frac{4}{3\ell^2} \int \ell\sigma^2 \, \mathrm{d}\ell \tag{106.73}$$

is the shear term, σ is shear, and

$$H_a^2 \equiv -\frac{2}{3\ell^2} \int \ell \dot{u}^a{}_{;a} \, \mathrm{d}\ell \tag{106.74}$$

is the acceleration term.

We can take vorticity to depend on the distance along flow lines ℓ as

$$\omega \propto \ell^{-m} \ , \tag{106.75}$$

and, for small vorticity, we can take $m = 1$ in the radiation era and $m = 2$ in the matter era [342, Table 6.1].[8] If we put the ℓ-variation of vorticity into (106.72), then we get

$$H_\omega^2 \approx H_3^2 f_\omega(a)\delta \ , \tag{106.76}$$

where δ is given by (106.41),

$$H_3 \approx \sqrt{\Lambda/3} \tag{106.77}$$

is the present value of the Hubble parameter, and

$$\begin{aligned} f_\omega(a) &= 4\left(\frac{a_2}{a_1}\right)^3 \left(\frac{a_1}{a}\right)^2 \left[2\ln\frac{a}{a_1} + \alpha_4 - 1\right] \text{ for } a \le a_1 \\ f_\omega(a) &= 4\left(\frac{a_2}{a_1}\right)^3 \left(\frac{a_1}{a}\right)^2 \left[\alpha_4 - \left(\frac{a_1}{a}\right)^2\right] \text{ for } a \ge a_1 \ , \end{aligned} \tag{106.78}$$

[8] Instead of this simple assumption, we could instead take into account the coupling between vorticity and shear in a combined way in terms of a vector perturbation [342, Chapter 10]. The final effect on the calculation of the total action, however, is not significant.

where α_4 is an arbitrary constant of integration in (106.72).

We can generalize (106.76) to second order by writing

$$H_\omega^2 = H_3^2\left[f_\omega(a)\delta + f_{\omega\omega}(a)\delta^2\right] , \tag{106.79}$$

where

$$\begin{aligned}
f_{\omega\omega}(a) &= 8\left(\frac{a_2}{a_1}\right)^3\left(\frac{a_1}{a}\right)^2\left[(2f_\ell(a_3)-f_\ell(a_1)-f_\ell(a))\left(2\ln\frac{a}{a_1}+\alpha_4-1\right)+f_\ell(a)-f_\ell(a_1)\right] \\
&\qquad \text{for } a\le a_1 \\
f_{\omega\omega}(a) &= 8\left(\frac{a_2}{a_1}\right)^3\left(\frac{a_1}{a}\right)^2\left[(2f_\ell(a_3)-f_\ell(a_1)-f_\ell(a))\,\alpha_4-2\,(f_\ell(a_3)-f_\ell(a))\left(\frac{a_1}{a}\right)^2\right] \\
&\qquad \text{for } a\ge a_1 \; .
\end{aligned} \tag{106.80}$$

We can use (106.64) for $f_\ell(a)$ in (106.80) to give

$$\begin{aligned}
f_{\omega\omega}(a) &= 8\left(\frac{a_2}{a_1}\right)^3\Bigg[2\,(2f_\ell(a_3)-1)\left(\frac{a_1}{a}\right)^2\ln\frac{a}{a_1}-2\ln\frac{a}{a_1} \\
&\quad +(2f_\ell(a_3)-1)(\alpha_4-1)\left(\frac{a_1}{a}\right)^2-\left(\frac{a_1}{a}\right)^2+2-\alpha_4\Bigg] \quad \text{for } a_0\le a\le a_1 \\
f_{\omega\omega}(a) &= 8\left(\frac{a_2}{a_1}\right)^3\left(\frac{a_1}{a}\right)^2\Bigg[(2f_\ell(a_3)-f_\ell(a_1)-27)\alpha_4-\alpha_4\left(-72\sqrt{\frac{a_1}{a}}+46\frac{a_1}{a}+12\frac{a_1}{a}\ln\frac{a}{a_1}\right) \\
&\quad +2\left(27-f_\ell(a_3)-72\sqrt{\frac{a_1}{a}}+46\frac{a_1}{a}+12\frac{a_1}{a}\ln\frac{a}{a_1}\right)\left(\frac{a_1}{a}\right)^2\Bigg] \quad \text{for } a_1\le a\le a_2 \\
f_{\omega\omega}(a) &= 8\left(\frac{a_2}{a_1}\right)^3\left(\frac{a_1}{a}\right)^2\Bigg\{\Bigg(2f_\ell(a_3)-f_\ell(a_1)-f_\ell(a_2)\frac{a_2}{a} \\
&\quad -3\frac{a_1}{a_2}\left[\left(3\sqrt{\frac{a_2}{a_1}}-\frac{3}{2}\right)^2\left(1-\frac{a_2}{a}\right)+\left(3\sqrt{\frac{a_2}{a_1}}-\frac{3}{2}\right)\left(\frac{a_2^2}{a^2}-\frac{a_2}{a}\right)-\frac{1}{12}\left(\frac{a_2^4}{a^4}-\frac{a_2}{a}\right)\right]\Bigg)\alpha_4 \\
&\quad -2\Bigg(f_\ell(a_3)-f_\ell(a_2)\frac{a_2}{a} \\
&\quad -3\frac{a_1}{a_2}\left[\left(3\sqrt{\frac{a_2}{a_1}}-\frac{3}{2}\right)^2\left(1-\frac{a_2}{a}\right)+\left(3\sqrt{\frac{a_2}{a_1}}-\frac{3}{2}\right)\left(\frac{a_2^2}{a^2}-\frac{a_2}{a}\right)-\frac{1}{12}\left(\frac{a_2^4}{a^4}-\frac{a_2}{a}\right)\right]\Bigg)\left(\frac{a_1}{a}\right)^2\Bigg\} \\
&\qquad \text{for } a_2\le a\le a_3 \; .
\end{aligned} \tag{106.81}$$

If the vorticity is small enough, then we can find a coordinate system with a cosmological scale factor a normal to surfaces of constant global time, that differ from flow lines by a very small amount. In that case, we can replace the scale factor ℓ along flow lines by the cosmological scale factor a, and we calculate the effect of that difference to first and second order. In making this calculation, we replace τ by the global time t.

We start by using $\dot{\ell}\approx\dot{a}/\cos\theta$ in (106.70) and neglect shear and acceleration to give

$$\frac{\dot{a}}{a} = \frac{\ell}{a}\cos\theta\sqrt{H^2+H_\omega^2+H_\sigma^2+H_a^2} \tag{106.82}$$

for the generalized Friedmann equation, where

$$H \equiv \sqrt{\frac{\Lambda}{3}+\frac{8\pi\rho}{3}-\frac{k}{a^2}} \to \sqrt{\frac{\Lambda}{3}+\frac{8\pi\rho}{3}} \tag{106.83}$$

is the Hubble parameter without vorticity, shear, or acceleration, and we neglect the spatial curvature term from here on because measurements show it to be nearly zero.

However, because the density ρ in (106.83) depends on ℓ rather than a, we have in analogy with (106.28) for the same effect on the Lagrangian

$$H^2 \approx H_0^2 \left[1 + f_L(a)\delta + f_{LL}(a)\delta^2\right] \ , \tag{106.84}$$

where

$$H_0 = \sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho_0}{3}} \tag{106.85}$$

is the Hubble parameter without the ℓ dependence and ρ_0 is the density without the ℓ dependence. We also have

$$\frac{1}{H^2} \approx \frac{1}{H_0^2}\left[1 - f_L(a)\delta + \left(f_L(a)^2 - f_{LL}(a)\right)\delta^2\right] \tag{106.86}$$

and

$$\frac{1}{H} \approx \frac{1}{H_0}\left[1 - \frac{1}{2}f_L(a)\delta + \left(\frac{3}{8}f_L(a)^2 - \frac{1}{2}f_{LL}(a)\right)\delta^2\right] \ . \tag{106.87}$$

Although in general, the vorticity propagation equation and the shear propagation equation are coupled [383], the coupling terms are second-order, so that for very small vorticity and very small shear, we can neglect the coupling. Therefore, we can neglect the shear term in (106.82). In addition, we neglect the acceleration term. Using (106.84) for H, (106.79) for H_ω^2, (106.60) for ℓ, and (106.54) for $\cos\theta$ gives

$$\begin{aligned}
&\frac{\dot a}{a} \approx \\
&H_0\left[1 + \frac{1}{2}\left(f_L(a) + \frac{H_3^2}{H_0^2}f_\omega(a)\right)\delta + \frac{1}{2}\left(f_{LL}(a) + \frac{H_3^2}{H_0^2}f_{\omega\omega}(a) - \frac{1}{4}\left(f_L(a) + \frac{H_3^2}{H_0^2}f_\omega(a)\right)^2\right)\delta^2\right] \times \\
&\left[1 + f_\ell(a)\delta + f_{\ell\ell}(a)\delta^2\right]\left[1 - f_\theta(a)\delta + \left(\frac{1}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right)\delta^2\right] \ .
\end{aligned} \tag{106.88}$$

Multiplying out and keeping terms through second order in δ gives

$$\begin{aligned}
\frac{\dot a}{a} \approx H_0 \Bigg\{ & 1 + \left[\frac{1}{2}f_L(a) + f_\ell(a) - f_\theta(a) + \frac{1}{2}\frac{H_3^2}{H_0^2}f_\omega(a)\right]\delta \\
& + \Bigg[\frac{1}{2}f_{LL}(a) + f_{\ell\ell}(a) + \frac{1}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) - \frac{1}{8}f_L(a)^2 + \frac{1}{2}f_L(a)f_\ell(a) - \frac{1}{2}f_L(a)f_\theta(a) - f_\ell(a)f_\theta(a) \\
& + \left(-\frac{1}{4}f_L(a)f_\omega(a) + \frac{1}{2}f_\ell(a)f_\omega(a) - \frac{1}{2}f_\theta(a)f_\omega(a) + \frac{1}{2}f_{\omega\omega}(a) - \frac{1}{8}\frac{H_3^2}{H_0^2}f_\omega(a)^2\right)\frac{H_3^2}{H_0^2}\Bigg]\delta^2\Bigg\} \ .
\end{aligned} \tag{106.89}$$

To the same degree of approximation, we have

$$\begin{aligned}
&\frac{1}{\dot a} \approx \\
&\frac{1}{aH_0}\left[1 - \frac{1}{2}\left(f_L(a) + \frac{H_3^2}{H_0^2}f_\omega(a)\right)\delta - \frac{1}{2}\left(f_{LL}(a) + \frac{H_3^2}{H_0^2}f_{\omega\omega}(a) - \frac{3}{4}\left(f_L(a) + \frac{H_3^2}{H_0^2}f_\omega(a)\right)^2\right)\delta^2\right] \times \\
&\left[1 - f_\ell(a)\delta + \left(f_\ell^2(a) - f_{\ell\ell}(a)\right)\delta^2\right]\left[1 + f_\theta(a)\delta + \left(\frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right)\delta^2\right] \ .
\end{aligned} \tag{106.90}$$

Multiplying out and keeping terms through second order in δ gives

$$\frac{1}{\dot{a}} \approx \frac{1}{aH_0}\left[1 + f_H(a)\delta + f_{HH}(a)\delta^2\right] \ , \tag{106.91}$$

where

$$f_H(a) = -\frac{1}{2}f_L(a) - f_\ell(a) + f_\theta(a) - \frac{1}{2}\frac{H_3^2}{H_0^2}f_\omega(a) \tag{106.92}$$

and

$$\begin{aligned} f_{HH}(a) &= -\frac{1}{2}f_{LL}(a) + f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6}f_\theta^2(a) - \sqrt{f_\theta(a)}f_{\theta\theta}(a) + \frac{3}{8}f_L(a)^2 \\ &+\frac{1}{2}f_L(a)f_\ell(a) - \frac{1}{2}f_L(a)f_\theta(a) - f_\ell(a)f_\theta(a) \\ &+ \left(\frac{3}{4}f_L(a)f_\omega(a) + \frac{1}{2}f_\ell(a)f_\omega(a) - \frac{1}{2}f_\theta(a)f_\omega(a) - \frac{1}{2}f_{\omega\omega}(a) + \frac{3}{8}\frac{H_3^2}{H_0^2}f_\omega(a)^2\right)\frac{H_3^2}{H_0^2} \ . \end{aligned} \tag{106.93}$$

We can consider that (106.89) gives the effective generalized Friedmann equation, including to second order the effect that flow lines are not normal to surfaces of constant global time.

106.14 Approximate action for small vorticity

The main effect of vorticity and shear on the action is through the generalized Friedmann equation (106.70). Starting with (106.25), change the t integration to an integration over a using (106.91) and use (106.28) for $\tilde{L}$ to give

$$I = \int \frac{V\tilde{L}\,\mathrm{d}\,a}{\dot{a}} \approx \int_{a_i}^{a_3}\frac{V(a)H_3^2F(a)}{aH_0}\left[1 + f_L(a)\overline{\delta} + f_{LL}(a)\overline{\delta^2}\right]\left[1 + f_H(a)\overline{\delta} + f_{HH}(a)\overline{\delta^2}\right]\mathrm{d}\,a \ , \tag{106.94}$$

where

$$V(a) = \frac{4}{3}\pi a^3 r_3^3 \tag{106.95}$$

is the approximate spatial volume[9], a is the cosmological scale factor, r_3 is the present radius of the cosmological horizon,

$$\overline{\delta} = \frac{1}{6}\overline{\left(\frac{\omega_3}{H_3}\right)^2}\left(\frac{a_3}{a_1}\right)\left(\frac{a_3}{a_2}\right)^3 = \frac{1}{6}\left(\frac{\langle\omega_3\rangle}{H_3}\right)^2\left(\frac{a_3}{a_1}\right)\left(\frac{a_3}{a_2}\right)^3 \tag{106.96}$$

is the spatial average of δ given by (106.41),

$$\overline{\delta^2} = \overline{\delta}^2 + \left[\frac{1}{6}\left(\frac{1}{H_3}\right)^2\left(\frac{a_3}{a_1}\right)\left(\frac{a_3}{a_2}\right)^3\right]^2\sigma_\omega^2 \tag{106.97}$$

is the spatial average of the square of δ, σ_ω^2 is the variance of ω_3^2, and we have neglected shear and acceleration, keeping only vorticity. Multiplying out, and keeping terms only up to second order in $\overline{\delta}$ gives

$$I \approx \int_{a_i}^{a_3}\frac{V(a)H_3^2F(a)}{aH_0}\left\{1 + \left[f_L(a) + f_H(a)\right]\overline{\delta} + \left[f_L(a)f_H(a) + f_{LL}(a) + f_{HH}(a)\right]\overline{\delta^2}\right\}\mathrm{d}\,a \ . \tag{106.98}$$

[9]Since our universe is approximately spatially flat. Although there are arguments that V should be infinite because this is an open cosmology [351], considering causality leads to restricting the spatial part of the action to the past light cone. Besides, an infinite action would give the result that the only cosmologies contributing significantly to the path integral had a vorticity exactly equal to zero.

We can write (106.98) as

$$I \approx I_0 + I_1\overline{\delta} + I_2\overline{\delta^2} \ , \tag{106.99}$$

where I_0 is the action for zero vorticity and zero shear,

$$I_1 = \int_{a_i}^{a_3} \frac{V(a)H_3^2F(a)}{aH_0} \left[f_L(a) + f_H(a)\right] \mathrm{d}\, a \ , \tag{106.100}$$

and

$$I_2 = \int_{a_0}^{a_3} \frac{V(a)H_3^2F(a)}{aH_0} \left[f_L(a)f_H(a) + f_{LL}(a) + f_{HH}(a)\right] \mathrm{d}\, a \ . \tag{106.101}$$

We can use (106.95) for $V(a)$, and (106.96) for $\overline{\delta}$ and (106.97) for $\overline{\delta^2}$ to write (106.99) as

$$I \approx I_0 + \hbar \left(\frac{\langle\omega_3\rangle}{\omega_m}\right)^2 \left(\frac{a_3}{a_1}\right) \left[C_I + \frac{\langle\omega_3\rangle^2 + \sigma_\omega^2/\langle\omega_3\rangle^2}{H_3^2} \left(\frac{a_3}{a_1}\right) C_{II}\right] \ , \tag{106.102}$$

where

$$\omega_m = \left(\frac{\hbar H_3}{r_3^3}\right)^{1/2} = \frac{T^*}{r_3}\sqrt{\frac{H_3}{r_3}} \approx T^* H_3^2 \approx 10^{-89}\mathrm{cm}^{-1} \approx 10^{-71}\mathrm{rad\ yr}^{-1} \ , \tag{106.103}$$

H_3 is the present value of the Hubble parameter, r_3 is the present radius of the universe (which we approximate by the inverse of the Hubble parameter), and T^* is the Planck time,

$$C_I = \frac{1}{6}\left(\frac{a_3}{a_2}\right)^3 \frac{4}{3}\pi \int_{a_i}^{a_3} a^2 \frac{H_3F(a)}{H_0} \left[f_L(a) + f_H(a)\right] \mathrm{d}\, a \ , \tag{106.104}$$

and

$$C_{II} = \frac{1}{36}\left(\frac{a_3}{a_2}\right)^6 \frac{4}{3}\pi \int_{a_0}^{a_3} a^2 \frac{H_3F(a)}{H_0} \left[f_L(a)f_H(a) + f_{LL}(a) + f_{HH}(a)\right] \mathrm{d}\, a \ . \tag{106.105}$$

We can use (106.92) for $f_H(a)$ in (106.104) to give

$$C_I = \frac{1}{6}\left(\frac{a_3}{a_2}\right)^3 \frac{4}{3}\pi \int_{a_i}^{a_3} a^2 \frac{H_3F(a)}{H_0} \left[f_\theta(a) - f_\ell(a) + \frac{1}{2}f_L(a) - \frac{1}{2}\frac{H_3^2}{H_0^2}f_\omega(a)\right] \mathrm{d}\, a \ . \tag{106.106}$$

We can use (106.92) for $f_H(a)$ and (106.93) for $f_{HH}(a)$ in (106.105) to give

$$\begin{aligned} C_{II} = \frac{1}{36}\left(\frac{a_3}{a_2}\right)^6 \frac{4}{3}\pi \int_{a_0}^{a_3} a^2 \frac{H_3F(a)}{H_0} \Big\{ f_\ell(a)^2 - f_{\ell\ell}(a) - f_\theta(a)f_\ell(a) \\ + \frac{5}{6}f_\theta(a)^2 - \sqrt{f_\theta(a)}f_{\theta\theta}(a) + \frac{1}{2}f_{LL}(a) + \frac{1}{2}f_L(a)f_\theta(a) - \frac{1}{2}f_L(a)f_\ell(a) - \frac{1}{8}f_L^2(a) \\ - \frac{1}{2}\frac{H_3^2}{H_0^2}\left[f_\omega(a)f_\theta(a) - f_\omega(a)f_\ell(a) - \frac{1}{2}f_L(a)f_\omega(a) + f_{\omega\omega}(a) - \frac{3}{4}f_\omega(a)^2\frac{H_3^2}{H_0^2}\right]\Big\} \mathrm{d}\, a \ . \end{aligned} \tag{106.107}$$

We can write (106.106) as

$$\begin{aligned} C_I = \left(\frac{a_3}{a_2}\right)^3 \frac{2\pi}{9} \Big\{ \int_{a_i}^{a_0} a^2 \frac{H_3F(a)}{H_0} \left[f_\theta(a) - f_\ell(a) + \frac{1}{2}f_L(a) - \frac{1}{2}\frac{H_3^2}{H_0^2}f_\omega(a)\right] \mathrm{d}\, a \\ + \int_{a_0}^{a_1} a^2 \frac{H_3F(a)}{H_0} \left[f_\theta(a) - f_\ell(a) + \frac{1}{2}f_L(a) - \frac{1}{2}\frac{H_3^2}{H_0^2}f_\omega(a)\right] \mathrm{d}\, a \\ + \int_{a_1}^{a_2} a^2 \frac{H_3F(a)}{H_0} \left[f_\theta(a) - f_\ell(a) + \frac{1}{2}f_L(a) - \frac{1}{2}\frac{H_3^2}{H_0^2}f_\omega(a)\right] \mathrm{d}\, a \\ + \int_{a_2}^{a_3} a^2 \frac{H_3F(a)}{H_0} \left[f_\theta(a) - f_\ell(a) + \frac{1}{2}f_L(a) - \frac{1}{2}\frac{H_3^2}{H_0^2}f_\omega(a)\right] \mathrm{d}\, a \Big\} \ . \end{aligned} \tag{106.108}$$

We can use (106.85) for H_0, (106.29) for $F(a)$, (106.52) for $f_\theta(a)$, (106.64) for $f_\ell(a)$, (106.32) for $f_L(a)$, and (106.78) for $f_\omega(a)$, in (106.108). Keeping only the dominant term in the Hubble factor in each era, using the appropriate dependence on a of ρ_0 in each era, performing the integrations, neglecting a_0 when appropriate, using (106.35) and (106.36) to allow us to neglect a_1 in some places, gives

$$\begin{aligned}C_I = {}&\tfrac{w}{12}\left(\tfrac{a_1}{a_2}\right)^{3/2} a_3^3\left[\left(\tfrac{103}{18}+\tfrac{4}{3}N-\tfrac{2}{3}\alpha_4\right)\beta^2+3\beta+41-\tfrac{2}{3}\alpha_4\right]\alpha_1\\
&+a_3^3\left[\tfrac{1}{12}\left(\tfrac{a_1}{a_2}\right)^{3/2}\left(\left(\tfrac{103}{18}+\tfrac{4}{3}N-\tfrac{2}{3}\alpha_4\right)\beta^2+3\beta\right)+\left(1-\left(\tfrac{a_1}{a_2}\right)^{1/2}\right)\beta\right.\\
&\qquad\left.+6\left(\tfrac{a_1}{a_2}\right)^{1/2}-\tfrac{1}{15}\tfrac{a_2}{a_1}\alpha_4\right]\alpha_2\\
&+2\pi a_3^3\left(\tfrac{a_3}{a_2}-1\right)\left[\left(2\left(\tfrac{a_3^2}{a_2^2}+\tfrac{a_3}{a_2}+1\right)-\left(\tfrac{a_1}{a_2}\right)^{1/2}\left(\tfrac{a_3}{a_2}\right)\left(\tfrac{a_3}{a_2}+1\right)\right)\beta\right.\\
&\qquad\left.+\tfrac{9}{2}\left(\tfrac{a_3}{a_2}+1\right)-\tfrac{2}{3}\tfrac{a_2}{a_1}\alpha_4\right]\alpha_3\ ,\end{aligned} \tag{106.109}$$

where β is given by (106.51), and gives the effect of inflation on the action.

Putting in values for w, a_1, a_2, and $a_3 = 1$ from appendix 106.10 gives

$$\begin{aligned}C_I \approx{}& \left(4\times10^{-6}-6\times10^{-8}\alpha_4\right)\alpha_1+\left(\beta+9\times10^{-2}-298\alpha_4\right)\alpha_2\\
&+2\pi\left(2.5\beta+3.3-941\alpha_4\right)\alpha_3\ .\end{aligned} \tag{106.110}$$

We can write (106.107) as

$$\begin{aligned}C_{II} = {}&\frac{1}{36}\left(\frac{a_3}{a_2}\right)^6\frac{4}{3}\pi\int_{a_0}^{a_1}a^2\frac{H_3F(a)}{H_0}\Big\{f_\ell(a)^2-f_{\ell\ell}(a)-f_\theta(a)f_\ell(a)\\
&+\frac{5}{6}f_\theta(a)^2-\sqrt{f_\theta(a)}f_{\theta\theta}(a)+\frac{1}{2}f_{LL}(a)+\frac{1}{2}f_L(a)f_\theta(a)-\frac{1}{2}f_L(a)f_\ell(a)-\frac{1}{8}f_L^2(a)\\
&-\frac{1}{2}\frac{H_3^2}{H_0^2}\left[f_\omega(a)f_\theta(a)-f_\omega(a)f_\ell(a)-\frac{1}{2}f_L(a)f_\omega(a)+f_{\omega\omega}(a)-\frac{3}{4}f_\omega(a)^2\frac{H_3^2}{H_0^2}\right]\Big\}\,\mathrm{d}\,a\\
&+\frac{1}{36}\left(\frac{a_3}{a_2}\right)^6\frac{4}{3}\pi\int_{a_1}^{a_2}a^2\frac{H_3F(a)}{H_0}\Big\{f_\ell(a)^2-f_{\ell\ell}(a)-f_\theta(a)f_\ell(a)\\
&+\frac{5}{6}f_\theta(a)^2-\sqrt{f_\theta(a)}f_{\theta\theta}(a)+\frac{1}{2}f_{LL}(a)+\frac{1}{2}f_L(a)f_\theta(a)-\frac{1}{2}f_L(a)f_\ell(a)-\frac{1}{8}f_L^2(a)\\
&-\frac{1}{2}\frac{H_3^2}{H_0^2}\left[f_\omega(a)f_\theta(a)-f_\omega(a)f_\ell(a)-\frac{1}{2}f_L(a)f_\omega(a)+f_{\omega\omega}(a)-\frac{3}{4}f_\omega(a)^2\frac{H_3^2}{H_0^2}\right]\Big\}\,\mathrm{d}\,a\\
&+\frac{1}{36}\left(\frac{a_3}{a_2}\right)^6\frac{4}{3}\pi\int_{a_2}^{a_3}a^2\frac{H_3F(a)}{H_0}\Big\{f_\ell(a)^2-f_{\ell\ell}(a)-f_\theta(a)f_\ell(a)\\
&+\frac{5}{6}f_\theta(a)^2-\sqrt{f_\theta(a)}f_{\theta\theta}(a)+\frac{1}{2}f_{LL}(a)+\frac{1}{2}f_L(a)f_\theta(a)-\frac{1}{2}f_L(a)f_\ell(a)-\frac{1}{8}f_L^2(a)\\
&-\frac{1}{2}\frac{H_3^2}{H_0^2}\left[f_\omega(a)f_\theta(a)-f_\omega(a)f_\ell(a)-\frac{1}{2}f_L(a)f_\omega(a)+f_{\omega\omega}(a)-\frac{3}{4}f_\omega(a)^2\frac{H_3^2}{H_0^2}\right]\Big\}\,\mathrm{d}\,a\ .\end{aligned} \tag{106.111}$$

We can use (106.85) for H_0, keep only the dominant term in the Hubble factor in each era, use (106.77) for H_3, use the appropriate dependence on a of ρ_0 in each era, and use (106.29) for $F(a)$ to give

$$C_{II} = \frac{\alpha_1 w+\alpha_2}{72}a_1^{1/2}a_2^{-9/2}a_3^6\int_{a_0}^{a_1}\Big\{f_\ell(a)^2-f_{\ell\ell}(a)-f_\theta(a)f_\ell(a)$$

$$+\frac{5}{6}f_\theta(a)^2 - \sqrt{f_\theta(a)}f_{\theta\theta}(a) + \frac{1}{2}f_{LL}(a) + \frac{1}{2}f_L(a)f_\theta(a) - \frac{1}{2}f_L(a)f_\ell(a) - \frac{1}{8}f_L^2(a)$$
$$-\frac{1}{2}\frac{H_3^2}{H_0^2}\left[f_\omega(a)f_\theta(a) - f_\omega(a)f_\ell(a) - \frac{1}{2}f_L(a)f_\omega(a) + f_{\omega\omega}(a) - \frac{3}{4}f_\omega(a)^2\frac{H_3^2}{H_0^2}\right]\Bigg\}\,\mathrm{d}\,a$$
$$+\frac{\alpha_2}{72}a_2^{-9/2}a_3^6\int_{a_1}^{a_2} a^{1/2}\Big\{f_\ell(a)^2 - f_{\ell\ell}(a) - f_\theta(a)f_\ell(a)$$
$$+\frac{5}{6}f_\theta(a)^2 - \sqrt{f_\theta(a)}f_{\theta\theta}(a) + \frac{1}{2}f_{LL}(a) + \frac{1}{2}f_L(a)f_\theta(a) - \frac{1}{2}f_L(a)f_\ell(a) - \frac{1}{8}f_L^2(a)$$
$$-\frac{1}{2}\frac{H_3^2}{H_0^2}\left[f_\omega(a)f_\theta(a) - f_\omega(a)f_\ell(a) - \frac{1}{2}f_L(a)f_\omega(a) + f_{\omega\omega}(a) - \frac{3}{4}f_\omega(a)^2\frac{H_3^2}{H_0^2}\right]\Bigg\}\,\mathrm{d}\,a$$
$$+\frac{\pi\alpha_3}{9}\left(\frac{a_3}{a_2}\right)^6\int_{a_2}^{a_3} a^2\Big\{f_\ell(a)^2 - f_{\ell\ell}(a) - f_\theta(a)f_\ell(a)$$
$$+\frac{5}{6}f_\theta(a)^2 - \sqrt{f_\theta(a)}f_{\theta\theta}(a) + \frac{1}{2}f_{LL}(a) + \frac{1}{2}f_L(a)f_\theta(a) - \frac{1}{2}f_L(a)f_\ell(a) - \frac{1}{8}f_L^2(a)$$
$$-\frac{1}{2}\frac{H_3^2}{H_0^2}\left[f_\omega(a)f_\theta(a) - f_\omega(a)f_\ell(a) - \frac{1}{2}f_L(a)f_\omega(a) + f_{\omega\omega}(a) - \frac{3}{4}f_\omega(a)^2\frac{H_3^2}{H_0^2}\right]\Bigg\}\,\mathrm{d}\,a\ . \qquad (106.112)$$

We can use (106.52) for $f_\theta(a)$, (106.64) for $f_\ell(a)$, (106.32) for $f_L(a)$, (106.78) for $f_\omega(a)$, (106.68) for $f_{\ell\ell}(a)$, (106.49) for $f_{\theta\theta}(a)$, (106.31) for $f_{LL}(a)$, (106.81) for $f_{\omega\omega}(a)$, neglect a_0, and use the fact that $a_1/a_2 \approx 10^{-4}$ to approximate (106.112) as

$$C_{II} \approx \frac{1}{72}\left(\frac{a_1}{a_2}\right)^{3/2}\left(\frac{a_3}{a_2}\right)^3 a_3^3\left[\frac{52007}{9000} - \frac{5}{12}f_\ell(a_2) + \frac{40}{9}f_\ell(a_3) + \frac{1}{2}f_{\ell\ell}(a_1) + \frac{3}{2}f_{\ell\ell}(a_2)\right.$$
$$\left.+\frac{3}{8}f_\ell^2(a_2) + \frac{3}{2}f_\ell(a_1)f_\ell(a_2) + \left(-\frac{211}{75} + f_\ell(a_2) - \frac{8}{3}f_\ell(a_3)\right)\alpha_4 + \frac{6}{5}\alpha_4^2\right](\alpha_1 w + \alpha_2)$$
$$+\frac{1}{72}\left(\frac{a_2}{a_1}\right)^{1/2}\left(\frac{a_3}{a_2}\right)^3 a_3^3\left[-144 - \frac{208}{5}\sqrt{\frac{a_2}{a_1}}\alpha_4 + \frac{12}{7}\frac{a_2}{a_1}\alpha_4^2\right]\alpha_2$$
$$+\frac{2\pi}{3}\left(\frac{a_3}{a_2}\right)^3 a_3^3\left(1 - \frac{a_2}{a_3}\right)\left[\frac{3}{2}\sqrt{\frac{a_2}{a_1}} - \frac{1}{3}\frac{a_3}{a_1}\alpha_4 + \left(\frac{a_2}{a_1}\right)^2\alpha_4^2\right]\alpha_3\ . \qquad (106.113)$$

Or,

$$C_{II} \approx \frac{1}{72}\left(\frac{a_1}{a_2}\right)^{3/2}\left(\frac{a_3}{a_2}\right)^3 a_3^3\left[1066 - 72\sqrt{\frac{a_2}{a_1}}\alpha_4 + \frac{6}{5}\alpha_4^2\right](\alpha_1 w + \alpha_2)$$
$$+\frac{1}{72}\left(\frac{a_2}{a_1}\right)^{1/2}\left(\frac{a_3}{a_2}\right)^3 a_3^3\left[-144 - \frac{208}{5}\sqrt{\frac{a_2}{a_1}}\alpha_4 + \frac{12}{7}\frac{a_2}{a_1}\alpha_4^2\right]\alpha_2$$
$$+\frac{2\pi}{3}\left(\frac{a_3}{a_2}\right)^3 a_3^3\left(1 - \frac{a_2}{a_3}\right)\left[\frac{3}{2}\sqrt{\frac{a_2}{a_1}} - \frac{1}{3}\frac{a_3}{a_1}\alpha_4 + \left(\frac{a_2}{a_1}\right)^2\alpha_4^2\right]\alpha_3\ . \qquad (106.114)$$

Putting in values for w, a_1, a_2, and $a_3 = 1$ from appendix 106.10 gives

$$C_{II} \approx \left(4\times 10^{-5} - 2\times 10^{-4}\alpha_4 + 4\times 10^{-8}\alpha_4^2\right)\alpha_1$$
$$+\left(-3\times 10^2 - 6\times 10^3\alpha_4 + 2\times 10^4\alpha_4^2\right)\alpha_2$$
$$+8\pi\left(5 - 10^2\alpha_4 + 10^6\alpha_4^2\right)\alpha_3\ . \qquad (106.115)$$

Chapter 107

The zero-point energy problem[1]

abstract

Although the existence of zero-point energy for electromagnetic radiation is well established in regions that contain a significant amount of matter and at wavelengths that can interact with such matter, there is no evidence for the existence of zero-point energy in regions where the matter is too sparse to interact with radiation or at wavelengths that cannot interact with such matter. We note that all of the evidence for zero-point energy, including Lamb shift, Casimir effect, and uncertainty relations, require matter for verification. In addition, the Casimir effect gives no evidence for zero-point energy at wavelengths smaller than the separation between the plates. Postulating the absence of zero-point energy in regions of sparse matter or at wavelengths that cannot interact with matter leads to a total zero-point energy less than the total rest-mass energy of ordinary matter, which solves the zero-point energy problem. Although the existence of zero-point energy greater than the energy associated with the cosmological constant would be a contradiction, zero-point energy less than that associated with the cosmological constant is no contradiction because zero-point energy is not necessarily associated with the cosmological constant.

107.1 Introduction

One of the elementary exercises in quantum mechanics is to calculate the wave functions and energy levels for a quantum harmonic oscillator. The wave functions are found in terms of Hermite polonomials, and the energy levels are given by [182, e.g. Schiff, Section 13, Chapter IV]

$$(n + 1/2)\hbar\omega \ , \tag{107.1}$$

where ω is the frequency of the oscillator, n is an integer, and $\hbar$ is Plank's constant. The smallest that the energy of the quantum harmonic oscillator can be is the ground-state energy of $\hbar\omega/2$.

It has been postulated that since electromagnetic radiation oscillates and because electromagnetic radiation seems to be quantized in steps of $\hbar\omega$ like the quantum harmonic oscillator, maybe electromagnetic radiation also has energy levels with the same form as (107.1), and therefore also a ground-state energy of $\hbar\omega/2$. This ground-state energy has been given the name "zero-point energy," and it has been enormously successful in explaining such diverse effects as the Lamb shift, spontaneous emission in atoms, and the Casimir effect.

There has been one drawback, however. If zero-point energy existed everywhere for all frequencies, then it would be infinite. To avoid that problem, it is usually assumed that there is a short-wavelength cutoff.

[1] 30 April 2021

Choosing the value of the cutoff is crucial. The most natural choice is the Plank length[226, e.g., Klauber] [225, e.g., Weinberg], which would give about 2×10^{71} GeV^4[225, Weinberg] $\approx 5 \times 10^{88}$ g cm^{-3} for the energy density of zero-point energy. That is about 119 orders of magnitude larger than the observed density of the vacuum of 8.6×10^{-31} g cm^{-3}, including baryonic matter, dark matter, and dark energy.[2]

"A peculiar and truly quantum mechanical feature of the quantum fields is that they exhibit zero-point fluctuations everywhere in space, even in regions which are otherwise 'empty' (i.e. devoid of matter and radiation)."[227, Rugh-Zinkernagel]

The main effect of such a large background mass density (if it actually had a gravitational effect) would be in terms of a cosmological model. According to Pauli, the zero-point energy evidently produces no gravitational field[229, p. 250].

The real problem is the large gravitational field that is not observed. This problem, sometimes called the "cosmological constant problem," [225, Weinberg] is one of the main outstanding problems in physical theory [231, Davies].

107.2 Possible Solutions

There have been some suggestions for a solution to the "zero-point energy problem."

1. Maybe zero-point energy does not gravitate. That is, maybe it does not produce a gravitational field, as does all other known forms of energy. [229, Pauli, p. 250]

2. Maybe there is some way to cancel the zero-point energy [226, Klauber].

3. Maybe choosing a different cutoff will solve the problem. Taking the cutoff to correspond to a larger wavelength, say the Compton wavelength of the proton ($k_{\text{max}} = m_{proton}c/\hbar$), would give a mass density of zero-point energy of $\approx 2 \times 10^{15}$ g cm^{-3}[21, Feynman-Hibbs, p. 246], which decreases the zero-point energy, but not enough. Although there is no theoretical justification for choosing the Compton wavelength of the proton as the cutoff, there may be justification for choosing some other cutoff.

4. Maybe using normal ordering would help. [228, Enz-Thellung, p. 842]

5. Maybe Fermion zero-point energy is negative and cancels Boson zero-point energy by super symmetry. [384, Shinji Iida and Hiroshi Kuratsuji, 1987]

So far, there has been no agreement on the solution.

Although some popular authors suggest ideas like using zero-point energy as a possible source of antigravity or similar things[232, Cook], serious scientists do not believe that zero-point energy is really as large as the usual estimates indicate. Instead, they consider it to be a problem that needs to be resolved. Weinberg[225] and Peebles and Ratra[224] summarizes the situation with zero-point energy and the cosmological constant, mentioning several possible solutions to the difficulty.

[2]equal to the critical density $3H_0^2/(8\pi G)$, where $H_0 = 67.74$ km s^{-1} Mpc^{-1} [382, Planck:Collaboration:2015] is the Hubble parameter and G is the Newtonian gravitational constant

Chapter 108

Are the wave functions associated with the four fundamental interactions of a different kind from each other?[1]

abstract

If the wave functions associated with the four fundamental interactions are of four different kinds, then they must be treated separately rather than added together. One possible way to to that while still keeping a unified approach, is to represent the system by complex quaternions (biquaternions). Using this procedure with path integrals requires a generalized analytic continuation, in which each "path" takes on complex quaternion values. It is necessary that the integral of the Lagrangian to give the action be independent of the contour, and that requires a certain amount of analyticity, which is more difficult to achieve with quaternions than with complex numbers.

108.1 Introduction

In the path integral formulation of quantum theory, we encounter formulas like

$$\int e^{iS(\mathrm{path})/\hbar}\,\mathcal{D}(\mathrm{path})\ , \tag{108.1}$$

where $\mathcal{D}(\mathrm{path})$ is the measure, which depends on the path, and $S(\mathrm{path})$ is the action, which also depends on the path. We write the action integral as

$$S = \int (-g)^{1/2} L\,\mathrm{d}^4 x\ , \tag{108.2}$$

It is implicitly assumed that all wave functions are of the same type, because the usual Lagrangian combines all terms from all four basic interactions into a single quantity to calculate the action. That is, the Lagrangian L is usually given by

$$L = L_{\mathrm{grav}} + L_{\mathrm{EM}} + L_{\mathrm{weak}} + L_{\mathrm{strong}}\ , \tag{108.3}$$

[1] 1 May 2012 - 25 Sep 2017, 14:37

where L_{grav}, L_{EM}, L_{weak}, and L_{strong}, are the parts of the Lagrangian that represent the four fundamental interactions (which may be complex). Explicitly,

$$L_{\mathrm{grav}} = \frac{R - 2\Lambda}{16\pi} \ , \tag{108.4}$$

$$L_{\mathrm{EM}} = \ , \tag{108.5}$$

$$L_{\mathrm{weak}} = \ , \tag{108.6}$$

and

$$L_{\mathrm{strong}} = \ . \tag{108.7}$$

Equation (108.3) implicitly assumes that the waves (or wave functions) for the four types of interactions are of the same type and can be added together.

Further, when we look for WKB approximations, we look for cases where the action is stationary for variation of the path:

$$\delta S = 0 \ . \tag{108.8}$$

Equation (108.8) implicitly assumes that the different kinds of waves associated with the four fundamental interactions are of the same type and can interfere with each other.

However, if the wave functions for the various interactions are really of a different type, then something different needs to be done. Instead of combining the phases for the four basic interactions, the four phases should be somehow treated separately. Instead of (108.2), we would have

$$S = S_{\mathrm{grav}} + S_{\mathrm{EM}} + S_{\mathrm{weak}} + S_{\mathrm{strong}} \ , \tag{108.9}$$

and to calculate a WKB approximation, instead of (108.8), we would have

$$\delta S_{\mathrm{grav}} = 0 \ , \tag{108.10}$$

$$\delta S_{\mathrm{EM}} = 0 \ , \tag{108.11}$$

$$\delta S_{\mathrm{weak}} = 0 \ , \tag{108.12}$$

and

$$\delta S_{\mathrm{strong}} = 0 \ . \tag{108.13}$$

Equations (108.10) through (108.13) now give us four conditions on the path instead of the one condition in (108.2). However, because the various fields and particles are interacting with one another, there is still only one path for each contribution to the path integral. That is, equations (108.10) through (108.13) now give us four conditions on one path instead of one condition on one path.

The problem, then, is to somehow generalize the path integral so that there is a single path to represent all four fundamental interactions, but we combine the contributions from each of the four fundamental interactions so that we are not trying to add together phases that do not represent actual phase interference.

Notice that in any situation where only one of the four fundamental interactions is significant, (108.2) with (108.3) would be perfectly satisfactory. Notice also that because gravitation is represented by geometry (not just curvature), the mere calculation of the action in (108.2) mixes gravitation with other three interactions.

108.2 A quaternion path integral

To try to resolve this problem, we realize that the four actions or Lagrangians should be combined in such a way that the distinction is kept rather than just adding the four phases together. Since there are four interactions and the phase (or action) for each interaction can be complex to represent phase and amplitude, then there are eight quantities that might be represented by an octonion, or perhaps by a biquaternion (which is a quaternion with complex coefficients.

Although The usual Lagrangian includes all four of the fundamental interactions, there are many applications where only one of the fundamental interactions is significant. We expect that in such cases, the methods I propose here would be equivalent to the usual methods. Situations where more than one of the fundamental interactions is significant would give a possibility of testing which method agrees better with experiment.

Let us represent the Lagrangian by a quaternion with complex coefficients. In so doing, we consider that the complex coefficient of each of the four quaternion basis vectors represents the phase and amplitude of each of the four basic interactions (gravitation, electromagnetic, weak nuclear, and strong nuclear). If the wave functions for the four interactions really are of a different type, then some kind of separation such as this is necessary.

Let us now consider the contribution of each path to the path integral to give a value of the wave function at a given spacetime point due to something happening at some other given spacetime point. The action for that path is the integral of the Lagrangian along the path connecting the two given spacetime points.

If this were a standard path-integral calculation, then this would give a single complex number for each path. We would then perform the path integral to get the complete wave function at that point. If we wanted to make a WKB approximation to the wave function, we would try to find out which paths contribute most to the path integral. It is well known now, that in the absence of significant amplitude dependence on the path, the paths that contribute most are those for which the real part of the action is an extremum, because such paths have several paths surrounding the extremal paths for which there is constructive phase interference.

If, however, attenuation along the various paths is such that attenuation is significantly dependent on path, then not only must we consider phase interference in determining which paths contribute significantly to the path integral, but we must also consider amplitude as well. This means that we now have two conditions to determine the optimum path, but we have no more freedom in adjusting the path unless we can enlarge the parameter space. The solution for that problem was to analytically continue the calculations so that each of the coordinates that determine the path be allowed to take on complex values. Only the endpoints were required to have real values. This method gave correct solutions because analyticity guaranteed that the various contour integrals used to calculate the action integral were independent of the contour (as long as the topology of the contour with respect to non-analytic points remained unchanged). The method led to ray tracing in complex space in which Hamilton's equations were applied to ray paths having complex coordinates. [385]

Instead, let us consider the action integral for a Lagrangian given in terms of complex quaternions. This action integral has eight values, a phase and amplitude for each of the four fundamental interactions. And because the weak and strong forces are short range, the dependence of amplitude on path will be significant. In the beginning, we start with a path having real coordinates, and calculate the action. To make WKB calculations, we want to find out which paths contribute the most to the path integral. We now have eight extremum conditions to determine the path, but that is seven conditions too many if we confine the path to have real coordinates. However, if we do a generalized analytic continuation by allowing each coordinate of each point of the path to take on complex quaternion values, then we will have just the exact number of parameters to determine the paths that contribute most to the path integral. However, for this method to work, the generalized analytic continuation must be such that the generalized contour integration gives results that are independent of the contour. That, in turn, requires a generalized analyticity for quaternion functions that may be difficult because quaternions are not commutative. That is, a product of two quaternions depends on the order of the factors in the product. This enters into the action integral because when we move the path into complex quaternion space, both the Lagrangian and the integrator are quaternions. The generalized analyticity requirements may give us a way to unite gravitation and quantum theory and the other three interactions.

108.3 Generalized analytic continuation

Let us write a Lagrangian in terms of a complex quaternion (also called a biquaternion).

$$L = L_0 e_0 + L_1 e_1 + L_2 e_2 + L_3 e_3 \ , \tag{108.14}$$

where L_0, L_1, L_2, and L_3, are the parts of the Lagrangian that represent the four fundamental interactions (which may be complex), e_0 is the quaternion identity, and e_1, e_2, and e_3, are the non-commuting quaternion elements.

For purposes of checking analyticity, it is sufficient to consider only one dimension. And, it is sufficient to consider only an infinitesimal section of the path. That is, we consider

$$L \, \mathrm{d}x = (L_0 e_0 + L_1 e_1 + L_2 e_2 + L_3 e_3) \, \mathrm{d}x \ , \tag{108.15}$$

When we do a generalized analytic continuation into the complex quaternion x direction, we get

$$L \, \mathrm{d}x = (L_0 e_0 + L_1 e_1 + L_2 e_2 + L_3 e_3) \, (\mathrm{d}x_0 e_0 + \mathrm{d}x_1 e_1 + \mathrm{d}x_2 e_2 + \mathrm{d}x_3 e_3) \ , \tag{108.16}$$

where $\mathrm{d}x_0$, $\mathrm{d}x_1$, $\mathrm{d}x_2$, and $\mathrm{d}x_3$, are infinitesimal complex displacements. In the process of generalized analytic continuation, L_0, L_1, L_2, and L_3 must also change.

Notice that there may already be a problem with trying to separate the contributions from each of the four fundamental interactions to the action. We know from General Relativity that the gravitational interaction is represented as spacetime geometry (not just curvature). But the calculation of the action involves an integration in spacetime. Because of that, it seems that we cannot really separate out the contribution of gravitation to the action. Maybe that is the way it has to be. Maybe that would be a good reason to represent the gravitational part of the action by the identity element in the complex quaternion.

Let us begin with a really simple case.

108.4 Two homogeneous blocks

The problem we consider is that of two homogeneous blocks separated by a plane surface. We have a source in medium 1, and an observer in medium 2. We can reduce the problem to two dimensions by considering the plane that contains the source and the observer, and is normal to the surface separating the two blocks. This plane is unique except for the uninteresting case that is easy to solve. We choose a coordinate system with z normal to the surface separating the two blocks, and choose z to be zero on the surface of discontinuity. We choose x to be along the surface of discontinuity and in our chosen two-dimensional plane.

In considering all paths that connect the source with the observer, we restrict ourselves to those paths that have a straight line from the source to the surface that separates the two homogeneous blocks, then another straight line from there to the observer. We can further restrict paths to those that remain in the chosen 2-dimensional plane.

The path is now determined by choosing a value for x. Thus, our path integral is an integral over x. This differs from the usual procedure in this case in that the Lagrangian takes on complex quaternion values and x takes on complex quaternion values. To find that path that contributes most to a WKB solution, we find an extremum for the action. That is, we look for a value of x such that

$$\delta(L_1\ell_1 + L_2\ell_2) = 0 \ , \tag{108.17}$$

where L_1 and L_2 are the values of the Lagrangian in blocks 1 and 2, respectively, and ℓ_1 and ℓ_2 are the lengths of the straight lines in blocks 1 and 2 respectively. When we vary x to satisfy (108.17), we get Snell's law

$$L_1 \sin\phi_1 = L_2 \sin\phi_2 \ , \tag{108.18}$$

except now, L_1, L_2, ϕ_1, and ϕ_2 take on complex quaternion (biquaternion) values. That is,

$$L_1 = L_{1\,0}e_0 + L_{1\,1}e_1 + L_{1\,2}e_2 + L_{1\,3}e_3 \ , \tag{108.19}$$

$$L_2 = L_{2\,0}e_0 + L_{2\,1}e_1 + L_{2\,2}e_2 + L_{2\,3}e_3 \ , \tag{108.20}$$

$$\sin\phi_1 = \sin\phi_{1\,0}e_0 + \sin\phi_{1\,1}e_1 + \sin\phi_{1\,2}e_2 + \sin\phi_{1\,3}e_3 \ , \tag{108.21}$$

and

$$\sin\phi_2 = \sin\phi_{2\,0}e_0 + \sin\phi_{2\,1}e_1 + \sin\phi_{2\,2}e_2 + \sin\phi_{2\,3}e_3 \ . \tag{108.22}$$

When we substitute (108.19) through (108.22) into (108.18), we get a complex quaternion equation. If we use the standard multiplication table for quaternions, we get four ordinary complex equations, that can be written as a matrix equation as

$$\hat{L}_1 \sin\hat{\phi}_1 = \hat{L}_2 \sin\hat{\phi}_2 \ , \tag{108.23}$$

where

$$\hat{L}_1 = \begin{pmatrix} L_{1\,0} & -L_{1\,1} & -L_{1\,2} & -L_{1\,3} \\ L_{1\,1} & L_{1\,0} & -L_{1\,3} & L_{1\,2} \\ L_{1\,2} & L_{1\,3} & L_{1\,0} & -L_{1\,1} \\ L_{1\,3} & -L_{1\,2} & L_{1\,1} & L_{1\,0} \end{pmatrix} \ , \tag{108.24}$$

$$\hat{L}_2 = \begin{pmatrix} L_{2\,0} & -L_{2\,1} & -L_{2\,2} & -L_{2\,3} \\ L_{2\,1} & L_{2\,0} & -L_{2\,3} & L_{2\,2} \\ L_{2\,2} & L_{2\,3} & L_{2\,0} & -L_{2\,1} \\ L_{2\,3} & -L_{2\,2} & L_{2\,1} & L_{2\,0} \end{pmatrix} \ , \tag{108.25}$$

$$\sin\hat{\phi}_1 = \begin{pmatrix} \sin\phi_{1\,0} \\ \sin\phi_{1\,1} \\ \sin\phi_{1\,2} \\ \sin\phi_{1\,3} \end{pmatrix} \ , \tag{108.26}$$

and

$$\sin\hat{\phi}_2 = \begin{pmatrix} \sin\phi_{2\,0} \\ \sin\phi_{2\,1} \\ \sin\phi_{2\,2} \\ \sin\phi_{2\,3} \end{pmatrix} \ . \tag{108.27}$$

For the special case that only one of the four fundamental forces contributes significantly to the Lagrangian, then (108.23) reduces to ordinary complex Snell's law.

108.5 Generalized Hamilton's equations

In the same way that we generalized the Lagrangian by a generalized analytic continuation to take on complex quaternion values and depend on coordinates that take on complex quaternion values, we do the same with the Hamiltonian and Hamilton's equations. The first difficulty is to consider derivatives.

In Hamiltonian ray tracing, the ray paths are determined by Hamilton's equations.

$$\frac{dx_i}{d\tau} = \frac{\partial H(t, x_i, \sigma, k_i)}{\partial k_i} , \tag{108.28}$$

$$\frac{dk_i}{d\tau} = -\frac{\partial H(t, x_i, \sigma, k_i)}{\partial x_i} , \tag{108.29}$$

$$\frac{dt}{d\tau} = -\frac{\partial H(t, x_i, \sigma, k_i)}{\partial \sigma} , \tag{108.30}$$

and

$$\frac{d\sigma}{d\tau} = \frac{\partial H(t, x_i, \sigma, k_i)}{\partial t} , \tag{108.31}$$

where i varies from 1 to 3, τ is an independent variable whose significance depends on the choice of Hamiltonian, $H(t, x_i, \sigma, k_i)$, and these equations are integrated numerically along the ray path. Equation (108.28) gives the progression of the ray, (108.29) gives the refraction of the wave normal, (108.30) gives the travel time of the time-maximum of a wave packet [23, 24], and (108.31) gives the frequency shift of the wave if the medium is changing with time.

[20, p. 488] refer to the Hamiltonian in (108.28) through (108.31) as a super-Hamiltonian. The difference is that a normal Hamiltonian is three-dimensional, represents energy, and varies along the path, whereas a super-Hamiltonian is four-dimensional and is a constant (equal to zero) along the path.

Because it is necessary to choose for the Hamiltonian something that should be constant along the ray path, it is usual to choose some form of the dispersion relation for a Hamiltonian.

108.6 Kaluza-Klein theory[2]

Another possibility is to use some of the results of Kaluza-Klein theory. In that theory, we would write Einstein's field equations in 5, 6, or 7 dimensions. The 5th dimension would add Maxwell's equations, and the 6th and 7th dimensions would add the Yang-Mills equations for the strong and weak interactions.[3]

I want to put all four interactions on the same footing. If gravitation can be expressed as geometry, then it must be possibility to express the other three interactions as geometry. Similarly, it should be possible to (in principle) have any of the four interactions form the background geometry, depending on the contents of the universe.

My idea now is to have each of the four interactions be expressed in a form similar to Einstein's field equations. The results of Kaluza-Klein theory suggest that something like this might be possible. Therefore, we would have the Lagrangian for each of the four interactions have the same form as that for General Relativity, namely $(R - 2\Lambda)/(16\pi)$, or something like that. But, each would be multiplied by a different bi-quaternion, as discussed above.

Probably the matter Lagrangian term for each of the other three interactions would be different from that for General Relativity. It would be difficult to figure out how to do that. That problem was discussed in Chapter 73.

[2] Added 21 September 2017

[3] I need to check on this.

Chapter 109

Representing Geometry as Gravity

abstract

General Relativity represents gravity as geometry. Another possibility is to represent geometry as gravity.

109.1 Introduction

General Relativity represents gravity as geometry. Another possibility is to represent geometry as gravity.

109.2 Gravitation

109.3 Geometry

Chapter 110

Gravity Without Geometry - III

abstract

It is possible to write field equations for gravitation in a way that is analogous to Maxwell's equations for electromagnetism. When the usual relation between the gravitational field and the metric tensor is kept, then the gravitational field equations are equivalent to Einstein's field equation for General Relativity. When that relation is not imposed, the gravitational field equations differ somewhat from the field equations for General Relativity. Even then, the equations give the same results as General Relativity for four of the Solar System tests of gravitation and for gravitaional waves.

110.1 Introduction

When gravitation is represented as geometry, it becomes the most fundamental of the interactions because everything else (including quantum theory) uses geometry as a basis on which it operates. That situation makes it difficult to quantize gravity (or to reconcile gravitation with quantum theory).

We experience the force of gravitation in the same way that we experience any other force. (Actually, the only other force that we experience personally is the electromagnetic force, since the strong and weak nuclear forces are short range and directly affect only sub-atomic particles.)

Because of that, it seems strange to represent gravitation as geometry. It would seem more natural to represent gravitation as just another force, as Newton did. The reason for the difference is that Einstein realized that there is no way to distinguish between an acceleration and a gravitational force. He referred to this as the equivalence principle, of which we now recognize several versions.

That is, Einstein recognized that inertial force is really a gravitational force. This led Einstein to formulate his General Theory of Relativity, in which gravitation was represented as geometry. However, that was not the only possibility. Gravitation can be represented as another force, just as electromagnetism is.

Three of the four fundamental interactions have been unified. Part of the difficulty with unifying gravitation with the others depends on two things:

1. The gravitational interaction is treated as geometry.

2. The gravitational interaction has not yet been quantized.

Let us consider the effect of the former. As long as the gravitational interaction is treated as a geometric arena upon which all other interactions are staged, it will be difficult to unify it with the other interactions on an equal basis. If the gravitational interaction is intrinsically geometrical in nature, then it will remain difficult to unify gravitation with the rest of the fundamental interactions.

If, however, the apparent geometrical character is an accident of the fact that there is so much matter in the universe, that the local geometry is induced by an interaction with rest of matter in the universe, then expressing the gravitational interaction in non-geometrical terms may simplify the unification process. (It may also simplify the quantization of the gravitational interaction.)

Ernst Mach[120, 102] argued in the last century, using different terminology, that the latter was the case. That is, that inertia was not an intrinsic property of matter, but due to an interaction with the rest of matter in the universe. Although inertia and geometry are not the same, similar arguments that Mach used for inertia can also be applied to geometry. Mach argued that the law of inertia was wrongly expressed. We can argue similarly that expressing gravitation in terms of geometry leads us in a wrong directions when we try to generalize laws of physics.

If the local geometrical character of space-time is due to gravitational interaction with matter in the universe, then if there were a net charge on matter in the universe the local geometrical character of space-time would be due to an electromagnetic interaction with matter in the universe. To support this view, we consider two examples in which electromagnetic interaction induces mass.

The first example comes from radiowave propagation in the ionosphere. Neglecting the effects of the Earth's magnetic field and collisions of electrons with neutral air molecules, the dispersion relation for radiowave photons in such a cold plasma is the same as the dispersion relation for free massive particles, in which the photon has an induced mass m such that $mc^2 = hf$, where c is the speed of light, h is Planck's constant, and f is plasma frequency (which depends on the density of ionized electrons in the ionosphere). Since electron density is position dependent in the ionosphere, the effective mass of the photon is position dependent.

The second example is from motion of electrons in a crystalline lattice, in which the lattice induces an effective mass on electrons due to electromagnetic interaction between the electron and the lattice. In this case, the effective mass is highly position dependent.

Having set up the motivation for why we should look for a non-geometric formulation for gravitation, let us proceed. Trying to express gravitation in non-geometrical terms is difficult, for at least two reasons:

1. Present gravitational formulation is so closely linked with geometry, it seems nearly impossible to separate the two.

2. But, without a geometric background, how can we express any laws of physics?

We shall take a less radical approach, by simply assuming a geometry exists, without asking where it came from, expressing it in terms of the metric tensor as usual, but trying to express local gravitational interaction in ways that are more analogous to the other long-range interaction, the electromagnetic interaction.

Here, we can use one method for generalizing physical laws that can be done in three steps:

1. Rewrite the existing laws in a different but equivalent form.

2. Alter the new form to one which is not equivalent, but indistinguishable.

3. Possibly, rewrite the new form in another equivalent form.

To do this, we must rewrite the following aspects of the gravitational interaction to conform to the form of the other interactions, especially the other long-range interaction, the electromagnetic interaction:

1. The form of the local force law (the geodesic equation)

2. The form of the action

3. The form of the field equations

110.2 Electromagnetism as an example

To show that, we begin with Maxwell's equations.

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu} \tag{110.1}$$

and

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \ . \tag{110.2}$$

110.3 Gravitation

Now we write Einstein's field equations for General Relativity [20, Misner, Thorne, and Wheeler, pp. 210, 341, 410][27, Adler, Bazzin, and Schiffer, pp. 277, 280, 334] in the following form.

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + \Lambda g_{\mu\nu} \ . \tag{110.3}$$

Using the definition of the Riemann tensor [20, Misner, Thorne, and Wheeler, Box 14.2, p. 340]

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} \tag{110.4}$$

in (110.3) gives

$$\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\alpha} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + \Lambda g_{\mu\nu} \ . \tag{110.5}$$

Or,

$$\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} = -\Gamma^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\alpha} + 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + \Lambda g_{\mu\nu} \ . \tag{110.6}$$

The similarity of (110.6) with the inhomogeneous Maxwell equation (110.1) is clear. Also, just as $F^{\mu\nu}$ represents the electromagnetic field, $\Gamma^{\alpha}{}_{\mu\nu}$ represents the gravitational field. The presence of the gravitational field terms on the right-hand-side of (110.6) shows that the gravitational field is a source for the gravitational field, a well-known property of gravitation. When Γ has the usual relationship with the metric tensor, then (110.6) is equivalent to Einstein's field equations in General Relativity. If that relationship is not imposed, then (110.6) represents the gravitational field equations in a way that is similar to Maxwell's equations for the electromagnetic field.

In analogy with the homogeneous Maxwell equation (110.2), we have homogeneous equations for the gravitational field.

The Riemann tensor normally satisfies certain symmetries. These are [18, Weinberg, (6.6.3)-(6.6.5), p. 141].

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \ , \tag{110.7}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \ , \tag{110.8}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \ , \tag{110.9}$$

and

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \ . \tag{110.10}$$

Imposing these symmetries as additional conditions gives us additional equations, possibly enough to determine the system.

Equations (110.9) and (110.10) are satisfied identically. The symmetries in (110.7) and (110.8) are true when Γ has the usual relation with the metric tensor[20, Misner, Thorne, and Wheeler, section 13.5, p. 325]. Using (110.4) in (110.7) and (110.8) gives

$$\Gamma_{\alpha\beta\delta,\gamma} - \Gamma_{\alpha\beta\gamma,\delta} - \Gamma_{\gamma\delta\beta,\alpha} + \Gamma_{\gamma\delta\alpha,\beta} + \Gamma_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} - \Gamma_{\gamma\epsilon\alpha}\Gamma^{\epsilon}{}_{\delta\beta} + \Gamma_{\gamma\epsilon\beta}\Gamma^{\epsilon}{}_{\delta\alpha} = 0 \tag{110.11}$$

and

$$\Gamma_{\alpha\beta\delta,\gamma} - \Gamma_{\alpha\beta\gamma,\delta} + \Gamma_{\beta\alpha\delta,\gamma} - \Gamma_{\beta\alpha\gamma,\delta} + \Gamma_{\alpha\epsilon\gamma}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma_{\alpha\epsilon\delta}\Gamma^{\epsilon}{}_{\beta\gamma} + \Gamma_{\beta\epsilon\gamma}\Gamma^{\epsilon}{}_{\alpha\delta} - \Gamma_{\beta\epsilon\delta}\Gamma^{\epsilon}{}_{\alpha\gamma} = 0\ . \quad (110.12)$$

Equations (110.11) and (110.12) give the homogeneous gravitational field equations analogous to (110.2). Just as (110.2) gives the equations for electromagnetic waves, (110.11) and (110.12) give the equations for gravitational waves.

Here, we postulate that (110.11) and (110.12) are the correct homogeneous gravitational field equations, without regard for where they may have come. However [20, Misner, Thorne, and Wheeler, section 13.5, p. 325] point out that (110.7) is not an independent symmetry, but follows from the others. Therefore (110.11) may also not be an independent set of equations.

When Γ has the usual relationship with the metric tensor, then we have the contracted Bianchi identity[20, Misner, Thorne, and Wheeler, section 13.5, p. 325]

$$G^{\mu\nu}{}_{;\nu} = 0\ , \quad (110.13)$$

where

$$G_{\mu\nu} = R^{\alpha}{}_{\beta\gamma\delta} - \frac{1}{2} g_{\mu\nu} R \quad (110.14)$$

is the Einstein tensor.Again, as above, we can postulate that (110.13) is correct in any case, which implies that the geodesic equation governs the motion of bodies.

110.4 Solar System tests

There are four main Solar System tests of General Relativity: deflection of starlight by the Sun, the perihelion shift of Mercury, Shapiro time delay, and gravitational redshift. For all of these Solar System tests (and others), requires only that (110.6) be solved with the Sun as the source of gravitation. That is, we need to solve

$$\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} = -\Gamma^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\alpha}\ , \quad (110.15)$$

where the cosmological constant Λ is also neglected, because it has no measurable effects on Solar System tests. Actually, the right-hand-side of (110.15) should include a delta function at the location of the Sun, but we can choose a solution that satisfies (110.15) everywhere except at the Sun with the correct parameters for the mass of the Sun to handle that.

Because the appropriate solution of (110.15) should be the same as the usual solution found from the Schwarzschild solution, and because motions of bodies (including null geodesics) and determined by Γ through the geodesic equation, it should not be necessary to actually check each Solar System test, but to make sure, I will actually check each Solar System test.

Actually, the above part of this section describes the usual way of looking at things. It ignores the origin of inertia and the origin of inertial frames. The correct way to do the calculation for Solar System tests is to use (110.6) in the frame of whatever body we are looking at (like Mercury). In that frame, the geodesic equation does not have the acceleration term. All terms include some component of the gravitational field Γ. In that frame, the universe is accelerating, and that is the source of inertia. It is like the calculation of Sciama [12, Sciama:1953], but done correctly, or by Davidson [14, Davidson:1957], but with more accuracy.

110.5 Electromagnetic waves

It is instructive to first write the equations for electromagnetic waves as an example of how to do that before writing the equations for gravitational wave. There are several ways to develop the appropriate equations for wave propagation. One method is to use all of the equations to eliminate all dependent variables but one to get a second-order differential equation for that dependent variable. Another way, which I shall use here, is to use an appropriate set of first-order differential equations, linearize those equations, and get the corresponding matrix equations to characterize the waves.

We can rewrite (110.1) as

$$\left(g^{\mu\mu'}g^{\nu\nu'}F_{\mu'\nu'}\right)_{;\nu} = g^{\mu\mu'}g^{\nu\nu'}\left(F_{\mu'\nu'}\right)_{;\nu} = g^{\mu\mu'}g^{\nu\nu'}\left(F_{\mu'\nu',\nu} - \Gamma^{\alpha}{}_{\mu'\nu}F_{\alpha\nu'} - \Gamma^{\alpha}{}_{\nu\nu'}F_{\mu'\alpha}\right) = 4\pi J^{\mu}\ . \tag{110.16}$$

Multiplying (110.16) by $g_{\beta\mu}$ gives

$$g^{\nu\nu'}\left(F_{\beta\nu',\nu} - \Gamma^{\alpha}{}_{\beta\nu}F_{\alpha\nu'} - \Gamma^{\alpha}{}_{\nu\nu'}F_{\beta\alpha}\right) = 4\pi J_{\beta}\ . \tag{110.17}$$

Continuing on, (110.2) is equivalent to

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0\ . \tag{110.18}$$

At this point, the next step would be to linearize (110.17) and (110.18). However, since both of those equations are already linear, we can skip that step here.

For the case of electromagnetic waves, the next step is to replace $_{,\nu}$ by ik_{ν}, where k_{ν} is a wavenumber, so that we can get a dispersion relation. This gives

$$ik^{\gamma}F_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta}{}^{\gamma}F_{\alpha\gamma} - \Gamma^{\alpha}{}_{\gamma}{}^{\gamma}F_{\beta\alpha} = 4\pi J_{\beta} \tag{110.19}$$

and

$$ik_{\gamma}F_{\alpha\beta} + ik_{\alpha}F_{\beta\gamma} + ik_{\beta}F_{\gamma\alpha} = 0\ , \tag{110.20}$$

where I have replaced the dummy variable ν' by γ, and $k^{\gamma} \equiv g^{\gamma\nu}k_{\nu}$.

For a wave equation, we set the source term on the right-hand-side of (110.19) to zero, and choose specific equations from those above. We need six equations for the six independent components of $F_{\beta\alpha}$. For the first three equations, we choose β to be 1, 2, and 3 in (110.19) and for the next three equations, we choose α, β, and γ to be 0, 2, and 3; 0, 3, and 1; and 0, 1, and 2. We can write these six equations as

$$-i(C + iD)\psi = 0\ , \tag{110.21}$$

where

$$C = \begin{pmatrix} -k^0 & 0 & 0 & 0 & k^3 & -k^2 \\ 0 & -k^0 & 0 & -k^3 & 0 & k^1 \\ 0 & 0 & -k^0 & k^2 & -k^1 & 0 \\ 0 & k_3 & -k_2 & -k_0 & 0 & 0 \\ -k_3 & 0 & k_1 & 0 & -k_0 & 0 \\ k_2 & -k_1 & 0 & 0 & 0 & -k_0 \end{pmatrix}\ , \tag{110.22}$$

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\ , \tag{110.23}$$

$$d_{ii} = -\Gamma^{i}{}_{i}{}^{0} - \Gamma^{0}{}_{\nu}{}^{\nu} + \Gamma^{0}{}_{i}{}^{i}\ ,\ \text{for } i = 1, 2, 3, \tag{110.24}$$

$$d_{ij} = \Gamma^{0}{}_{i}{}^{j} - \Gamma^{j}{}_{i}{}^{0}\ ,\ \text{for } i, j = 1, 2, 3, i \neq j, \tag{110.25}$$

$$d_{i,i+3} = \Gamma^{i+2\text{Mod } 3}{}_{i}{}^{i+1\text{Mod } 3} - \Gamma^{i+1\text{Mod } 3}{}_{i}{}^{i+2\text{Mod } 3} \text{, for } i = 1, 2, 3, \tag{110.26}$$

$$d_{i,i+1\text{Mod } 3} = \Gamma^{i}{}_{i}{}^{i-1\text{Mod } 3} + \Gamma^{i-1\text{Mod } 3}{}_{\nu}{}^{\nu} - \Gamma^{i-1\text{Mod } 3}{}_{i}{}^{i} \text{, for } i = 1, 2, 3, \tag{110.27}$$

$$d_{i,i+2\text{Mod } 3} = -\Gamma^{i}{}_{i}{}^{i+1\text{Mod } 3} - \Gamma^{i+1\text{Mod } 3}{}_{\nu}{}^{\nu} - \Gamma^{i+1\text{Mod } 3}{}_{i}{}^{i} \text{, for } i = 1, 2, 3, \tag{110.28}$$

and

$$\psi = \begin{pmatrix} F_{10} \\ F_{20} \\ F_{30} \\ F_{23} \\ F_{31} \\ F_{12} \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} , \tag{110.29}$$

where the latter definition is in terms of electric and magnetic fields.

For a dispersion relation, we have

$$|C + iD| = 0 , \tag{110.30}$$

and for a Hamiltonian to use in Hamilton's equations to calculate the path of electromagnetic waves, we can use

$$H = |C + iD| . \tag{110.31}$$

Locally, we can always choose a freely falling frame, in which all Γs are zero. In that case, the dispersion relation becomes

$$\begin{vmatrix} -k^0 & 0 & 0 & 0 & k^3 & -k^2 \\ 0 & -k^0 & 0 & -k^3 & 0 & k^1 \\ 0 & 0 & -k^0 & k^2 & -k^1 & 0 \\ 0 & k_3 & -k_2 & -k_0 & 0 & 0 \\ -k_3 & 0 & k_1 & 0 & -k_0 & 0 \\ k_2 & -k_1 & 0 & 0 & 0 & -k_0 \end{vmatrix} = k^0 k_0 (g^{\mu\nu} k_\mu k_\nu)^2 = 0 . \tag{110.32}$$

Or, equivalently,

$$H = g^{\mu\nu} k_\mu k_\nu = 0 . \tag{110.33}$$

However, It would be difficult to use (110.33) as a Hamiltonian to calculate the path of a light ray because it would require continually changing coordinate system to be continually in a freely falling frame. It is more practical to choose a convenient coordinate frame and use (110.31) as the Hamiltonian in Hamiltonian's equations. However, it is not clear if the determinant in (110.31) is real or complex. If it is complex, it signifies that the wave will be attenuated or amplified. It might depend on whether the gravitational field is time dependent.

Equations (110.32) and (110.33) show explicitly and (110.30) implicitly that a better interpretation of General Relativity is not as geometry but that it provides the background medium for the propagation of light. Similar equations for massive particles should presumably show the same for the waves representing massive particles.

110.6 Gravitational waves

Just as (110.1) and (110.2) give the equations for electromagnetic waves, (110.6), (110.11), and (110.12) give the equations for gravitational waves. It might be necessary to include (110.13), but I will try without it first. Or, if (110.11) is redundant, then maybe we need only (110.6) and (110.12) to calculate gravitational waves.

110.7 Comments from Dave Peterson[1]

A few thoughts on Chapter 110

Turning coefficients might make more sense using differential forms. The usual form of the Bianchi identities eqn (110.2) have always looked too complex for me, but in differential forms they are simply $dF = ddA = 0$ (at least the notation is simpler). Curvature is given by 2-form $\Omega = D\omega = d\omega + \omega \wedge \omega$ where ω's are connection 1-forms and D is covariant derivative. For electromagnetism, A is a connection correction for a covariant derivative $D = \partial - eA$, and $A \wedge A = 0$ so that $DA = dA = F$. For Yang-Mills, the full new $F = dA + A \wedge A$ is used (my chapter on Covariant Derivative Issues helps a bit). In general relativity, $R^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu$ [MTW page 351] where Ricci is defined from Riemann on page 352.

Mach's principle is tricky now that we have to include dark matter and possibly dark energy Λ as well.

I've always been confused about the EM field Lagrangian $L = (B^2 - E^2)/2 + jA/c$ and whether it can be considered as "wave-counting" like for particles where momentum is now vague and what to do with accelerated expansion.

For a wave, if $B = k \times E/c$ then $E = cB$. For the Riemann-Silberstein $F = E + icB$, $F * F = E^2 + c^2B^2 \approx 2u_e/\epsilon + 2\mu u_B c^2 = (2/\epsilon)(u_e + u_B)$ where $\mu\epsilon = 1/c^2$ and u is field energy density. Energy has mass equivalence – is there a vibration? I don't know (this is classical stuff).

So, Electricity and Magnetism is a theory of a 4-vector field A_μ, Vary L wrt ϕ and A. I don't know intuitively how to get classical physics from quantum waves or how this pertains to gravitation.

[1] 4 March 2020

Chapter 111

Refractive index for a particle wave in a gravitational field

abstract

Representing gravitation as a refractive index in a propagation medium instead of as geometry is a representation that is more intuitive. Starting from either the Klein-Gordon wave equation or the Dirac wave equation, we get the same classical Hamiltonian to calculate ray paths for those particle waves, which could be interpreted as particle trajectories. In addition, taking $m = 0$ gives the corresponding values for electromagnetic waves. To convert to usual units, multiply all velocities by c. The formulas for refractive index are still valid, since they are dimensionless. In the same way that using a refractive index to represent propagation of a wave in a medium is coarse graining while fine graining considers scattering of the wave by each atom in the medium, representing gravitation by a refractive index or by geometry is coarse graining. Fine graining would consider the inductive gravitational interaction of a wave with each gravitating body in the universe.

111.1 Introduction

Representing gravitation as a refractive index in a propagation medium instead of as geometry is a representation that is more intuitive.

111.2 Hamilton's equations

The Hamiltonian for a particle wave (either a Boson or a Fermion if we neglect spin) of mass m in a gravitational field is

$$H = g^{\mu\nu} k_\mu k_\nu + m^2 = 0 \ , \tag{111.1}$$

where $g^{\mu\nu}$ is the metric, k_μ gives components of the wavenumber, $x^0 = t$, and $k_0 = \omega$. Hamilton's equations are

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial k_\mu} = 2g^{\mu\nu} k_\nu = 2k^\mu \tag{111.2}$$

and

$$\frac{dk_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu} = -g^{\alpha\beta}_{,\mu} k_\alpha k_\beta \ , \tag{111.3}$$

where the meaning of τ depends on the definition of H, and has no particular significance. Notice that Hamilton's equations require that

$$\frac{dH}{d\tau} = \frac{\partial H}{\partial k_\mu}\frac{dk_\mu}{d\tau} + \frac{\partial H}{\partial x^\mu}\frac{dx^\mu}{d\tau} = 0 \ . \tag{111.4}$$

111.3 Phase velocity and group velocity

Phase velocity is given by

$$v_p = v_p^i = \frac{\omega}{k_i} = \frac{k_0}{k_i} \ . \tag{111.5}$$

Group velocity is given by

$$v_g = v_g^i = \frac{\partial \omega}{\partial k_i} = \frac{\partial k_0}{\partial k_i} = \frac{\partial H/\partial k_i}{\partial H/\partial k_0} = \frac{g^{i\nu}k_\nu}{g^{0\mu}k_\mu} = \frac{k^i}{k^0} \ . \tag{111.6}$$

111.4 Phase refractive index and group refractive index

Refractive index is the reciprocal of velocity. The phase refractive index is

$$n = \frac{|k|}{|k_0|} = \sqrt{\frac{-g^{00}k_0k_0 - m^2}{g^{\ell\ell}k_0k_0}} \ , \tag{111.7}$$

where I have used (111.1) and isotropic coordinates. The group refractive index is given by

$$n' = \frac{|k^0|}{|k|} = \sqrt{\frac{g^{\ell\ell}k_0k_0}{-g^{00}k_0k_0 - m^2}} \ . \tag{111.8}$$

111.5 Derivation of the geodesic equation

Taking the derivative of (111.2) with respect to τ gives

$$\frac{d^2x^\mu}{d\tau^2} = 2g^{\mu\nu}{}_{,\alpha}\frac{dx^\alpha}{d\tau}k_\nu + 2g^{\mu\nu}\frac{dk_\nu}{d\tau} = 2k_\nu k_\beta(2g^{\alpha\beta}g^{\mu\nu}{}_{,\alpha} - g^{\mu\alpha}g^{\nu\beta}{}_{,\alpha}) \ . \tag{111.9}$$

Taking the derivative of

$$g^{\alpha\beta}g_{\alpha\gamma} = \delta^\beta_\gamma \tag{111.10}$$

gives

$$g^{\epsilon\beta}{}_{,\delta} = -g^{\gamma\epsilon}g^{\alpha\beta}g_{\alpha\gamma,\delta} \ . \tag{111.11}$$

Substituting (111.11) into (111.9) and changing indexes gives

$$\frac{d^2x^\mu}{d\tau^2} = -2k^\alpha k^\beta g^{\mu\gamma}(2g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}) = -2k^\alpha k^\beta g^{\mu\gamma}(g_{\beta\gamma,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}) \ . \tag{111.12}$$

Substituting (111.2) into (111.12) gives

$$\frac{d^2x^\mu}{d\tau^2} = -\frac{1}{2}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\Gamma^\mu{}_{\alpha\beta} \ , \tag{111.13}$$

where

$$\Gamma^\mu{}_{\alpha\beta} = g^{\mu\gamma}(g_{\beta\gamma,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}) \tag{111.14}$$

is the Affine connection.

111.6 Inertia

Inertia appears on the left side of the geodesic equation (111.13). Inertial force is experienced as any other force. In fact, there is no way to distinguish between an inertial force and a gravitational force. This is known as the equivalence principle. However, whereas the gravitational force on the right side of (111.13) is proportional to ${\Gamma^\mu}_{\alpha\beta}$, which could be interpreted as a gravitational field, there is nothing on the left side of (111.13) that could be interpreted as a gravitational field. It seems that inertia appears as a gravitational force only if we choose a coordinate system fixed on the body in question, as Sciama did in his 1953 paper on the origin of inertia.[12]. Davidson made the same calculation in his 1957 paper using General Relativity, also in the frame fixed on the body in question. [14]

The problem of having an inertial term not being proportional to anything that could be interpreted as a gravitational field already exists in Hamilton's equations, since (111.3) is already equivalent to Newton's second law, in that the left side for the $\mu \neq 0$ cases is proportional to the time derivative of momentum.

More generally, Hamilton's equations are expressed in terms of q_μ and p_μ instead of x_μ and k_μ, where q_μ and p_μ are canonical variables. This doesn't change anything. I think the problem of having an inertial term not being proportional to anything that could be interpreted as a gravitational field may originate with Hamilton's equations. Somehow, we need to find a set of equations that do not require having to choose a coordinate system. Maybe that is taken into account using the notation of differential forms. That would take care of the problem of the same geometry being expressed differently in different coordinate systems (the diffeomorphism problem).

111.7 Gravitational redshift

The simple (but wrong) way to calculate gravitational redshift gives the correct answer. If we consider that as a photon climbs up in a gravitational field, it converts kinetic energy to potential energy, then the frequency will drop accordingly. However, the $\mu = 0$ case in (111.3) of Hamilton's equations shows that the frequency of a wave cannot change unless the metric (more generally, the medium) varies with time.

The correct explanation for gravitational redshift is that atoms oscillate at higher frequencies if they are at a higher gravitational potential, so that as a photon travels to a higher gravitational potential (keeping at the same frequency), it will compare itself to an atom whose spectra are at a higher frequency in the higher potential. However, I cannot see how to show this, either from Hamilton's equations, or from Einstein's field equations.

111.8 Coarse graining versus fine graining

In the same way that using a refractive index to represent propagation of a wave in a medium is coarse graining while fine graining considers scattering of the wave by each atom in the medium, representing gravitation by a refractive index or by geometry is coarse graining. Fine graining would consider the inductive gravitational interaction of a wave with each gravitating body in the universe.

Chapter 112

The quantum N-body problem[1]

abstract

I formulate the quantum N-body problem in a way that allows approximate separation of variables in some cases to calculate the wave function for any atom or molecule. In some cases, iteration can lead to exact solutions. There may be applications in quantum gravity.

112.1 Introduction

Formulating and solving the quantum N-body problem is difficult. For example, atoms and molecules. Also, trying to formulate quantum gravity.

112.2 Classical formulation

I generalize the formulation by [182, (Schiff, 1955, section 16)] where he separates the Schrödinger equation for the 2-body problem (for a hydrogen atom, for example) into an equation for the center-of-mass motion and an equation for the relative position of the two particles (proton and electron, for example). This results in the reduced mass of $m_1 m_2/(m_1 + m_2)$ as an effective mass for the relative wave function.

Here, I generalize this result from $N = 2$ to arbitrarily large N, and show that the resulting equation is approximately separable into a center-of-mass equation and a relative equation for each pair of particles.

The formulation begins with classical equations. First, the total mass of the system of particles is

$$M = \sum_{i=1}^{N} m_i \ , \tag{112.1}$$

where m_i is the mass of the ith particle. The coordinate of the center of mass is

$$\mathbf{X} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{x}_i \ , \tag{112.2}$$

where $\mathbf{x}_i$ is the coordinate of the ith particle. I define relative coordinates between each pair of particles as

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j \ . \tag{112.3}$$

[1] 21 July 2020

Equations (112.2) and (112.3) are a set of linear equations that can be solved for the coordinates $\mathbf{x}_i$. The solution is

$$\mathbf{x}_i = \mathbf{X} + \frac{1}{M} \sum_{j \neq i}^{N} m_j \mathbf{x}_{ij} \ . \tag{112.4}$$

We need the time derivative of (112.4). This is

$$\dot{\mathbf{x}}_i = \dot{\mathbf{X}} + \frac{1}{M} \sum_{j \neq i}^{N} m_j \dot{\mathbf{x}}_{ij} \ . \tag{112.5}$$

The kinetic energy of the system is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 \ . \tag{112.6}$$

To calculate (112.6), we need to square (112.5). This gives squared terms plus cross terms. The cross terms involving $\dot{X}$ cancel because $\dot{\mathbf{x}}_{ij} + \dot{\mathbf{x}}_{ji} = 0$. The other cross terms can be canceled by adding terms of the form

$$\frac{1}{2} \frac{m_i m_j m_k}{M^2} (\dot{\mathbf{x}}_{ij} + \dot{\mathbf{x}}_{jk} + \dot{\mathbf{x}}_{ki})^2 = 0 \ . \tag{112.7}$$

Adding those terms that cancel the cross terms alter the squared terms. The result is

$$T = \frac{1}{2} M \dot{\mathbf{X}}^2 + \sum_{i<j}^{N} \frac{1}{2} m_{ij} \dot{\mathbf{x}}_{ij}^2 \ , \tag{112.8}$$

where

$$m_{ij} = \frac{m_i m_j}{M} \tag{112.9}$$

is the generalization of the reduced mass for the N-body case.

We define the relative momentum as

$$\mathbf{p}_{ij} = \mathbf{p}_i - \mathbf{p}_j = m_{ij} \dot{\mathbf{x}}_{ij} \ . \tag{112.10}$$

This allows us to write (112.8) as

$$T = \frac{\mathbf{P}^2}{2M} + \sum_{i<j}^{N} \frac{\mathbf{p}_{ij}^2}{2m_{ij}} \ , \tag{112.11}$$

where

$$\mathbf{P} = M \dot{\mathbf{X}} \ , \tag{112.12}$$

is the momentum of the center-of-mass system.

112.3 Quantum formulation

The correct formulation for quantum mechanics generalizes the development by [182, (Schiff, 1955, section 16)] from the 2-body case to the N-body case as follows.

$$\frac{\partial}{\partial x_i} = \frac{\partial X}{\partial x_i} \frac{\partial}{\partial X} + \sum_{j \neq i}^{N} \frac{\partial x_{ij}}{\partial x_i} \frac{\partial}{\partial x_{ij}} \ . \tag{112.13}$$

Using (112.2 and (112.3) gives

$$\frac{\partial}{\partial x_i} = \frac{m_i}{M}\frac{\partial}{\partial X} + \sum_{j\neq i}^{N} \frac{\partial}{\partial x_{ij}} \ . \tag{112.14}$$

Changing notation in (112.14) gives

$$\hat{\mathbf{p}}_i = \frac{m_i}{M}\hat{\mathbf{P}} + \sum_{j\neq i}^{N} \hat{\mathbf{p}}_{ij} \ . \tag{112.15}$$

The kinetic energy term in Schrödinger's equation is

$$\hat{T} = \sum_{i=1}^{N} \frac{\hat{\mathbf{p}}_i^2}{2m_i} = \sum_{i=1}^{N} \frac{(\frac{m_i}{M}\hat{\mathbf{P}} + \sum_{j\neq i}^{N} \hat{\mathbf{p}}_{ij})^2}{2m_i} \ . \tag{112.16}$$

Expanding (112.16) gives

$$\hat{T} = \frac{\hat{\mathbf{P}}^2}{2M} + \sum_{i=1}^{N}\sum_{j\neq i}^{N} \frac{\hat{\mathbf{p}}_{ij}^2}{2m_i} + \frac{\hat{\mathbf{P}}}{M}\sum_{i=1}^{N}\sum_{j\neq i}^{N} \hat{\mathbf{p}}_{ij} + \sum_{i=1}^{N}\sum_{j\neq i}^{N}\sum_{k\neq i,k>j}^{N} \frac{\hat{\mathbf{p}}_{ij}\hat{\mathbf{p}}_{ik}}{m_i} \ . \tag{112.17}$$

Collecting terms and simplifying (112.17) gives

$$\hat{T} = \frac{\hat{\mathbf{P}}^2}{2M} + \sum_{i<j}^{N} \frac{\hat{\mathbf{p}}_{ij}^2}{2m_{ij}} + \sum_{i=1}^{N}\sum_{j\neq i}^{N}\sum_{k\neq i,k>j}^{N} \frac{\hat{\mathbf{p}}_{ij}\hat{\mathbf{p}}_{ik}}{m_i} \ , \tag{112.18}$$

where

$$m_{ij} = \frac{m_i m_j}{m_i + m_j} \tag{112.19}$$

is the same as the reduced mass in the 2-body formulation.

112.4 Quantum mechanics

We convert (112.18) to a Schrödinger equation in the usual way by letting

$$\hat{\mathbf{p}}_{ij} = -i\hbar\nabla_{ij} \ . \tag{112.20}$$

This gives

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi - \sum_{i<j}^{N}\frac{\hbar^2}{2m_{ij}}\nabla_{ij}^2\psi - \sum_{i=1}^{N}\sum_{j\neq i}^{N}\sum_{k\neq i,k>j}^{N}\frac{\hbar^2}{m_i}\nabla_{ij}\nabla_{ik}\psi + V\psi \ , \tag{112.21}$$

where

$$V = V(\mathbf{X}, \mathbf{x}_{ij}) \tag{112.22}$$

is the potential.

112.5 Atoms and molecules

We now consider a special case, in which

$$V = \sum_{i<j}^{N} V_{ij}(\mathbf{x}_{ij}) \ . \tag{112.23}$$

That is, we consider the special case where the potential depends only on the sum of potentials for the relative coordinates for each pair of particles. This is the case for any atom or molecule when there are no external fields applied.

We now take

$$\psi = \psi_X(t, \mathbf{X}) \prod_{i<j}^{N} \psi_{ij}(t, \mathbf{x}_{ij}) \ . \tag{112.24}$$

Substituting (112.24) and (112.23) into (112.21) allows us to separate the Schrödinger equation. This gives

$$i\hbar \frac{\partial \psi_{ij}(t, \mathbf{x}_{ij})}{\partial t} = -\frac{\hbar^2}{2m_{ij}} \nabla^2 \psi_{ij}(t, \mathbf{x}_{ij}) + (C_{ij} + V_{ij}(\mathbf{x}_{ij}) + \lambda_{ij})\psi_{ij}(t, \mathbf{x}_{ij}) \ , \tag{112.25}$$

where

$$C_{ij} = -\sum_{k \neq i,j} \frac{\hbar^2}{m_k} \left(\frac{\nabla_{ki}\psi_{ki}(t, \mathbf{x}_{ki})}{\psi_{ki}(t, \mathbf{x}_{ki})} \right) \cdot \left(\frac{\nabla_{kj}\psi_{kj}(t, \mathbf{x}_{kj})}{\psi_{kj}(t, \mathbf{x}_{kj})} \right) \tag{112.26}$$

is a coupling term,

$$i\hbar \frac{\psi_X(t, \mathbf{X})}{\partial t} = -\frac{\hbar^2}{2m_{ij}} \nabla^2 \psi_X(t, \mathbf{X}) + \lambda_X \psi_X(t, \mathbf{X}) \ , \tag{112.27}$$

and

$$\lambda_X + \sum_{i<j}^{N} \lambda_{ij} = 0 \tag{112.28}$$

are separation constants.

For the time independent case, we have

$$(E_{ij} - \lambda_{ij})\psi_{ij}(\mathbf{x}_{ij}) = -\frac{\hbar^2}{2m_{ij}} \nabla^2 \psi_{ij}(\mathbf{x}_{ij}) + (C_{ij} + V_{ij})(\mathbf{x}_{ij})\psi_{ij}(\mathbf{x}_{ij}) \tag{112.29}$$

and

$$(E_X - \lambda_X)\psi_X(t, \mathbf{X}) = -\frac{\hbar^2}{2m_{ij}} \nabla^2 \psi_X(t, \mathbf{X}) \ . \tag{112.30}$$

For the case of a many-electron atom, the potential between each pair of particles is simply

$$V_{ij}(\mathbf{x}_{ij}) = \frac{q_i q_j}{|\mathbf{x}_{ij}|} \ . \tag{112.31}$$

In that case, the solution to (112.29) for each pair of particles that have opposite charge is simply a lot of hydrogen atom wave functions for each pair of particles with generalized reduced mass for each pair of particles if we neglect the coupling terms. So, we have a wave function for the nucleus and each electron. Finding a wave function for each pair of electrons is more difficult, however.

112.6 Helium atom ground state

For the helium atom, we have three particles, a helium nucleus and two electrons. We have $m_1 = m_\alpha$ for the mass of the helium nucleus, and we have $m_2 = m_3 = m_e$ for the mass of each electron. The total mass is $M = m_\alpha + 2m_e$. $m_{12} = m_{13} = m_\alpha m_e/(m_\alpha + m_e)$. $m_{23} = m_e/2$. We take e to be the positive charge on the proton. Then $Z = 2$, $q_1 = Ze$, $q_2 = q_3 = -e$.

Ignoring the wave function for the center-of-mass system, the total wave function is

$$\psi = \psi_{12}(\mathbf{x}_{12})\psi_{31}(\mathbf{x}_{31})\psi_{23}(\mathbf{x}_{23}) \exp\left(-i(E_{12} + E_{31} + E_{23})t/\hbar\right) . \qquad (112.32)$$

For the ground state, we take $\psi_{12}(\mathbf{x}_{12})$ and $\psi_{31}(\mathbf{x}_{31})$ to be in the $1s$ state. For $\psi_{23}(\mathbf{x}_{23})$, we have a case where the charge on both electrons is the same, so the usual 2-body solutions with Laguerre polynomials do not work.

112.6.1 1-2 wave function

We have

$$(E_{12} - \lambda_{12})\psi_{12}(\mathbf{x}_{12}) = -\frac{\hbar^2}{2m_{12}}\nabla^2\psi_{12}(\mathbf{x}_{12}) + (C_{12} + V_{12})(\mathbf{x}_{12})\psi_{12}(\mathbf{x}_{12}) , \qquad (112.33)$$

where

$$C_{12} = \frac{\hbar^2}{m_3}\left(\frac{\nabla_{31}\psi_{31}(t,\mathbf{x}_{31})}{\psi_{31}(t,\mathbf{x}_{31})}\right)\cdot\left(\frac{\nabla_{23}\psi_{23}(t,\mathbf{x}_{23})}{\psi_{23}(t,\mathbf{x}_{23})}\right) \qquad (112.34)$$

is a coupling term. If as a first approximation, we neglect the coupling term, then we get

$$\psi_{12}(\mathbf{x}_{12}) = a_{012}^{-3/2} e^{-r_{12}/a_{012}} , \qquad (112.35)$$

where

$$a_{012} = \frac{\hbar^2}{Zm_{12}e^2} = \frac{\hbar^2(m_1 + m_2)}{Zm_1m_2e^2} = \frac{m_\alpha + m_e}{Zm_\alpha}\frac{\hbar^2}{m_ee^2} = \frac{m_\alpha + m_e}{Zm_\alpha}a_0 \qquad (112.36)$$

is the generalized reduced Bohr radius for the mass m_{12} and $a_0 = \frac{\hbar^2}{m_ee^2}$ is the Bohr radius. The energy Eigenvalues are given by

$$E_{12} - \lambda_{12} - C_{12} = -\frac{1}{2}m_{12}c^2\frac{\alpha^2}{n^2}Z^2 = -\frac{1}{2}\frac{m_em_\alpha c^2}{m_\alpha + m_e}\frac{\alpha^2}{n^2}Z^2 = -\frac{m_\alpha}{m_\alpha + m_e}\frac{Z^2}{n^2}hcR_\infty , \qquad (112.37)$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant, $hcR_\infty = m_ee^4/(2\hbar^2) = m_ec^2\alpha^2/2 = 13.605693122994(26)$ eV is the Rydberg energy, and $R_\infty = m_ec\alpha^2/(2h) = 109737.31568160(21)$ cm^{-1}.

For the second approximation, we calculate the values of the coupling terms found from the first approximation, and use those to get the second approximation. For the ground state, we have

$$C_{12} = -\frac{\hbar^2}{m_3}\frac{1}{a_{031}}\frac{1}{a_{023}} = -Z\frac{m_\alpha}{m_\alpha + m_e}hcR_\infty . \qquad (112.38)$$

Using (112.38) and (112.37) with $Z = 2$ gives

$$E_{12} = \lambda_{12} - 6\frac{m_\alpha}{m_\alpha + m_e}hcR_\infty . \qquad (112.39)$$

112.6.2 3-1 wave function

Similarly,

$$(E_{31}-\lambda_{31})\psi_{31}(\mathbf{x}_{31}) = -\frac{\hbar^2}{2m_{31}}\nabla^2\psi_{31}(\mathbf{x}_{31}) + (C_{31}+V_{31})(\mathbf{x}_{31})\psi_{31}(\mathbf{x}_{31}) \;, \tag{112.40}$$

where

$$C_{31} = \frac{\hbar^2}{m_2}\left(\frac{\nabla_{23}\psi_{23}(t,\mathbf{x}_{23})}{\psi_{23}(t,\mathbf{x}_{23})}\right)\cdot\left(\frac{\nabla_{12}\psi_{12}(t,\mathbf{x}_{12})}{\psi_{12}(t,\mathbf{x}_{12})}\right) \tag{112.41}$$

is a coupling term. If as a first approximation, we neglect the coupling term, then we get

$$\psi_{31}(\mathbf{x}_{31}) = a_{031}^{-3/2} e^{-r_{31}/a_{031}} \;, \tag{112.42}$$

where

$$a_{031} = \frac{\hbar^2}{Zm_{31}e^2} = \frac{\hbar^2(m_1+m_3)}{Zm_1m_3e^2} = \frac{m_\alpha+m_e}{Zm_\alpha}\frac{\hbar^2}{m_ee^2} = \frac{m_\alpha+m_e}{Zm_\alpha}a_0 \tag{112.43}$$

is the generalized reduced Bohr radius for the mass m_{31}. The energy Eigenvalues are given by

$$E_{31}-\lambda_{31}-C_{31} = -\frac{1}{2}m_{31}c^2\frac{\alpha^2}{n^2}Z^2 = -\frac{1}{2}\frac{m_em_\alpha c^2}{m_\alpha+m_e}\frac{\alpha^2}{n^2}Z^2 = -\frac{m_\alpha}{m_\alpha+m_e}\frac{Z^2}{n^2}hcR_\infty \;, \tag{112.44}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant.

For the second approximation, we calculate the values of the coupling terms found from the first approximation, and use those to get the second approximation. For the ground state, we have

$$C_{31} = -\frac{\hbar^2}{m_2}\frac{1}{a_{012}}\frac{1}{a_{023}} = -Z\frac{m_\alpha}{m_\alpha+m_e}hcR_\infty \;. \tag{112.45}$$

Using (112.45) and (112.44) with $Z=2$ gives

$$E_{31} = \lambda_{31} - 6\frac{m_\alpha}{m_\alpha+m_e}hcR_\infty \;. \tag{112.46}$$

112.6.3 2-3 wave function

For the solution for $\psi_{23}(\mathbf{x}_{23})$, we have to do something different because the two electrons have the same charge. The usual development [182, (Schiff, 1955, section 16)] that leads to Laguerre polynomials does not work. We start with

$$(E_{23}-\lambda_{23})\psi_{23}(\mathbf{x}_{23}) = -\frac{\hbar^2}{2m_{23}}\nabla^2\psi_{23}(\mathbf{x}_{23}) + (C_{23}+V_{23})(\mathbf{x}_{23})\psi_{23}(\mathbf{x}_{23}) \;, \tag{112.47}$$

where

$$C_{23} = \frac{\hbar^2}{m_1}\left(\frac{\nabla_{31}\psi_{31}(t,\mathbf{x}_{31})}{\psi_{31}(t,\mathbf{x}_{31})}\right)\cdot\left(\frac{\nabla_{12}\psi_{12}(t,\mathbf{x}_{12})}{\psi_{12}(t,\mathbf{x}_{12})}\right) \tag{112.48}$$

is a coupling term. If as a first approximation, we neglect the coupling term, then we get the radial wave equation [182, (Schiff, 1955, section 16)]

$$-\frac{\hbar^2}{2m_{23}}\frac{1}{r_{23}^2}\frac{d}{dr_{23}}\left(r_{23}^2\frac{dR_{23}}{dr_{23}}\right) + \frac{q_2q_3}{r_{23}}R_{23}(r_{23}) + \frac{\ell(\ell+1)\hbar^2}{2m_{23}r_{23}^2}R_{23}(r_{23}) = (E_{23}-\lambda_{23})R_{23}(r_{23}) \;, \tag{112.49}$$

modified for our case. Further following [182, (Schiff, 1955, section 16)], we change the independent variable to

$$\rho = \frac{\sqrt{8m_{23}|E_{23}-\lambda_{23}|}}{\hbar}r_{23} \;. \tag{112.50}$$

Then we can write (112.49) as

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR_{23}}{d\rho}\right)+\left(\frac{\lambda}{\rho}-\frac{1}{4}-\frac{\ell(\ell+1)}{\rho^2}\right)R_{23}=0\ , \tag{112.51}$$

where

$$\lambda=-\frac{q_2q_3}{\hbar}\left(\frac{m_{23}}{2|E_{23}-\lambda_{23}|}\right)^{1/2} \tag{112.52}$$

has nothing to do with λ_{ij} or λ_X. The negative sign in front of the $\frac{1}{4}$ in (112.51) assumes that $E_{23}-\lambda_{23}$ is negative.

Continuing to follow [182, (Schiff, 1955, section 16)], but with a positive exponential, we try the solution

$$R_{23}(\rho)=F(\rho)e^{+\frac{1}{2}\rho}\ . \tag{112.53}$$

This leads to the following equation:

$$F''+\left(\frac{2}{\rho}+1\right)F'+\left[\frac{\lambda+1}{\rho}-\frac{\ell(\ell+1)}{\rho^2}\right]F=0\ . \tag{112.54}$$

Then we try

$$F(\rho)=\rho^sL(\rho)=\rho^s(a_0+a_1\rho+a_2\rho^2+\ .\ .\ .\ )\ . \tag{112.55}$$

This leads to

$$\rho^2L''+\rho\left[2(s+1)+\rho\right]L'+\left[\rho\left(\lambda+s+1\right)+s(s+1)-\ell(\ell+1)\right]L=0\ . \tag{112.56}$$

When $\rho=0$, we must have $s(s+1)-\ell(\ell+1)=0$. This requires that either $s=\ell$ or $s=-\ell-1$. For the wave function to be finite at $\rho=0$ requires that we choose $s=\ell$. That gives

$$\rho L''+\left[2(\ell+1)+\rho\right]L'+\left(\lambda+\ell+1\right)L=0\ . \tag{112.57}$$

Substituting (112.55) into (112.57) gives a recursion relation to determine the coefficients of the power series. This is

$$a_{\nu+1}=\frac{\nu+\ell+1+\lambda}{(\nu+1)(\nu+2\ell+2)}a_\nu\ . \tag{112.58}$$

If the power series does not end, it will approach an exponential, which does not satisfy the boundary conditions. So the series must end. That requires that the numerator of (112.58) be zero for some value of ν. That leads to the usual solution in terms of Laguerre polynomials. That can happen if λ is a negative integer, which is true in this case, since q_2 and q_3 have the same sign. Taking $\lambda=-n$, a negative integer, gives

$$E_{23}-\lambda_{23}-C_{23}=-\frac{m_{23}q_2^2q_3^2}{2\hbar^2n^2}=-\frac{m_{23}e^4}{2\hbar^2n^2}=-\frac{1}{2}m_{23}c^2\frac{e^4}{\hbar^2c^2n^2}=-\frac{1}{2}\frac{hcR_\infty}{n^2}\ . \tag{112.59}$$

Thus, the solution, from (112.53) is

$$R_{23}=\rho^\ell e^{+\frac{1}{2}\rho}L(\rho)=\left(\frac{r_{23}}{a_{023}n}\right)^\ell e^{+\frac{1}{2}\frac{r_{23}}{a_{023}n}}L\left(\frac{r_{23}}{a_{023}n}\right)\ , \tag{112.60}$$

and

$$a_{023}=\frac{\hbar^2}{m_{23}e^2}=\frac{m_2+m_3}{m_2m_3}\frac{\hbar^2}{e^2}=2\frac{\hbar^2}{m_ee^2}=2a_0 \tag{112.61}$$

is the generalized reduced Bohr radius for the mass m_{23}.

For the ground state, we take $n = 1$ to give

$$R_{23} = e^{+\frac{r_{23}}{a_{023}}} \ . \tag{112.62}$$

For the second approximation, we calculate the values of the coupling terms found from the first approximation, and use those to get the second approximation. For the ground state, we have

$$C_{23} = \frac{\hbar^2}{m_1}\frac{1}{a_{012}}\frac{1}{a_{031}} = 2\frac{m_\alpha m_e}{(m_\alpha + m_e)^2}Z^2 hcR_\infty \ . \tag{112.63}$$

Using (112.63) and (112.59) with $Z = 2$ gives

$$E_{23} = \lambda_{23} + 2\left(4\frac{m_\alpha}{m_\alpha + m_e}\frac{m_e}{m_\alpha + m_e} - \frac{1}{4}\right) hcR_\infty \ . \tag{112.64}$$

Unfortunately, this does not agree with the measured ground state for Helium. The problem is that the solutions for $\psi_{12}(\mathbf{x}_{12})$, $\psi_{31}(\mathbf{x}_{31})$, and $\psi_{23}(\mathbf{x}_{23})$ are not independent, because $\mathbf{x}_{31}+\mathbf{x}_{12}+\mathbf{x}_{23} = 0$. We also have $\nabla_{31} + \nabla_{12} + \nabla_{23} = 0$. I am not sure how to fix this, though.

112.6.4 Total wave function

Putting it all together gives

$$\psi = (a_{012}a_{031})^{-3/2} \exp\left(-\frac{r_{12}}{a_{012}} - \frac{r_{31}}{a_{031}} + \frac{r_{23}}{a_{023}}\right) \exp\left(-iEt/\hbar\right) \ , \tag{112.65}$$

where

$$\begin{aligned} E = E_{12} + E_{31} + E_{23} &= -m_e c^2 \frac{m_\alpha}{m_\alpha + m_e}\alpha^2 Z^2 - \frac{1}{4}m_e c^2\alpha^2 + \lambda_{12} + \lambda_{31} + \lambda_{23} \\ +C_{12} + C_{31} + C_{23} &= 2\left(-6\frac{m_\alpha}{m_\alpha + m_e} + 4\frac{m_\alpha}{m_\alpha + m_e}\frac{m_e}{m_\alpha + m_e} - \frac{1}{4}\right) hcR_\infty \ . \end{aligned} \tag{112.66}$$

I still have to normalize the wave function.

112.7 H_2^+ ground state

For the H_2^+ ground state, we have three particles, two protons and an electron. We have $m_1 = m_2 = m_p$ for the mass of each of the two protons, and we have $m_3 = m_e$ for the mass of the electron. The total mass is $M = 2m_p + m_e$. $m_{12} = m_p^2/(2m_p)$. $m_{13} = m_{23} = m_p m_e/(m_p + m_e)$. We take e to be the positive charge on the proton. Then $q_1 = q_2 = e$, $q_3 = -e$.

Ignoring the wave function for the center-of-mass system, the total wave function is

$$\psi = \psi_{12}(\mathbf{x}_{12})\psi_{13}(\mathbf{x}_{13})\psi_{23}(\mathbf{x}_{23}) \exp\left(-i(E_{12} + E_{13} + E_{23})t/\hbar\right) \ . \tag{112.67}$$

For the ground state, we take $\psi_{13}(\mathbf{x}_{13})$ and $\psi_{23}(\mathbf{x}_{23})$ to be in the $1s$ state.

For $\psi_{12}(\mathbf{x}_{12})$, we have the same situation as with Helium because the charge on both protons is the same, so the usual 2-body solutions with Laguerre polynomials do not work.

112.7.1 3-1 wave function

We start with

$$(E_{31} - \lambda_{31})\psi_{31}(\mathbf{x}_{31}) = -\frac{\hbar^2}{2m_{31}}\nabla^2\psi_{31}(\mathbf{x}_{31}) + (C_{31} + V_{31})(\mathbf{x}_{31})\psi_{31}(\mathbf{x}_{31}) \ , \tag{112.68}$$

where

$$C_{31} = \frac{\hbar^2}{m_2}\left(\frac{\nabla_{23}\psi_{23}(t,\mathbf{x}_{23})}{\psi_{23}(t,\mathbf{x}_{23})}\right)\cdot\left(\frac{\nabla_{12}\psi_{12}(t,\mathbf{x}_{12})}{\psi_{12}(t,\mathbf{x}_{12})}\right) \tag{112.69}$$

is a coupling term. If as a first approximation, we neglect the coupling term, then we get

$$\psi_{31}(\mathbf{x}_{31}) = \pi^{-1/2}a_{031}^{-3/2}e^{-r_{31}/a_{031}} \ , \tag{112.70}$$

where

$$a_{031} = \frac{\hbar^2}{m_{31}e^2} \tag{112.71}$$

is the generalized reduced Bohr radius for the mass m_{31}. The energy Eigenvalues are given by

$$E_{31} - \lambda_{31} - C_{31} = -\frac{1}{2}m_{31}c^2\frac{\alpha^2}{n^2} = -\frac{1}{2}\frac{m_p m_e c^2}{m_p + m_e}\frac{\alpha^2}{n^2} = -\frac{m_p}{m_p + m_e}\frac{hcR_\infty}{n^2} \ , \tag{112.72}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant.

For the second approximation, we calculate the values of the coupling terms found from the first approximation, and use those to get the second approximation. For the ground state, we have

$$C_{31} = -\frac{\hbar^2}{m_2}\frac{1}{a_{012}}\frac{1}{a_{023}} = -\frac{m_p}{m_p + m_e}hcR_\infty \ . \tag{112.73}$$

Using (112.73) and (112.72) with $n = 1$ gives

$$E_{31} = \lambda_{31} - 2\frac{m_p}{m_p + m_e}hcR_\infty \ . \tag{112.74}$$

112.7.2 2-3 wave function

We start with

$$(E_{23} - \lambda_{23})\psi_{23}(\mathbf{x}_{23}) = -\frac{\hbar^2}{2m_{23}}\nabla^2\psi_{23}(\mathbf{x}_{23}) + (C_{23} + V_{23})(\mathbf{x}_{23})\psi_{23}(\mathbf{x}_{23}) \ , \tag{112.75}$$

where

$$C_{23} = \frac{\hbar^2}{m_1}\left(\frac{\nabla_{31}\psi_{31}(t,\mathbf{x}_{31})}{\psi_{31}(t,\mathbf{x}_{31})}\right)\cdot\left(\frac{\nabla_{12}\psi_{12}(t,\mathbf{x}_{12})}{\psi_{12}(t,\mathbf{x}_{12})}\right) \tag{112.76}$$

is a coupling term. If as a first approximation, we neglect the coupling term, then we get

$$\psi_{23}(\mathbf{x}_{23}) = \pi^{-1/2}a_{023}^{-3/2}e^{-r_{23}/a_{023}} \ , \tag{112.77}$$

where

$$a_{023} = \frac{\hbar^2}{m_{23}e^2} \tag{112.78}$$

is the generalized reduced Bohr radius for the mass m_{23}. The energy Eigenvalues are given by

$$E_{23} - \lambda_{23} - C_{23} = -\frac{1}{2}m_{23}c^2\frac{\alpha^2}{n^2} = -\frac{1}{2}\frac{m_p m_e c^2}{m_p + m_e}\frac{\alpha^2}{n^2} = -\frac{m_p}{m_p + m_e}\frac{hcR_\infty}{n^2} \ , \tag{112.79}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant.

For the second approximation, we calculate the values of the coupling terms found from the first approximation, and use those to get the second approximation. For the ground state, we have

$$C_{23} = -\frac{\hbar^2}{m_1}\frac{1}{a_{012}}\frac{1}{a_{031}} = -\frac{m_p}{(m_p+m_e)}hcR_\infty \ . \tag{112.80}$$

Using (112.80) and (112.79) with $n = 1$ gives

$$E_{23} = \lambda_{23} - 2\frac{m_p}{m_p+m_e}hcR_\infty \ . \tag{112.81}$$

112.7.3 1-2 wave function

For the $\psi_{12}(\mathbf{x}_{12})$ wave function for the two protons, the usual formulation in terms of Laguerre polynomials does not work.

We have

$$(E_{12}-\lambda_{12})\psi_{12}(\mathbf{x}_{12}) = -\frac{\hbar^2}{2m_{12}}\nabla^2\psi_{12}(\mathbf{x}_{12}) + (C_{12}+V_{12})(\mathbf{x}_{12})\psi_{12}(\mathbf{x}_{12}) \ , \tag{112.82}$$

where

$$C_{12} = \frac{\hbar^2}{m_3}\left(\frac{\nabla_{31}\psi_{31}(t,\mathbf{x}_{31})}{\psi_{31}(t,\mathbf{x}_{31})}\right)\cdot\left(\frac{\nabla_{23}\psi_{23}(t,\mathbf{x}_{23})}{\psi_{23}(t,\mathbf{x}_{23})}\right) \tag{112.83}$$

is a coupling term. If as a first approximation, we neglect the coupling term, then we get

$$\psi_{12}(\mathbf{x}_{12}) = e^{+r_{12}/a_{012}} \ , \tag{112.84}$$

where

$$a_{012} = \frac{\hbar^2}{m_{12}e^2} = \frac{2\hbar^2}{m_p e^2} = 2\frac{m_p}{m_e}a_0 \tag{112.85}$$

is the generalized reduced Bohr radius for the mass m_{12}. I assume the two protons have opposite spin. The energy Eigenvalues are given by

$$E_{12}-\lambda_{12}-C_{12} = -\frac{1}{2}m_{12}c^2\frac{\alpha^2}{n^2} = -\frac{1}{2}\frac{m_p^2c^2}{2m_p}\frac{\alpha^2}{n^2} = -\frac{1}{2}\frac{m_p}{m_e}\frac{hcR_\infty}{n^2} \ , \tag{112.86}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant.

For the second approximation, we calculate the values of the coupling terms found from the first approximation, and use those to get the second approximation. For the ground state, we have

$$C_{12} = \frac{\hbar^2}{m_3}\frac{1}{a_{031}}\frac{1}{a_{023}} = 2\left(\frac{m_p}{m_p+m_e}\right)^2 hcR_\infty \ . \tag{112.87}$$

Using (112.87) and (112.86) with $n = 1$ gives

$$E_{12} = \lambda_{12} + 2\left(\frac{m_p}{m_p+m_e}\right)^2 hcR_\infty - \frac{1}{2}\frac{m_p}{m_e}hcR_\infty \ . \tag{112.88}$$

112.7.4 Total wave function

Putting it all together gives

$$\psi = \frac{1}{\pi}(a_{031}a_{023})^{-3/2}\exp\left(\frac{r_{12}}{a_{012}} - \frac{r_{31}}{a_{031}} - \frac{r_{23}}{a_{023}}\right)\exp\left(-iEt/\hbar\right) , \tag{112.89}$$

where

$$\begin{aligned} E = E_{12} + E_{31} + E_{23} &= -2\frac{m_p}{m_p + m_e}hcR_\infty - \frac{1}{2}\frac{m_p}{m_e}hcR_\infty + \lambda_{12} + \lambda_{31} + \lambda_{23} \\ +C_{12} + C_{31} + C_{23} &= -4\frac{m_p}{m_p + m_e}hcR_\infty + 2\left(\frac{m_p}{m_p + m_e}\right)^2 hcR_\infty - \frac{1}{2}\frac{m_p}{m_e}hcR_\infty . \end{aligned} \tag{112.90}$$

The normalization may be not quite right, but we definitely have a bound state.

112.8 Limitations

- This development is non-relativistic. No relativistic effects. Including relativity may be hard.
- The effective mass for the whole center-of-mass system should include binding energy, not simply the sum of the individual masses. There may be similar effects on the individual wave functions for each pair of particles. Maybe this is taken care of by simply including binding energy in M wherever it appears in any equation, but this is just a guess.
- Spin is not included. It might be possible to add spin.

112.9 Spin

THIS IS NOT RIGHT. I STILL NEED TO CORRECT IT.

To put in spin, we do two things. First, we replace $\mathbf{p}_i$ by $\sigma \cdot \mathbf{p}_i$ everywhere, where the σ are the Pauli spin matrices. Second, we replace $\mathbf{p}_i$ by $\mathbf{p}_i - q_i\mathbf{A}_i$ everywhere. This means that $\mathbf{p}_{ij}$ is replaced by $\mathbf{p}_{ij} - q_i\mathbf{A}_i + q_j\mathbf{A}_j$ everywhere. Thus, (112.18) becomes

$$T = \frac{(\sigma \cdot (\mathbf{P} - q\mathbf{A}))^2}{2M} + \sum_{i<j}^{N} \frac{(\sigma \cdot (\mathbf{p}_{ij} - q_i\mathbf{A}_i + q_j\mathbf{A}_j))^2}{2m_{ij}} , \tag{112.91}$$

To include spin, we use (112.20) in (112.91) to give

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{(\sigma \cdot (-i\hbar\nabla - q\mathbf{A}))^2}{2M}\psi + \sum_{i<j}^{N} \frac{(\sigma \cdot (-i\hbar\nabla_{ij} - q_i\mathbf{A}_i + q_j\mathbf{A}_j))^2}{2m_{ij}}\psi + V\psi , \tag{112.92}$$

112.10 Relativity

Once we have recognized that we can split any N-body problem into a lot of 2-body problems using the generalized reduced mass for that reduced system, then it may be possible at that point to use the Dirac equation or the Klein-Gordon equation for that reduced 2-body system, depending on the spin.

However, for the 2-body subsystem, there are 3 possibilities: 2 Bosons, 2 Fermions, or a Boson and a Fermion. In the case of the helium atom, we have a Boson (the helium nucleus) and a Fermion (twice) and the 2 electrons is a 2-Fermion case. Since both the Dirac equation and the Klein-Gordon equation are 1-body equations, it is not clear how to correctly make the generalization.

112.11 Future

In principle, we could use these equations for all of the particles in the universe. That might have some use in quantum gravity.

Chapter 113

When did our universe become classical?[1]

abstract

Our universe seems classical (non-quantum) because it is a superposition of cosmologies indistinguishable from one another. The cosmologies allowed by constructive phase interference in any reasonable theory of quantum gravity would have parameters restricted to a narrow range of values. For example, our universe is a quantum superposition of cosmologies that have rms vorticity less than about 10^{-73} radians per year ["The rotation problem", General Relativity and Gravitation **52** (5), 1-35 (2020]. Although in the past, the cosmologies allowed by constructive interference would have less restrictive rotation rates, calculations show that even as early as the end of the inflation era phase interference would limit rms relative rotation to be less than about 10^{-20} radians per year for inflation having 50 to 60 e-foldings. Because the rms vorticity was limited to a narrow range, our universe was already classical at the end of inflation, at least with regard to vorticity. But even without inflation, our universe would have been classical within a fraction of a second after the initial singularity. Other parameters, such as shear, may be similarly restricted by phase interference, so our universe may have been classical in several aspects since the end of inflation.

113.1 Introduction

From the viewpoint of quantum gravity or quantum cosmology, our universe is a quantum superposition of cosmologies that have various values of the parameters that define a cosmology. Our universe seems classical (non-quantum) because it is a superposition of cosmologies indistinguishable from one another. That is, the values for each of the parameters defining a cosmology must be restricted to such a narrow range that it is impossible to distinguish them. The best explanation is that phase interference in any reasonable theory of quantum gravity would restrict all of the parameters to a narrow range of values.

For example, consider the rms vorticity of the universe as one of those parameters. Various measurements, e.g. [307, 305, 303, 302, 300, 311, 301, 308, 304, 164, 309, 310, 306, 369, 312, 299] have shown that the relative rotation of the average inertial frame and the average matter distribution is very small, the most restrictive being less than about 10^{-20} radians per year. Explaining these measurements has been referred to as "The Rotation Problem" e.g. [298].

This problem was resolved by showing that any reasonable theory of quantum gravity could explain (by phase interference using a path-integral calculation) why measurements would show

[1]submitted to General Relativity and Gravitation 23 March 2021; rejected 8 April 2021

a lack of relative rotation ["The Rotation Problem" (Jones [386], hereafter referred to as TRP)]. Cosmologies with present relative rms rotation larger than about $T^*H^2a_1^{1/2} \approx 10^{-73}$ radians per year would cancel each other out by phase interference, where $T^* \approx 10^{-51}$ years is the Planck time, $H \approx 10^{-10}$ yr^{-1} is the present value of the Hubble parameter, and $a_1 \approx 10^{-4}$ is the cosmological scale factor at the time of radiation and matter equality. Cosmologies with relative rotation less than that would have constructive interference.

Our universe is classical with respect to rms vorticity not because the value is so small, but because it is restricted to such a narrow range of values. However, in the past the rms vorticity would have been less restricted. The purpose of the present paper is to estimate how the range of allowed values for rms vorticity would evolve as a function of the cosmological time to indicate when in the distant past our universe became classical, that is, non-quantum.

Section 113.2 discusses path integrals in quantum cosmology. Section 113.3 calculates the action. Section 113.4 takes vorticity into account, and shows that our universe was already classical at the end of inflation with regard to both vorticity and shear. As will be seen, this result is valid for a range of values for the rate at which vorticity and shear decrease as the cosmological scale factor grows with time. Section 113.5 suggests that these calculations concerning when our universe became classical might apply to other parameters in addition to vorticity and shear.

Appendix 113.6 summarizes the background cosmology for an inflation era with 50, 55, or 60 e-foldings. Appendix 113.7 makes the detailed calculations based on vorticity propagation [342, Section 6.2.1]. Appendix 113.8 gives formulas for $F(a)$, $f_\theta(a)$, $f_\ell(a)$, $f_L(a)$, and $f_\omega(a)$ from [TRP] that are required to calculate the action. Appendix 113.9 makes the detailed calculations based on vector-mode perturbation calculations [342, Chapter 10][387, Chapter 29] that include the coupling between vorticity and shear.

We take the speed of light c and Newton's gravitational constant G to be 1 throughout.

113.2 Path integrals in quantum cosmology

There are strong reasons why a theory of quantum gravity should exist, e.g.[372], and it is generally believed that such a theory exists. There are many difficulties with formulating a theory of quantum gravity, some of which are discussed in [373, 374]. Although we do not have a final theory of quantum gravity, and therefore, no universally accepted theory of quantum cosmology, we have some speculations for theories of quantum gravity, e.g. [62, 63, 64, 375, 376, 377, 19].

However, some calculations (including the present one) can be made without having a full theory of quantum gravity by using a path-integral representation because the action is most likely to dominate over the measure (which we do not know), and the action in the case of vorticity depends only weakly on the exact form of the Lagrangian.

One of the standard formulations of quantum cosmology is in terms of path integrals [123], in which an initial 3-geometry changes to some final 3-geometry along a "path" that is a 4-geometry (i.e., a spacetime, or cosmology). Thus, each "path" is one cosmology. This is related to a sum-over-histories approach [317, 165, 316, 318, 319].[2]

Although in general, the 4-geometries considered in a path integral do not have to be classical cosmologies (that is, solutions of Einstein's field equations), it is known that classical cosmologies usually dominate the path integral, and therefore, here, we shall consider only classical cosmologies in the path integral.

We further restrict cosmologies in our path integral to those cosmologies that differ from the standard cosmological model only in that they have vorticity (or, vorticity and shear in the case of vector mode perturbations). Because vorticity is a known function of cosmological time for

[2]also called decoherent histories [320, 321, 322, 324, 323, 325, 326, 327] or consistent histories [328, 329, 330, 331]

solutions of Einstein's field equations (e.g. [342, Table 6.1] in the case of vorticity propagation), it is possible to designate each "path" (cosmology) by the initial rms vorticity, the rms vorticity now, or the rms vorticity at any designated time in between.

Appendix A in [TRP] sets up a calculation of the amplitude for measuring a rotation of the universe in terms of a path-integral calculation in quantum cosmology. For the specific case in which each "path" is a classical cosmology with a specific rms vorticity, we have

$$\psi \propto \int A(\langle\omega_f\rangle)\exp[iI(\langle\omega_f\rangle)/\hbar]\mathrm{d}\,\langle\omega_f\rangle\ , \tag{113.1}$$

where $A(\langle\omega_f\rangle)$ is a slowly varying function of $\langle\omega_f\rangle$, and since for classical spacetimes, the vorticity is a known function of cosmological time, we can consider the action I to depend on the rms vorticity at any cosmological time we choose, say t_f, which we designate as $\langle\omega_f\rangle$, where $\langle\omega_f\rangle^2 \equiv \overline{\omega_f^2}$, where the average is a spatial average over the volume within the past light cone.

[TRP] calculates the action for path integrals whose endpoints were the beginning of inflation and the present time. Here, we want to do that calculation with the final endpoint at any specified value of cosmological time.

113.3 Calculating the action

The action for a perfect fluid involves both a volume integral over spacetime plus a surface term [Appendix B in TRP]. In some cases, the surface term in the formula for action could be included in an effective Lagrangian so that the surface term in the action could be eliminated, and the action could be written as only an integral over the spacetime of the effective Lagrangian.

$$I = \int (-g^{(4)})^{1/2}\tilde{L}\mathrm{d}^4x = \int_{t_i}^{t_f}\int (-g^{(4)})^{1/2}\tilde{L}\mathrm{d}^3x dt\ , \tag{113.2}$$

where t_i is the initial time (which we take to be the beginning of the inflation era), t_f is the final time, and when only classical cosmologies are considered,

$$\tilde{L} = \alpha_1 p + \alpha_2\rho + \alpha_3\Lambda \tag{113.3}$$

can be considered to be an effective Lagrangian, where p is the pressure, ρ is the density, Λ is the cosmological constant, and α_1, α_2, and α_3 are dimensionless constants of order unity. Although including vorticity or shear adds terms to the energy-momentum tensor, these terms do not contribute to the action because the one remaining term in the energy frame is trace free. Some possibilities for the parameters for the effective Lagrangian are given in table 113.1.

For an equation of state, we take

$$p = w\rho\ , \tag{113.4}$$

where $w = 1/3$ in the radiation-dominated era, and $w = 0$ in the matter-dominated era. The variation of density ρ with cosmological scale factor a is given by [342, Table 6.1]

$$\rho = \rho_1(a/a_1)^{-3(1+w)}\ , \tag{113.5}$$

where ρ_1 is the value of ρ at the boundary between the radiation era and the matter era where $a = a_1$.

We can convert the time integral to an integral over the cosmological scale factor a

$$I = \int_{a_i}^{a_f}\int \frac{(-g^{(4)})^{1/2}\tilde{L}\mathrm{d}^3x\,\mathrm{d}\,a}{\dot{a}}\ , \tag{113.6}$$

Table 113.1: Parameters for the effective Lagrangian. The parameter α is defined by the matter Lagrangian $L_{\text{matter}} = \rho + \alpha(p - \rho)$. The parameters α_1, α_2, and α_3 are defined in (113.3).

References	Including the surface term?	α	α_1	α_2	α_3	$\alpha_1 w + \alpha_2$
[161][a]	No	0	$-3/2$	$3/2$	$1/8\pi$	1
[161]	Yes	0	0	0	$-1/4\pi$	0
[334][b], [162][c]	No	1	$-1/2$	$1/2$	$1/8\pi$	$1/3$
[334], [162]	Yes	1	1	-1	$-1/4\pi$	$-2/3$

[a] [Schutz & Sorkin, 1977]
[b] [MacCallum & Taub, 1972]
[c] [Schutz, 1976]

where $\dot{a} = \mathrm{d}a/\mathrm{d}t$, a_i is the value of the cosmological scale factor at $t = t_i$, and a_f is the value of the cosmological scale factor at $t = t_f$.

We can express the volume integral in (113.6) as a product of the spatial volume and the spatial average.

$$I = \int_{a_i}^{a_f} V(a) \overline{\left(\frac{\tilde{L}}{\dot{a}}\right)} \, \mathrm{d}\, a \ , \tag{113.7}$$

where we have used the knowledge that our universe is spatially flat, an overbar indicates a spatial average,

$$V(a) = \frac{4}{3}\pi a^3 r_3^3 \tag{113.8}$$

is the approximate spatial volume,[3] and r_3 is the present radius of the cosmological horizon.

113.4 Including vorticity

If there were no vorticity, then we could use the Friedmann equation to calculate $\dot{a}$ in (113.7). However, with vorticity, the Friedmann equation, generalized to include vorticity, shear, and acceleration gives [Appendix F of TRP]

$$\dot{\ell} = \ell \sqrt{H(a)^2 + H_\omega^2 + H_\sigma^2 + H_a^2} \ , \tag{113.9}$$

where ℓ is a scale factor along lines of cosmic flow. In the presence of vorticity, a and ℓ differ. $H(a)$ [given by (113.14) in Appendix 113.6] is the Hubble parameter without vorticity, shear, or acceleration,

$$H_\omega^2 \equiv \frac{4}{3\ell^2} \int \ell \omega^2 \, \mathrm{d}\ell \tag{113.10}$$

is the vorticity term, ω is vorticity,

$$H_\sigma^2 \equiv -\frac{4}{3\ell^2} \int \ell \sigma^2 \, \mathrm{d}\ell \tag{113.11}$$

[3] Since our universe is approximately spatially flat. Although there are arguments that V should be infinite because this is an open cosmology [351], considering causality leads to restricting the spatial part of the action to the past light cone.

is the shear term, σ is shear, and

$$H_a^2 \equiv -\frac{2}{3\ell^2}\int \ell \dot{u}^a{}_{;a}\,\mathrm{d}\ell \tag{113.12}$$

is the acceleration term.

We can take vorticity to depend on the distance along flow lines ℓ as

$$\omega \propto \ell^{-m} , \tag{113.13}$$

where the value of m depends on the assumptions we make. Based on vorticity propagation [342, Section 6.2.1] for small vorticity, we can take $m = 1$ in the radiation era and $m = 2$ in the matter era [342, Table 6.1]. Appendix 113.7 makes the calculations based on vorticity propagation. These results are in Tables 113.2, 113.3, and 113.4.

On the other hand, we could instead take into account the coupling between vorticity and shear in a combined way in terms of a vector mode perturbation [342, Chapter 10][387, Chapter 29], which leads to $m = 2$ or $m = 3$. Appendix 113.9 makes the calculations based on vector mode perturbations. These results are also in Tables 113.2, 113.3, and 113.4.

As can be seen, our universe was already classical (non-quantum) at the end of inflation, before the electroweak transition. Any value of m equal or greater than one will give the same result.

Table 113.2: Cosmological scale factor a, Hubble parameter H, and maximum rms vorticity $\langle\omega_f\rangle$max as a function of global time t for an inflation era with 50 e-foldings. Everything but the last column is calculated using the formulas in Appendix 113.6. Calculations in the last column are for $m = 1$ except as noted. The $m = 1$ results are based on vorticity propagation. The $m = 2$ or $m = 3$ results are based on vector mode perturbations. Vorticity $\omega \propto a^{-m}$ for the $m = 2$ or $m = 3$ results in the last column. Vorticity $\omega \propto \ell^{-m}$ for all other results in the last column, where ℓ is a scale factor along lines of cosmic flow. Note: The 0.24-second time in the radiation era was chosen arbitrarily. Other times in the radiation era could be added using the formulas given.

era	t	a	H yr^{-1}	$\langle\omega_f\rangle$max rad/yr
beginning of inflation	10^{-36} s	$a_i = 3.9 \times 10^{-42}$		
inflation era			1.6×10^{43}	
end of inflation	10^{-34} s	$a_0 = 2 \times 10^{-20}$	1.6×10^{41}	8×10^{-20} for $m = 1$ 7×10^{-28} for $m = 2$ 4×10^{-71} for $m = 3$
electroweak transition	2.4×10^{-11} s	10^{-15}	6.6×10^{17}	10^{-47}
radiation era	0.24 s	10^{-10}	6.6×10^{7}	3×10^{-55}
radiation/ matter equality	50×10^{3} yr	$a_1 = 3 \times 10^{-4}$	10^{-5}	5.8×10^{-65}
recombination	380×10^{3} yr	$.9 \times 10^{-3}$	1.6×10^{-6}	
matter/ dark energy equality	10×10^{9} yr	$a_2 = 0.76$	8×10^{-11}	
today	13.8×10^{9} yr	$a_3 = 1.0$	6.928×10^{-11}	

Table 113.3: Cosmological scale factor a, Hubble parameter H, and maximum rms vorticity $\langle\omega_f\rangle\text{max}$ as a function of global time t for an inflation era with 55 e-foldings. Everything but the last column is calculated using the formulas in Appendix 113.6. Calculations in the last column are for $m = 1$ except as noted. The $m = 1$ results are based on vorticity propagation. The $m = 2$ or $m = 3$ results are based on vector mode perturbations. Vorticity $\omega \propto a^{-m}$ for the $m = 2$ or $m = 3$ results in the last column. Vorticity $\omega \propto \ell^{-m}$ for all other results in the last column, where ℓ is a scale factor along lines of cosmic flow. Note: The 0.24-second time in the radiation era was chosen arbitrarily. Other times in the radiation era could be added using the formulas given.

era	t	a	H yr^{-1}	$\langle\omega_f\rangle\text{max}$ rad/yr
beginning of inflation	10^{-36} s	$a_i = 2.6 \times 10^{-44}$		
inflation era			1.8×10^{43}	
end of inflation	10^{-34} s	$a_0 = 2 \times 10^{-20}$	1.6×10^{41}	6×10^{-23} for $m = 1$ 1×10^{-30} for $m = 2$ 3×10^{-78} for $m = 3$
electroweak transition	2.4×10^{-11} s	10^{-15}	6.6×10^{17}	10^{-47}
radiation era	0.24 s	10^{-10}	6.6×10^{7}	3×10^{-55}
radiation/ matter equality	50×10^{3} yr	$a_1 = 3 \times 10^{-4}$	10^{-5}	5.8×10^{-65}
recombination	380×10^{3} yr	$.9 \times 10^{-3}$	1.6×10^{-6}	
matter/ dark energy equality	10×10^{9} yr	$a_2 = 0.76$	8×10^{-11}	
today	13.8×10^{9} yr	$a_3 = 1.0$	6.928×10^{-11}	

113.5 Discussion

The calculations here show that our universe was already classical with regard to vorticity and shear at the end of inflation. The calculations also show that even without inflation our universe would have already been classical early in the radiation era.

Ashtekar et al. [388], using different methods and different criteria, reach a similar conclusion as that reached here, namely that our universe became classical very early. They also reach the conclusion that the emergence of classicality does not depend on inflation, which agrees with the calculations here. Kiefer [389] considers the emergence of classicality through decoherence.

Although the present paper deals with the emergence of classicality in a quantum universe, a related topic deals with the emergence of geometry in a quantum universe. For example, Sorkin [367] considers the emergence of geometry through causal sets.

We have shown that our universe has been classical (non-quantum) at least since the end of inflation with regard to vorticity and shear. But, is it possible to generalize that concept of classicality to other parameters in addition to vorticity and shear?

To include other parameters as perturbations on the standard Robertson-Walker cosmology, such parameters would most likely enter the Friedmann equation in the same way as vorticity, shear, and acceleration. It would then be possible to include such parameters in the same way as was done for vorticity and shear. Although this is speculation, I would be surprised if we did not

Table 113.4: Cosmological scale factor a, Hubble parameter H, and maximum rms vorticity $\langle\omega_f\rangle$max as a function of global time t for an inflation era with 60 e-foldings. Everything but the last column is calculated using the formulas in Appendix 113.6. Calculations in the last column are for $m = 1$ except as noted. The $m = 1$ results are based on vorticity propagation. The $m = 2$ or $m = 3$ results are based on vector mode perturbations. Vorticity $\omega \propto a^{-m}$ for the $m = 2$ or $m = 3$ results in the last column. Vorticity $\omega \propto \ell^{-m}$ for all other results in the last column, where ℓ is a scale factor along lines of cosmic flow. Note: The 0.24-second time in the radiation era was chosen arbitrarily. Other times in the radiation era could be added using the formulas given.

era	t	a	H yr^{-1}	$\langle\omega_f\rangle$max rad/yr
beginning of inflation	10^{-36} s	$a_i = 1.8 \times 10^{-46}$		
inflation era			1.9×10^{43}	
end of inflation	10^{-34} s	$a_0 = 2 \times 10^{-20}$	1.6×10^{41}	3×10^{-26} for $m = 1$ 1×10^{-33} for $m = 2$ 2×10^{-85} for $m = 3$
electroweak transition	2.4×10^{-11} s	10^{-15}	6.6×10^{17}	2×10^{-48}
radiation era	0.24 s	10^{-10}	6.6×10^{7}	6×10^{-56}
radiation/ matter equality	50×10^3 yr	$a_1 = 3 \times 10^{-4}$	10^{-5}	1.2×10^{-65}
recombination	380×10^3 yr	$.9 \times 10^{-3}$	1.6×10^{-6}	
matter/ dark energy equality	10×10^9 yr	$a_2 = 0.76$	8×10^{-11}	
today	13.8×10^9 yr	$a_3 = 1.0$	6.928×10^{-11}	

get similar results for other parameters. Therefore, I speculate that our universe has been classical (non-quantum) with regard to nearly all parameters since at least the end of the inflation era.

113.6 Background cosmology

We start with the formula for the Hubble parameter neglecting vorticity, shear, and acceleration

$$\frac{1}{a}\frac{da}{dt} = H = H_0\sqrt{\Omega_\Lambda + \Omega_m/a^3 + \Omega_r/a^4}\ , \tag{113.14}$$

where t is global time, $H_0 = 6.928 \times 10^{-11}$ yr^{-1} $\approx 2.195 \times 10^{-18}$ s^{-1} [390] is the present value of the Hubble parameter, $a = 1/(z+1)$ is the cosmological scale factor, whose present value is 1, z is the redshift factor, and $\Omega_\Lambda = 0.6911$ [390] is the dark energy density divided by the critical density today. Using $z_{\rm eq} = 3402$ [391] for the redshift at radiation/matter equality with $\Omega_m = 0.3089$ [390] for the matter density today divided by the critical density gives $\Omega_r = 9.077 \times 10^{-5} \approx 9 \times 10^{-5}$ for the radiation energy density divided by the critical density today. Equation (113.14) is not valid during inflation, but that era is considered later.

We can convert (113.14) into an integral to get

$$t = \frac{1}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_\Lambda a^2 + \Omega_m/a + \Omega_r/a^2}}\ . \tag{113.15}$$

Equation (113.15) is a well-defined integral to give the global time t as a function of the cosmological scale factor a. Although it is not easy to calculate in closed form, there is no region where more than two terms in the radical are significant. That allows a very good approximate evaluation of the integral in closed form. We have

$$t = \frac{2}{3H_0} \frac{\Omega_r^{3/2}}{\Omega_m^2} \left[2 - \left(2 - \frac{\Omega_m}{\Omega_r} a \right) \sqrt{1 + \frac{\Omega_m}{\Omega_r} a} \right] \text{ for } a \leq a_m \approx 10^{-2},$$

and

$$t = \frac{1}{3H_0\sqrt{\Omega_\Lambda}} \ln \frac{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda} a^{-3}} + 1}{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda} a^{-3}} - 1} \text{ for } a \geq a_m, \tag{113.16}$$

where

$$a_1 = \frac{\Omega_r}{\Omega_m} \approx 3 \times 10^{-4} \ll a_m \ll a_2 = \left(\frac{\Omega_m}{\Omega_\Lambda} \right)^{1/3} \approx 0.76 \; . \tag{113.17}$$

Using (113.14) and (113.16) allows us to make a table that gives H as a function of global time t. When the cosmological scale factor a is very small, we can make some approximations. These approximations are valid in the very early universe when our universe becomes classical.

$$a^2 \approx 2H_0\sqrt{\Omega_r}t, \text{ that is, } t \approx 2.4 \times 10^{19} a^2 \text{ seconds} \;\text{ for } a \ll \Omega_r/\Omega_m. \tag{113.18}$$

$$H \approx \frac{1}{2t} \text{ for } a \ll \Omega_r/\Omega_m. \tag{113.19}$$

For the inflation era (from 10^{-36} seconds to 10^{-34} seconds), we choose a constant value for the Hubble parameter H that will give 50, 55, or 60 e-foldings. Reference [390] estimates that there were about 50 to 60 e-foldings during inflation.

Tables 113.2, 113.3, and 113.4 give the results of these calculations for an inflation era with 50, 55, or 60 e-foldings for selected values of global time corresponding to values of the cosmological scale factor a. The formulas above can be used to fill out the table for other values of the cosmological scale factor.

113.7 Vorticity propagation

This appendix calculates the path integral based on vorticity propagation [342, Section 6.2.1]. In (113.13), for small vorticity, we can take $m = 1$ in the radiation era and $m = 2$ in the matter era [342, Table 6.1]. We neglect the shear term and the acceleration term.[4]

To find the connection between a and ℓ, we use

$$\ell \approx \int \frac{\mathrm{d}a}{\cos\theta} \tag{113.20}$$

and

$$\theta = \int_{t_i}^{t} \omega \, \mathrm{d}t = \int_{a_i}^{a} \frac{\omega}{\dot{a}} \, \mathrm{d}a \; . \tag{113.21}$$

[TRP] expands the above equations to second order in a small parameter proportional to ω_3^2/H_3^2, where ω_3 is the vorticity now and H_3 is the value of the Hubble parameter now. The formulas

[4] Because vorticity and shear are strongly coupled, neglecting shear might result in an error, however appendix 113.9 considers coupling of vorticity and shear and gives the result that agrees with the calculations here that our universe was already classical at the end of inflation with regard to both vorticity and shear.

derived by [TRP] make it straightforward to make the correct calculation for the early universe. The result from [TRP, equations (102), (103), and (106)] is

$$I \approx I_0 + \hbar \left(\frac{\langle \omega_3 \rangle}{\omega_m} \right)^2 \left(\frac{a_3}{a_1} \right) [C_I + \text{second-order terms}] \ , \tag{113.22}$$

where $a_1 \approx 3 \times 10^{-4}$ is the value of the cosmological scale factor at radiation/matter equality, $a_3 = 1$ is the present value of the cosmological scale factor, I_0 is the value of the action without vorticity,

$$\omega_m = \left(\frac{\hbar H_3}{r_3^3} \right)^{1/2} = \frac{T^*}{r_3} \sqrt{\frac{H_3}{r_3}} \approx T^* H_3^2 \approx 10^{-71} \text{rad yr}^{-1} \ , \tag{113.23}$$

r_3 is the present radius of the cosmological horizon (which we approximate by the inverse of the Hubble parameter), T^* is the Planck time, and

$$C_I(a_f) = \frac{2\pi}{9} \frac{a_3^3}{a_2^3} \int_{a_i}^{a_f} a^2 \frac{H_3 F(a)}{H(a)} \left[f_\theta(a) - f_\ell(a) + \frac{1}{2} f_L(a) - \frac{1}{2} \frac{H_3^2}{H(a)^2} f_\omega(a) \right] \mathrm{d}\, a \ , \tag{113.24}$$

where $F(a)$, $f_\theta(a)$, $f_\ell(a)$, $f_L(a)$, and $f_\omega(a)$ are defined in appendix 113.8. $a_2 \approx 0.76$ is the value of the cosmological scale factor at the transition from the matter era to the dark energy era. $H(a)$ is the Hubble parameter. Tables 113.2, 113.3, and 113.4 give the background cosmology for 50, 55, and 60 inflation-era e-foldings.

113.7.1 The path-integral in the early universe

For a_f in the inflationary era or the radiation era, we have

$$\omega_f = \omega_3 \left(\frac{a_3}{a_1} \right) \left(\frac{a_3}{a_f} \right) \ . \tag{113.25}$$

Using (113.25) in (113.22) gives

$$I \approx I_0 + \hbar \left(\frac{\langle \omega_f \rangle}{\omega_m} \right)^2 \left(\frac{a_1}{a_3} \right) \left(\frac{a_f}{a_3} \right)^2 [C_I + \text{second-order terms}] \ , \tag{113.26}$$

As can be seen, using (113.26) in (113.1) shows that (113.1) has a saddlepoint at $\langle \omega_f \rangle = 0$.

[TRP] showed that the saddlepoint is isolated from any other saddlepoints and isolated from any non-analytic points in the path integral. Therefore, we can make a saddlepoint approximation in (113.1) to give

$$\psi \propto A(0) \omega_m \sqrt{\frac{\pi}{|C_I|}} \sqrt{\frac{a_3}{a_1}} \frac{a_3}{a_f} e^{i\pi/4} \tag{113.27}$$

for

$$|\langle \omega_f \rangle| < \frac{\omega_m}{\sqrt{|C_I|}} \sqrt{\frac{a_3}{a_1}} \frac{a_3}{a_f} \approx \frac{T^* H_3^2}{\sqrt{|C_I|}} \sqrt{\frac{a_3}{a_1}} \frac{a_3}{a_f} \approx \frac{10^{-69}}{\sqrt{|C_I|}} \frac{a_3}{a_f} \text{ rad/year,} \tag{113.28}$$

$\psi \approx 0$ otherwise.

At the end of inflation

At the end of inflation, we can take $a_f = a_0 \approx 2 \times 10^{-20}$, and from (113.24) using the formulas in Appendix 113.8, we have

$$C_I(a_0) \approx \frac{5}{12} (\alpha_1 w + \alpha_2) \frac{t_0}{t_2} \beta^3 \approx 10^{-52} \beta^3 \ , \tag{113.29}$$

where

$$\beta = \frac{H_3}{H(t_0)}(\frac{a_1}{a_i})\left(\frac{a_2}{a_1}\right)^{3/2} = H_3\frac{t_0 - t_i}{N}a_1\frac{e^N}{a_0}\left(\frac{a_2}{a_1}\right)^{3/2} \approx H_3 t_0\left(\frac{a_2}{a_1}\right)^{3/2}(\frac{a_1}{a_0})\frac{e^N}{N}$$
$$= H_3 t_1\left(\frac{a_2}{a_1}\right)^{3/2}(\frac{a_0}{a_1})\frac{e^N}{N} = H_3 t_2(\frac{a_0}{a_1})\frac{e^N}{N} \approx 1.5\times 10^{-23}\frac{e^N}{N}\ , \tag{113.30}$$

and N is the number of e-foldings during inflation. Using (113.30) and (113.29) in (113.28) gives

$$|\langle\omega_f\rangle| <\approx 8.6\times 10^{10}N^{3/2}e^{-3N/2}\ . \tag{113.31}$$

The Planck Collaborative [390] estimates that $50 \le N \le 60$. $N = 50$ gives $\beta \approx 1.6\times 10^{-3}$, $N = 55$ gives $\beta \approx 0.2$, and $N = 60$ gives $\beta \approx 28.6$. The result, for the rms vorticity at the end of inflation, is

$$\begin{aligned}
&|\langle\omega_f\rangle| <\approx 8\times 10^{-20} \text{ radians per year for } N = 50 \text{ e-foldings,}\\
&|\langle\omega_f\rangle| <\approx 6\times 10^{-23} \text{ radians per year for } N = 55 \text{ e-foldings, and}\\
&|\langle\omega_f\rangle| <\approx 3\times 10^{-26} \text{ radians per year for } N = 60 \text{ e-foldings.}
\end{aligned} \tag{113.32}$$

In the radiation era

Although it is now clear that the universe was already classical at the end of inflation, it is useful to make the calculation during the radiation era. In the radiation era, (113.24) and the formulas in Appendix 113.8 gives

$$\begin{aligned}
C_I(a_f) \approx\ & \frac{5}{12}(\alpha_1 w+\alpha_2)\frac{t_0}{t_2}\beta^3 + \frac{\alpha_1 w+\alpha_2}{12}\frac{a_1^{1/2}a_f}{a_2^{3/2}}a_3^3\left\{\left[\frac{3}{2}\beta^2+3\beta+\frac{85}{2}\right]\left(1-\frac{a_0}{a_f}\right)\right.\\
&+9\beta^2\frac{a_0}{a_f}\ln\frac{a_f}{a_0} - \frac{3}{2}\beta\frac{a_f}{a_1}\left(1-\frac{a_0^2}{a_f^2}\right) - \frac{4}{3}\frac{a_f^2}{a_1^2}\left(\ln\frac{a_f}{a_1} - \frac{a_0^3}{a_f^3}\ln\frac{a_0}{a_1}\right)\\
&\left.-\frac{2}{3}\frac{a_f^2}{a_1^2}\left(\alpha_4-\frac{5}{3}\right)\left(1-\frac{a_0^3}{a_f^3}\right)\right\}\ ,
\end{aligned} \tag{113.33}$$

For $a_f \gg a_0$, we have

$$\begin{aligned}
C_I(a_f) \approx\ & \frac{\alpha_1 w+\alpha_2}{12}\frac{a_1^{1/2}a_f}{a_2^{3/2}}a_3^3\left\{5\frac{t_0}{t_2}\beta^3\frac{a_2^{3/2}}{a_1^{1/2}a_f a_3^3} + \frac{3}{2}\beta^2+3\beta+\frac{85}{2}\right.\\
&\left.-\frac{3}{2}\beta\frac{a_f}{a_1} - \frac{4}{3}\frac{a_f^2}{a_1^2}\ln\frac{a_f}{a_1} - \frac{2}{3}\frac{a_f^2}{a_1^2}\left(\alpha_4-\frac{5}{3}\right)\right\}\\
\approx\ & \frac{\alpha_1 w+\alpha_2}{12}\frac{a_1^{1/2}a_f}{a_2^{3/2}}a_3^3\left\{\frac{3}{2}\beta^2+3\beta+\frac{85}{2}-\frac{3}{2}\beta\frac{a_f}{a_1}\right.\\
&\left.+\left(\frac{10}{9}-\frac{2}{3}\alpha_4-\frac{4}{3}\ln\frac{a_f}{a_1}\right)\frac{a_f^2}{a_1^2}\right\}\ .
\end{aligned} \tag{113.34}$$

Or,

$$\begin{aligned}
C_I(a_f) &\approx \frac{\alpha_1 w+\alpha_2}{12}\frac{a_1^{1/2}a_f}{a_2^{3/2}}a_3^3\left\{42.5+\left(\frac{10}{9}-\frac{2}{3}\alpha_4\right)\frac{a_f^2}{a_1^2}\right\}\ , \text{ for } N = 50\to 55, \text{ and}\\
C_I(a_f) &\approx \frac{\alpha_1 w+\alpha_2}{12}\frac{a_1^{1/2}a_f}{a_2^{3/2}}a_3^3\,\{1355\}\ , \text{ for } N = 60.
\end{aligned} \tag{113.35}$$

Or,

$$\begin{aligned} C_I(a_f) &\approx 9.3(\alpha_1 w + \alpha_2)a_f \text{ , for } N = 50 \to 55, \text{ and} \\ C_I(a_f) &\approx 295(\alpha_1 w + \alpha_2)a_f \text{ , for } N = 60. \end{aligned} \tag{113.36}$$

Or, using (113.28) gives

$$\begin{aligned} |\langle\omega_f\rangle| < &\approx \frac{10^{-69}}{\sqrt{9.3(\alpha_1 w + \alpha_2)}} a_f^{-3/2} \text{ radians per year, for } N = 50 \to 55, \text{ and} \\ |\langle\omega_f\rangle| < &\approx \frac{10^{-69}}{\sqrt{295(\alpha_1 w + \alpha_2)}} a_f^{-3/2} \text{ radians per year, for } N = 60. \end{aligned} \tag{113.37}$$

The quantity $\alpha_1 w + \alpha_2$ is of order unity except for the one case where it is zero, as can be seen from table 113.1. For the case where it is zero, the above analysis does not apply. So, for the cases where the above analysis applies, we can write (113.37) as

$$\begin{aligned} |\langle\omega_f\rangle| < &\approx 3 \times 10^{-70} a_f^{-3/2} \text{ radians per year, for } N = 50 \to 55, \text{ and} \\ |\langle\omega_f\rangle| < &\approx 6 \times 10^{-71} a_f^{-3/2} \text{ radians per year, for } N = 60, \\ & \text{where } a_0 = 2 \times 10^{-20} \ll a_f < a_1 = 3 \times 10^{-4}. \end{aligned} \tag{113.38}$$

Clearly, for any value of a_f within the range, $|\langle\omega_f\rangle|$ is restricted to be such a small number that we can consider the universe to be classical with regard to vorticity. Notice however, that the value given for N from 50 to 55 actually corresponds to the value without inflation. Thus, even without inflation, our universe would have been classical within a fraction of a second after the initial singularity.

In the matter era or in the dark-energy era

We could also calculate $|\langle\omega_f\rangle|$ during the matter era and the dark-energy era using the formulas in [TRP], but it is not necessary since we only needed to find out when our universe became classical, and we now know that our universe was classical at the end of inflation.

113.8 Formulas for $F(a)$, $f_\theta(a)$, $f_\ell(a)$, $f_L(a)$, and $f_\omega(a)$

From [TRP, equation (29)],

$$\begin{aligned} F(a) = &\ \tfrac{3}{8\pi}(\alpha_1 w + \alpha_2)\tfrac{a_1}{a_2}\left(\tfrac{a_2}{a}\right)^4 && \text{for } a \le a_1, \\ F(a) = &\ \tfrac{3}{8\pi}\alpha_2\left(\tfrac{a_2}{a}\right)^3 && \text{for } a_1 \le a \le a_2, \\ F(a) = &\ 3\alpha_3 && \text{for } a_2 \le a \le a_3, \end{aligned} \tag{113.39}$$

where a_1 is the value of a at radiation/matter equality, a_2 is the value of a at the boundary between the matter era and the dark energy era, and $a_3 = 1$ is the present value of the cosmological scale factor.

From [TRP, equation (50)],

$$f_\theta(a) = 3\beta^2\left(1 - \frac{a_0}{a}e^{-N}\right)^2 \text{ for } a_i \le a \le a_0,$$

$$f_\theta(a) = 3\left[\beta + \left(1 - \frac{a_0}{a}\right)\left(\frac{a}{a_1}\right)\right]^2 \text{ for } a_0 \le a \le a_1,$$

$$
\begin{aligned}
f_\theta(a) &= 3\left[\beta + 3 - 2\left(\frac{a_1}{a}\right)^{1/2} - \frac{a_0}{a_1}\right]^2 \quad \text{for } a_1 \le a \le a_2, \text{ and} \\
f_\theta(a) &= 3\left[\beta + 3 - \frac{a_0}{a_1} - \left(\frac{a_1}{a_2}\right)^{1/2}\left(\frac{3}{2} + \frac{1}{2}\left(\frac{a_2}{a}\right)^2\right)\right]^2 \quad \text{for } a_2 \le a \le a_3. \qquad (113.40)
\end{aligned}
$$

We can use (113.40) in

$$
f_\ell(a) = \frac{1}{a}\int_{a_i}^{a} f_\theta(a)\,\mathrm{d}a \qquad (113.41)
$$

and neglect a_0 in some places to give a correction for [TRP, equation (64)] as

$$
\begin{aligned}
f_\ell(a) &= 3\beta^2\left(1 - 2\frac{a_0}{a}Ne^{-N} - 2\frac{a_0}{a}e^{-N}\ln\frac{a}{a_0} - \frac{a_0^2}{a^2}e^{-2N}\right) \quad \text{for } a_i \le a \le a_0, \\
f_\ell(a) &= 3\beta^2\left(2 - \frac{a_0}{a}\right) + 3\beta\frac{a}{a_1}\left(1 - \frac{a_0^2}{a^2}\right) + \frac{a^2}{a_1^2}\left(1 - \frac{a_0^3}{a^3}\right) \quad \text{for } a_0 \le a \le a_1, \\
f_\ell(a) &= f_\ell(a_1) + 3\left(\beta + 3 - \frac{a_0}{a_1}\right)^2\left(1 - \frac{a_1}{a}\right) - 24\left(\beta + 3 - \frac{a_0}{a_1}\right)\left(\sqrt{\frac{a_1}{a}} - \frac{a_1}{a}\right) \\
&-12\frac{a_1}{a}\ln\frac{a_1}{a} \quad \text{for } a_1 \le a \le a_2, \text{ and} \\
f_\ell(a) &= f_\ell(a_2) + 3\frac{a_1}{a_2}\left\{\left[\left(\beta + 3 - \frac{a_0}{a_1}\right)\sqrt{\frac{a_2}{a_1}} - \frac{3}{2}\right]^2\left(1 - \frac{a_2}{a}\right)\right. \\
&\left.+\left[\left(\beta + 3 - \frac{a_0}{a_1}\right)\sqrt{\frac{a_2}{a_1}} - \frac{3}{2}\right]\left(\frac{a_2^2}{a^2} - \frac{a_2}{a}\right) - \frac{1}{12}\left(\frac{a_2^4}{a^4} - \frac{a_2}{a}\right)\right\} \quad \text{for } a_2 \le a \le a_3. \qquad (113.42)
\end{aligned}
$$

From (113.42), we have

$$
f_\ell(a_1) = 3\beta^2\left(2 - \frac{a_0}{a_1}\right) + 3\beta\left(1 - \frac{a_0^2}{a_1^2}\right) + \left(1 - \frac{a_0^3}{a_1^3}\right) \qquad (113.43)
$$

and

$$
f_\ell(a_2) = f_\ell(a_1) + 3\left(\beta + 3 - \frac{a_0}{a_1}\right)^2\left(1 - \frac{a_1}{a_2}\right) + 72\left(\beta + 3 - \frac{a_0}{a_1}\right)\left(\sqrt{\frac{a_1}{a_2}} - \frac{a_1}{a_2}\right) + 36\frac{a_1}{a_2}\ln\frac{a_1}{a_2}\,. \qquad (113.44)
$$

From [TRP, equation (30)],

$$
\begin{aligned}
f_L(a) &= f_\ell(a_1) + 3f_\ell(a_2) - 4f_\ell(a) \quad &&\text{for } a \le a_1, \\
f_L(a) &= 3f_\ell(a_2) - 3f_\ell(a) &&\text{for } a_1 \le a \le a_2, \\
f_L(a) &= 0 &&\text{for } a_2 \le a \le a_3. \qquad (113.45)
\end{aligned}
$$

From [TRP, equation (78)],

$$
\begin{aligned}
f_\omega(a) &= 4\left(\frac{a_2}{a_1}\right)^3\left(\frac{a_1}{a}\right)^2\left[2\ln\frac{a}{a_1} + \alpha_4 - 1\right] \quad \text{for } a \le a_1 \\
f_\omega(a) &= 4\left(\frac{a_2}{a_1}\right)^3\left(\frac{a_1}{a}\right)^2\left[\alpha_4 - \left(\frac{a_1}{a}\right)^2\right] \quad \text{for } a \ge a_1\,, \qquad (113.46)
\end{aligned}
$$

where α_4 is an arbitrary constant of integration in (113.10).

113.9 Vector mode perturbations

Vorticity and shear are coupled into a vector mode [342, Chapter 10][387, Chapter 29]. Depending on what kind of assumptions are made regarding sources of vorticity and shear, we get

$$\omega_{ab} + \sigma_{ab} \propto a^{-m} \ , \tag{113.47}$$

where the value of m depends on the assumptions made. This gives

$$\omega^2 - \sigma^2 = \left(\omega_f^2 - \sigma_f^2\right)\left(\frac{a_f}{a}\right)^{2m} \ . \tag{113.48}$$

To first order, $\ell = a$, so that using (113.48) in (113.10) and (113.11) gives

$$H_\omega^2 + H_\sigma^2 = \frac{4}{3}\left(\omega_f^2 - \sigma_f^2\right)\left(\frac{a_f}{a}\right)^{2m}\left[\frac{1}{2(1-m)} + \alpha_4 a^{-2(1-m)}\right] \ , \tag{113.49}$$

where the result is not valid for $m = 1$.

We are particularly interested in values at the end of inflation. We therefore take $a_f = a_0$. In the inflation era, $H = N/(t_0 - t_i) \approx N/t_0$ is a constant, where N is the number of e-foldings during inflation. Again taking $\ell = a$ to first order in (113.9), assuming $H_\omega^2 + H_\sigma^2 \ll H$, then (113.7) becomes

$$I \approx I_0 - \hbar\frac{\langle\omega_f\rangle^2 - \langle\sigma_f\rangle^2}{\omega_{\text{ref}}^2}\left(\frac{a_0}{a_3}\right)^3 D_I \ , \tag{113.50}$$

where I_0 is the value of the action without vorticity, shear, or acceleration,

$$\omega_{\text{ref}} = \left(\frac{\hbar}{r_3^3 t_0}\right)^{1/2} = \frac{T^*}{r_3}\sqrt{\frac{1}{r_3 t_0}} \approx T^* H_3^2\sqrt{\frac{1}{H_3 t_0}} \approx 7\times 10^{-46}\text{rad yr}^{-1} \ , \tag{113.51}$$

and

$$\begin{aligned} D_I &\approx \frac{1}{3}\frac{8\pi}{3}\rho(a_0)t_0^2\frac{e^{(2m+2w)N}}{N^3}(\alpha_1 w + \alpha_2)a_3^3\left[-\frac{1}{2(m-1)(2m+3w)}\right. \\ &\left. + \frac{\alpha_4}{2+3w}a_0^{2(m-1)}e^{-2N(m-1)}\right] \\ &\approx -\frac{8\pi}{3}\rho(a_0)t_0^2\frac{e^{(2m+2w)N}}{N^3}\frac{1}{6(m-1)(2m+3w)} \ , \end{aligned} \tag{113.52}$$

since $\alpha_1 w + \alpha_2$ is roughly about one and $a_3 = 1$.

The parameters $\langle\omega_f\rangle$ and $\langle\sigma_f\rangle$ each depend on initial conditions. They are varied independently in the path integral. They each have a saddlepoint at zero. As before, they contribute significantly to the path integral only within the first Fresnel zone. That is,

$$|\langle\omega_f\rangle| <\approx \frac{\omega_{\text{ref}}}{\sqrt{|D_I|}}\left(\frac{a_3}{a_0}\right)^{3/2} \approx T^* H_3^2\sqrt{\frac{1}{|D_I| H_3 t_0}}\left(\frac{a_3}{a_0}\right)^{3/2} \tag{113.53}$$

and

$$|\langle\sigma_f\rangle| <\approx \frac{\omega_{\text{ref}}}{\sqrt{|D_I|}}\left(\frac{a_3}{a_0}\right)^{3/2} \approx T^* H_3^2\sqrt{\frac{1}{|D_I| H_3 t_0}}\left(\frac{a_3}{a_0}\right)^{3/2} \ . \tag{113.54}$$

We put in $w = 1/3$ to give

$$|\langle\omega_f\rangle| <\approx T^* H_3^2\left(\frac{N}{H_3 t_0 a_2}\right)^{3/2} e^{-(m+1/3)N}\sqrt{\frac{6a_0(m-1)(2m+1)}{a_1}} \text{ rad/year} \tag{113.55}$$

and

$$|\langle\sigma_f\rangle| <\approx T^* H_3^2 \left(\frac{N}{H_3 t_0 a_2}\right)^{3/2} e^{-(m+1/3)N} \sqrt{\frac{6a_0(m-1)(2m+1)}{a_1}} \text{ meters/year per meter.} \tag{113.56}$$

To estimate the correct value for m, we use some of the equations in [387, Chapter 29]. Specifically, using equations (29.24b), (29.40c), (29.40d), and (29.50b) from [387] gives

$$\left(\frac{\partial}{\partial\eta} + \frac{2}{a}\frac{\partial a}{\partial\eta}\right)[a(\omega_{ab}+\sigma_{ab})] = \frac{1}{a^2}\frac{\partial}{\partial\eta}\left[a^3(\omega_{ab}+\sigma_{ab})\right] = 8\pi G a^2 T^{ab}_{\text{vector}} = 0\ , \tag{113.57}$$

where $a\mathrm{d}\eta = \mathrm{d}t$, and the right hand side of (113.57) is set to zero to indicate no sources of vorticity or shear. Although equation (29.40c) from [387] sets the vorticity to zero when considering the classical situation, in quantum cosmology, there is an initial amplitude for every possible value of vorticity. Equation (113.57) shows that the vector mode is proportional to $a(\omega_{ab}+\sigma_{ab})$ and varies as a^{-2}, and shows that $a^3(\omega_{ab}+\sigma_{ab})$ does not change with time, which indicates that the most likely value for m is 3. Similar equations in [342, Chapter 10] support the same conclusions. That the vector mode varies as a^{-2} is also supported by [392]. This gives

$$|\langle\omega_f\rangle| <\approx 0.28 N^{3/2} e^{-10N/3} \text{ rad/year} \tag{113.58}$$

and

$$|\langle\sigma_f\rangle| <\approx 0.28 N^{3/2} e^{-10N/3} \text{ meters/year per meter.} \tag{113.59}$$

The result, for the rms vorticity at the end of inflation for $m = 3$, is

$$\begin{aligned}
|\langle\omega_f\rangle| &<\approx 4\times 10^{-71} \text{ radians per year for } N = 50 \text{ e-foldings,}\\
|\langle\omega_f\rangle| &<\approx 3\times 10^{-78} \text{ radians per year for } N = 55 \text{ e-foldings,}\\
|\langle\omega_f\rangle| &<\approx 2\times 10^{-85} \text{ radians per year for } N = 60 \text{ e-foldings,}
\end{aligned} \tag{113.60}$$

also shown in Tables 113.2, 113.3, and 113.4. The result, for the rms shear at the end of inflation for $m = 3$, is

$$\begin{aligned}
|\langle\sigma_f\rangle| &<\approx 4\times 10^{-71} \text{ meters/year per meter for } N = 50 \text{ e-foldings,}\\
|\langle\sigma_f\rangle| &<\approx 3\times 10^{-78} \text{ meters/year per meter for } N = 55 \text{ e-foldings,}\\
|\langle\sigma_f\rangle| &<\approx 2\times 10^{-85} \text{ meters/year per meter for } N = 60 \text{ e-foldings.}
\end{aligned} \tag{113.61}$$

Thus, our universe was very classical (non-quantum) at the end of inflation with respect to both vorticity and shear.

Notice, however, (113.55) and (113.56) show that our universe was classical at the end of inflation for any value of m greater than one. Negative values of m are unphysical.

It is not entirely clear whether $m = 2$ or $m = 3$ for vector mode perturbations, so to be more complete, for $m = 2$, we have

$$|\langle\omega_f\rangle| <\approx 0.17 N^{3/2} e^{-4N/3} \text{ rad/year} \tag{113.62}$$

and

$$|\langle\sigma_f\rangle| <\approx 0.17 N^{3/2} e^{-4N/3} \text{ meters/year per meter.} \tag{113.63}$$

For example, from (113.62) and (113.63), for the rms vorticity at the end of inflation for $m = 2$, we have

$$\begin{aligned}
|\langle\omega_f\rangle| &<\approx 7\times 10^{-28} \text{ radians per year for } N = 50 \text{ e-foldings,}\\
|\langle\omega_f\rangle| &<\approx 1\times 10^{-30} \text{ radians per year for } N = 55 \text{ e-foldings,}\\
|\langle\omega_f\rangle| &<\approx 1\times 10^{-33} \text{ radians per year for } N = 60 \text{ e-foldings,}
\end{aligned} \tag{113.64}$$

also shown in Tables 113.2, 113.3, and 113.4. The result, for the rms shear at the end of inflation for $m = 2$, is

$$\begin{aligned}
|\langle\sigma_f\rangle| &<\approx 7 \times 10^{-28} \text{ meters/year per meter for } N = 50 \text{ e-foldings,} \\
|\langle\sigma_f\rangle| &<\approx 1 \times 10^{-30} \text{ meters/year per meter for } N = 55 \text{ e-foldings,} \\
|\langle\sigma_f\rangle| &<\approx 1 \times 10^{-33} \text{ meters/year per meter for } N = 60 \text{ e-foldings.}
\end{aligned} \tag{113.65}$$

Chapter 114

A realistic quantum model[1]

abstract

A physically realistic interpretation of quantum theory is presented in which the wave function gives the average behavior of a physical field, but the actual field is continuously fluctuating, not just during "collapse of the wave function" at the time of a measurement. The faster-than-light propagation of such field fluctuations are in the same category as phase velocity because no signals can be sent.

However, because quantum theory based on a real physical field makes the same predictions as quantum theory based on a wave function that represent only knowledge, there is no known measurement that can tell us whether or not the wave function represents a physically real field.

114.1 Introduction

I think I have a realistic model for quantum theory now:

1. "Wave function collapse" is not a problem because it is just like a phase velocity, which can be faster light, but carries no information.

2. The actually quantum wave is always fluctuating and "collapsing."

3. The wave function we calculate is the average of the quantum wave that is always fluctuating.

4. The actual "collapse" occurs when the fluctuating wave happens to have all of the wave in a localized region and can interact with something in that region.

5. Looked at in the right way, the "paths" in the path-integral formalism are equivalent to "many worlds."

6. On the other hand, there are no measurements that can distinguish between these "paths" or "worlds" actually existing or whether that is just something that can be used for making calculations.

If the wave function represents a physical field, then how can we explain faster-than-light propagation in the case of wave-function collapse, including when this happens for entangled particles?

There is no general agreement on whether the wave function represents a physical field or simply a way for us to make calculations.

Can a wave function represent a physical field without violating causality during wave-function collapse?

Actually, faster-than-light propagation of a wave function during wave-function collapse is no stranger than phase speeds faster than light, since neither carry information. Thus, there is no real problem with assuming that the wave function represents a physical field. More specifically,

[1] 6 December 2021

the wave function gives the average of the physical field. The actual physical field fluctuates. The fluctuation for one quanta is about 100%. So, a quantum is continually collapsing. The collapse does not just happen at the time of measurement.

Can an experiment tell whether a wave function represents a real physical field?

Notice that the path-integral formulation of quantum theory is to a great extent equivalent to the many-worlds interpretation of quantum theory. In making a path-integral calculation, the result is independent of whether we actually have a quantum superposition of universes or whether that is simply a useful assumption for making the calculation. Therefore, there is no known experiment that can tell whether a wave function represents a real physical field.

Chapter 115

The acceleration problem[1]

abstract

Any reasonable form of quantum gravity can explain (by phase interference) why, on a large scale, inertial frames seem not to accelerate relative to the average matter distribution without the need for absolute space, finely tuned initial conditions, or giving up on independent degrees of freedom for the gravitational field. A simple saddlepoint approximation to a path-integral calculation for a perfect fluid cosmology is used to estimate the limits on the relative acceleration as a function of cosmic time for inflation rates of 50, 55, and 60 e-foldings, and for values of the dependence of relative acceleration on cosmological scale factor a as a^{-m} for various values of m.

Inflation dominates the calculation and gives a limit of relative acceleration at about 10^{-11} s of about 10^{-44} m s^{-2} for $m = 1$ and for 50 e-foldings, with tighter restrictions for larger m and for 55 and 60 e-foldings, and even tighter restrictions today.

However, even if there were no inflation, relative acceleration for $m = 1$ in the radiation era and $m = 2$ in the matter era would be restricted to be no larger than about L^*H^2, where $L^* \approx 2\times10^{-35}$ m is the Planck length, and H is the Hubble parameter, which varies from about 2×10^{10} s^{-1} at about 10^{-11} s to about 2×10^{-18} s^{-1} today.

Although the calculations are based on solutions to Einstein's field equations, the results are valid for a more general dependence of the Lagrangian on the pressure, density, and cosmological constant.

115.1 Introduction

The "acceleration problem" considered here refers to the absence of observable significant acceleration of the "average inertial frame" relative to the matter distribution in the universe. The calculations here ignore any acceleration caused by inhomogeneities in the background matter distribution because that is a different problem. The "acceleration problem" is independent of cosmic expansion acceleration represented by any scalar term Λ in the usual Einstein field equations.

We might expect an acceleration of the average inertial frame relative to the average matter distribution to show up as an anisotropy of the acceleration of the global expansion of the universe. An anisotropy of cosmic acceleration was observed [393] with an alignment nearly perpendicular to the CMB dipole, suggesting that this observed anisotropy might be anomalous. Another search for anisotropy [394] found no significant anisotropy. Further evidence for anisotropy [395] showed an anisotropy that was nearly aligned with the CMB dipole, that may also be an artifact.

[1] Submitted to General Relativity and Gravitation on 6 December 2021, rejected on 15 December. This gives the details of the calculation. Subsection 115.13.2 added later.

Although the absence of an observed acceleration of the average inertial frame relative to the matter distribution may not be surprising, there are many solutions of Einstein's field equations for General Relativity that have large-scale relative acceleration of matter and inertial frames, e.g. [396, 397, 398, 399, 400, 401, 402, 403, 404]

It is difficult to explain the absence of relative acceleration in our universe classically without absolute space (as proposed by Newton) or without assuming very finely tuned initial conditions for the universe, because in General Relativity, gravitation (including inertia) (as expressed by the metric tensor) is determined not only by the distribution of matter (in terms of the stress-energy tensor), but also by initial and boundary conditions.

Newton used his famous rotating bucket experiment to argue for the existence of absolute space. He started with a bucket nearly filled with water hanging by its handle from a rope. The surface of the water in the bucket was flat. He started the bucket rotating about a vertical axis. The water remained flat. Eventually, because of friction, the water was rotating with the bucket and the surface of the water became concave upward because of centrifugal force. According to Newton, this demonstrated the existence of absolute space because the centrifugal force on the water did not depend on the motion of the water relative to the bucket. Ernst Mach [15, page 284] pointed out that "No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick." Mach [120, 102, 122, 15] argued further that because absolute motion is unobservable, only relative motion should appear in the laws of physics. He suggested that inertia might be determined by distant matter. He argued further that the observed lack of acceleration and rotation of what we now call inertial frames relative to the "fixed stars" is not accidental.

Einstein tried to incorporate Mach's ideas in General Relativity, but only partially succeeded. Although General Relativity incorporates frame dragging, because of the arbitrariness of initial conditions, there are solutions of Einstein's field equations that have acceleration or rotation of inertial frames relative to matter. Various versions of the proposal to require inertia to be determined solely by the matter distribution have come to be known as Mach's principle. Since we now know (from General Relativity) that inertia is a gravitational force, such an implementation of Mach's principle would require that the gravitational field (or at least part of it) be determined only by its sources (matter) rather than having independent degrees of freedom (in terms of initial and boundary conditions).

If the many proposals to implement Mach's principle for General Relativity, e.g. [11, 16, 156, 109, 315, 159] were correct, then gravitation would behave very differently from the electromagnetic interaction[2], in that electric and magnetic fields are determined not only from sources (charges and currents), but also from initial and boundary conditions.

A possible explanation for such a small value of initial acceleration may come from some form of quantum gravity. That is, quantum gravity may provide the mechanism for implementing Mach's principle. [386] has shown that any reasonable form of quantum gravity can explain (in terms of phase interference) the lack of observed rotation of inertial frames relative to the matter distribution. Here, we show that any reasonable form of quantum gravity can also explain the lack of observed acceleration of inertial frames relative to the matter distribution. Making that calculation as a function of cosmic time shows that relative acceleration would be restricted to small values even as early as the electroweak transition ($\approx 2.4 \times 10^{-11}$ s).

[2] Only certain potentials permit a uniqueness theorem [405]

115.2 Path integrals in quantum cosmology

There are strong reasons why a theory of quantum gravity should exist, e.g.[406], and it is generally believed that such a theory exists. There are many difficulties with formulating a theory of quantum gravity, some of which are discussed in [373, 374]. Although we do not have a final theory of quantum gravity, and therefore, no universally accepted theory of quantum cosmology, we have some speculations for theories of quantum gravity, e.g. [62, 63, 64, 19, 375, 376, 377].

However, some calculations (including the present one) can be made without having a full theory of quantum gravity by using a path-integral representation because the action is most likely to dominate over the measure (which we do not know), and the action in the case of acceleration depends only weakly on the exact form of the Lagrangian.

It is likely that some form of quantum cosmology can explain through phase interference the lack of acceleration. One of the standard formulations of quantum cosmology is in terms of path integrals [123], in which an initial 3-geometry changes to some final 3-geometry along a "path" that is a 4-geometry (i.e., a spacetime, or cosmology). Thus, each "path" is one cosmology. This is related to a sum-over-histories approach [316, 165, 317, 318, 319].[3]

Although in general, the 4-geometries considered in a path integral do not have to be classical cosmologies (that is, solutions of Einstein's field equations), it is known that classical cosmologies usually dominate the path integral, and therefore, here, we shall consider only classical cosmologies in the path integral.

We further restrict cosmologies in our path integral to those cosmologies that differ from the standard cosmological model only in that they have acceleration of inertial frames relative to the matter distribution. We take relative acceleration to vary with cosmological scale factor a as a^{-m}, and take m to have various values. This gives relative acceleration as a known function of cosmological time for solutions of Einstein's field equations, so that it is possible to designate each "path" (cosmology) by the initial rms acceleration, the rms acceleration now, or the rms acceleration at any designated time in between. There is a complex amplitude associated with each "path", that is, with each cosmology. The complex amplitude has a magnitude and a phase, whose values can be calculated from a correct theory of quantum gravity, once that theory is known. Usually, we expect the value of the phase to dominate the path integral.

Appendix 115.7 sets up a calculation of the amplitude for measuring acceleration of the universe in terms of a path-integral calculation in quantum cosmology. For the specific case in which each "path" is a classical cosmology with a specific rms acceleration, we have

$$\psi \propto \int A(\langle \dot{u}_f \rangle) \exp[iI(\langle \dot{u}_f \rangle)/\hbar] \mathrm{d} \langle \dot{u}_f \rangle \ , \tag{115.1}$$

where $A(\langle \dot{u}_f \rangle)$ is a slowly varying function of $\langle \dot{u}_f \rangle$, and since for classical spacetimes, the acceleration is a known function of cosmological time, we can consider the action I to depend on the rms acceleration at any cosmological time we choose, say t_f, which we designate as $\langle \dot{u}_f \rangle$, where $\langle \dot{u}_f \rangle^2 \equiv \overline{\dot{u}_f^2}$, where the average is a spatial average over the volume within the past light cone.

Appendix 115.8 discusses the background cosmology, including the inflation, radiation, matter, and dark-energy eras because the effective Lagrangian would depend differently on the cosmological scale factor in the four eras.

115.3 Saddlepoint

We expect that $I(\langle \dot{u}_f \rangle)$ should be a smoothly varying function of $\langle \dot{u}_f \rangle$. Also, we expect that when the rms acceleration $\langle \dot{u}_f \rangle$ is zero, the cosmology will be the standard cosmological model (the

[3] also called decoherent histories [320, 321, 322, 323, 324, 325, 326, 327] or consistent histories [328, 329, 330, 331]

standard Robertson-Walker cosmology), and will have the action I_0 associated with that model. Further, because of the symmetry that the action will be unchanged if the acceleration changes sign at each location[4], the action $I(\langle \dot{u}_f \rangle)$ should be an even function of $\langle \dot{u}_f \rangle$. Therefore, the action $I(\langle \dot{u}_f \rangle)$ should look roughly like

$$I(\langle \dot{u}_f \rangle) \approx I_0 + \hbar \left(\frac{\langle \dot{u}_f \rangle}{\dot{u}_m} \right)^2 f_I(\langle \dot{u}_f \rangle) \ , \tag{115.2}$$

where $f_I(\langle \dot{u}_f \rangle)$ is some slowly varying dimensionless even function of $\langle \dot{u}_f \rangle$. Notice that (115.2) reduces to I_0, the action for the standard cosmological model, when the rms acceleration $\langle \dot{u}_f \rangle$ is zero, and that (115.2) is an even function of $\langle \dot{u}_f \rangle$ if $f_I(\langle \dot{u}_f \rangle)$ is an even function of $\langle \dot{u}_f \rangle$, as required. Direct calculation in the appendices shows that (115.2) is correct if $\langle \dot{u}_f \rangle$ is the rms value for $\dot{u}_f$ when taking a spatial average over the past light cone.

Comparing (115.1) and (115.2) shows that (115.1) has a saddlepoint at $\langle \dot{u}_f \rangle = 0$. If that saddlepoint is the only significant saddlepoint (and other criteria are satisfied), then the only significant contributions to the path integral (115.1) comes from values of the rms acceleration at the time t_f of

$$\langle \dot{u}_f \rangle = 0 \pm \dot{u}_m / \sqrt{f_I(0)} \ . \tag{115.3}$$

Thus, there is no doubt that (115.2) gives the correct behavior for the action for small rms acceleration, nor no doubt that (115.3) gives the limits for the rms acceleration at time t_f of cosmologies that contribute significantly to the path integral in (115.1). The only question is in the values for $\dot{u}_m$ and for the function $f_I(\langle \dot{u}_f \rangle)$.

115.4 Evaluation of the action

Estimating the value of $\dot{u}_m$ is straightforward. Estimating the function $f_I(\langle \dot{u}_f \rangle)$ is straightforward, but tedious, as is the task of estimating the dependence of the action on the rms acceleration for all values of the rms acceleration. The appendices make such estimates.

Appendix 115.9 calculates the action in terms of a 4-volume integral of the Lagrangian for a perfect fluid plus a surface term. For some cosmological models, the surface term can be absorbed into the 4-volume integral of an effective Lagrangian. It is argued that a general effective Lagrangian for solutions of Einstein's field equations can be expressed as a linear combination of pressure, density, and the cosmological constant with constant coefficients. Specifically, equation (115.35) in appendix 115.9 gives an effective Lagrangian of the form

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda \ , \tag{115.4}$$

where p is pressure, ρ is density, Λ is the cosmological constant, and α_1, α_2, and α_3 are dimensionless constants of order unity.

If there is relative acceleration of the matter distribution and the inertial frame, then density and pressure will depend on the scale length ℓ along flow lines rather than depend on the cosmological scale factor a. The significance is that the cosmological scale factor a is a function of global time, but scale length ℓ along flow lines is not normal to surfaces of constant global time. This causes the calculation to be much more complicated. Appendices 115.10 and 115.11 take this effect into account.

The main effect of acceleration on the action can be found by a straightforward calculation of the action integral. The time integral to give the action is converted to an integral over the cosmological

[4]A cosmology with an average present acceleration equal to $-\dot{u}_f$ in some given direction should be equivalent to a cosmology with an average present acceleration equal to $+\dot{u}_f$ in that same direction, and should therefore have the same action.

scale factor using a generalization of the Friedmann equation that includes acceleration (appendix 115.12). The spatial integral to give the action is approximated by averaging the integrand times the spatial volume within the past light cone, which is proportional to the cube of the radius of the visible universe. The calculation is expanded to first order in the mean square acceleration.

The result is given by (115.88) in appendix 115.13 as

$$\dot{u}_m = \left(\frac{\hbar H_f}{r_0^3 a_f^3}\right)^{1/2} = L^* \sqrt{\frac{H_f}{r_f^3}} = L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}}\ , \tag{115.5}$$

H_f is the value of the Hubble parameter at $t = t_f$, r_f is the radius of the universe at $t = t_f$, $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds, and $L^* \approx 2 \times 10^{-35}$ m is the Planck length.

Including relative acceleration in the calculation of the action (even approximately) is tedious, even though straightforward. Appendix 115.13 calculates the total action to second order in the mean square of the acceleration at $t = t_f$.

The result is given by (115.87) in appendix 115.13, which shows that to lowest order in $\langle \dot{u}_f \rangle / H_f$

$$\begin{aligned} f_I(\langle \dot{u}_f \rangle) &\approx \left[C_I(a_f) + \tfrac{\langle \dot{u}_f \rangle^2 + \sigma_a^2 / \langle \dot{u}_f \rangle^2}{H_f^2} C_{II}(a_f)\right] \\ &\approx C_I(a_f) \text{ for small } \langle \dot{u}_f \rangle\ , \end{aligned} \tag{115.6}$$

where σ_a^2 is the variance of $\dot{u}_f^2$, and $C_I(a_f)$ and $C_{II}(a_f)$ are dimensionless functions.

115.5 Saddlepoint approximation

The saddlepoint at $\langle \dot{u}_f \rangle = 0$ in (115.1) is isolated from other saddlepoints and any possible non-analytic points as shown by (115.6). The integral in (115.1) can be approximated by a saddlepoint integration to give

$$\begin{aligned} &A(0)\dot{u}_m \sqrt{\pi} / \sqrt{f_I(0)} e^{i\pi/4} \\ &\text{for } \langle \dot{u}_f \rangle < \frac{\dot{u}_m}{\sqrt{|\ f_I(0)\ |}} \approx L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \frac{1}{\sqrt{|\ f_I(0)\ |}} \approx \frac{L^*}{r_0^{3/2}} \sqrt{\frac{H_f}{a_f^3}} \frac{1}{\sqrt{|\ C_I(a_f)\ |}}, \\ &\approx 0, \text{ otherwise,} \end{aligned} \tag{115.7}$$

where $r_0 \approx 46.5 \times 10^9$ light years is the present radius of the universe and $L^* \approx 2 \times 10^{-35}$ m is the Planck length. Appendix 115.13 calculates $C_I(a_f)$, which allows estimates for allowed upper limits on the values for the rms relative acceleration $\langle \dot{u}_f \rangle$ from (115.7).

Specifically, (115.93) gives formulas to calculate $C_I(a_f)$ for each of the eras listed in Tables 115.1, 115.2, and 115.3, including the effects of inflation. In addition, (115.95) gives approximate formulas to calculate $C_I(a_f)$ that are valid for all of the eras after the end of inflation. These values for $C_I(a_f)$ are then used to give the limits on the rms relative acceleration as a function of cosmic time for inflation with 50, 55, and 60 e-foldings given in Tables 115.1, 115.2, and 115.3. Calculations are included for $m = 1$ in the radiation era and $m = 2$ in the matter era and also for $m = 2$, 3, and 4 in all eras, where m gives the dependence of relative acceleration on cosmological scale factor a as a^{-m}. As listed in Tables 115.1, 115.2, and 115.3, m_r is the value in the radiation era, and m_m is the value in the matter era.

Inflation dominates the calculation and gives a limit of relative acceleration at the electroweak transition[5] ($\approx 2.4 \times 10^{-11}$ s) of 10^{-45}, 10^{-67}, 10^{-88}, and 10^{-110} m s^{-2} for $m = 1$, 2, 3, and 4 for

[5] Whittle [407, page 230] gives $\approx 10^{-15}$ for the value of the cosmological scale factor at the electroweak transition. Using that value in (115.25) gives $\approx 2.4 \times 10^{-11}$ s for the approximate time of the electroweak transition.

50 e-foldings, with tighter restrictions for 55 and 60 e-foldings. The corresponding values at the present time are 10^{-100}, 10^{-122}, 10^{-155}, and 10^{-196} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

However, although inflation dominates the calculation, inflation is not necessary to restrict the relative acceleration to small values. Even if there were no inflation, relative acceleration would be restricted to be not much larger than about $L^* \sqrt{H/(r_0 a)^3}$, where $L^* \approx 2 \times 10^{-35}$ m is the Planck length, $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds is the approximate radius of the observable universe today, H is the Hubble parameter, which varies from about 2×10^{10} s^{-1} at the electroweak transition to $\approx 2 \times 10^{-18}$ s^{-1} today, and a is the cosmological scale factor, which varies from about 10^{-15} s^{-1} at the electroweak transition to 1 today.

These calculations used (115.97) to calculate $C_I(a_f)$ for each of the eras listed in Table 115.4, which then allowed the estimation of upper limits on the rms relative acceleration $\langle \dot{u}_f \rangle$ from (115.7). In addition, (115.113) gives approximate formulas to calculate $C_I(a_f)$ that are valid for all of the eras after the end of inflation.

Table 115.4 gives the limits on the rms relative acceleration as a function of cosmic time neglecting inflation. As shown by table 115.4, even in the absence of inflation, relative acceleration at the electroweak transition would have been limited to about 10^{-34}, 10^{-35}, 10^{-39}, and 10^{-23} m s^{-2} for $m = 1$, 2, 3, and 4. The corresponding values at the present time without inflation are 10^{-73}, 10^{-74}, 10^{-92}, and 10^{-112} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

115.6 Summary and Discussion

The calculations here show that nearly any reasonable model for quantum gravity would explain by phase interference the small observed values of relative acceleration of matter and inertial frames. The calculations giving upper limits on relative acceleration as a function of global time show that phase interference limits relative acceleration to very small values already within a fraction of a second after the initial singularity. Although inflation dominates the calculation, even without inflation, phase interference would restrict the relative acceleration to very small values.

That relative acceleration is associated with the scalar mode [387, Chapter 29] suggests that $m = 4$ should be the correct value for the dependence of relative acceleration on cosmological scale factor a as a^{-m}, but until we have a generally accepted theory of quantum gravity, it is useful to know how the results depend on various values of m.

The numerical estimates given in Tables 115.1, 115.2, 115.3, and 115.4 take $\alpha_1 w + \alpha_2$ and α_3 to be of order unity, where α_1, α_2, and α_3 are defined in (115.35), and w is defined in (115.34). However, different values for those quantities could be used in the formulas given in the appendices to take into account different models for quantum gravity.

The calculations here allow for initial conditions in the cosmologies in which the relative acceleration of matter and inertial frames can be inhomogeneous, and shows that the action depends on the rms value of the initial relative acceleration. However, the inhomogeneities in the background have been ignored. As pointed out by David Peterson [408], the actual acceleration of the Earth calculated due to Newtonian gravity from the Andromeda galaxy and our supercluster are much larger than the values for today in Tables 115.1, 115.2, 115.3, and 115.4. However, the reason that those actual accelerations are much larger than the values calculated in the tables is because phase interference in quantum cosmology causes those values in the tables to be so small. Without phase interference in quantum cosmology, the acceleration due to relative acceleration in the initial conditions might dominate.

115.7 Amplitude for measuring acceleration of the universe

The amplitude for measuring a particular value for some quantity is equal to the amplitude for measuring that value given a particular 4-geometry times the amplitude for that 4-geometry, and then we sum over all 4-geometries.

For example, following [124], the amplitude for the 3-geometry and matter field to be fixed at specified values on two spacelike hypersurfaces is

$$\langle {}^{(3)}\mathcal{G}_f, \phi_f \,|^{(3)}\, \mathcal{G}_i, \phi_i\rangle = \int \psi[{}^{(4)}\mathcal{G}, \phi]\, \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi \;\; , \tag{115.8}$$

where the integral is over all 4-geometries and field configurations that match the given values on the two spacelike hypersurfaces, and

$$\psi[{}^{(4)}\mathcal{G}, \phi] \equiv \exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right) \tag{115.9}$$

is the contribution of the 4-geometry ${}^{(4)}\mathcal{G}$ and matter field ϕ on that 4-geometry to the path integral, where $I[{}^{(4)}\mathcal{G}, \phi]$ is the action. The proper time between the two hypersurfaces is not specified. A correct theory of quantum gravity would be necessary to specify the measures $\mathcal{D}^{(4)}\mathcal{G}$ and $\mathcal{D}\phi$, but that will not be necessary for the purposes here. Hartle and Hawking [124] restricted the integration in (115.8) to compact (closed) 4-geometries, but (115.8) can be applied to open 4-geometries if that is done carefully.

Equation (115.8) is a path integral. In this case, the "path" is the sequence of 3-geometries that form the 4-geometry ${}^{(4)}\mathcal{G}$. Thus, each 4-geometry is one "path." The space in which these paths exist is often referred to as superspace, e.g. [20]. As pointed out by Hajicek [217], there are two kinds of path integrals: those in which the time is specified at the endpoints, and those in which the time is not specified. The path integral in (115.8) is the latter. References [217] and [221] consider refinements to the path integral in (115.8), but such refinements are not necessary here.

Because of diffeomorphisms, a given 4-geometry can be specified by different metrics that are connected by coordinate transformations. This makes it difficult to avoid duplications when making path integral calculations. We avoid that difficulty here by considering only simple models.

Let $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ be the amplitude that the 3-geometry was ${}^{(3)}\mathcal{G}_i$ on some initial space-like hypersurface and that the matter fields on that 3-geometry were ϕ_i. Let $\psi_f({}^{(3)}\mathcal{G}_f, \phi_f)$ be the amplitude that the 3-geometry is ${}^{(3)}\mathcal{G}_f$ on some final space-like hypersurface and that the matter fields on that 3-geometry are ϕ_f. Then, we have

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int \langle {}^{(3)}\mathcal{G}_f, \phi_f \,|^{(3)}\, \mathcal{G}_i, \phi_i\rangle \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \;\; . \tag{115.10}$$

Substituting (115.8) and (115.9) into (115.10) gives

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int\int \exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right) \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi\, \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \;\; . \tag{115.11}$$

Although in (115.11), the integration is over all possible 4-geometries, not just classical 4-geometries, the main contribution to the integral (in most cases) comes from classical 4-geometries, e.g. [220, 221]. Thus, we shall now restrict (115.11) to be an integration over classical 4-geometries. This is appropriate for our purposes, in any case, since we are trying to explain why we do not measure relative acceleration of matter and inertial frames in what appears to be a classical universe.

In principle, the idea is very simple. Any measurement to determine the inertial frame will give a result that depends on the 4-geometry. If several 4-geometries contribute significantly to an amplitude, such as in (115.11), then any measurement to determine an inertial frame might give the inertial frame corresponding to any one of those 4-geometries. However, the probability for the result being a particular inertial frame will depend on the contribution of the corresponding 4-geometry to calculations such as that in (115.11).

The condition that there are not finely tuned initial conditions is equivalent to $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ being a broad wave function. That allows us to neglect the effect of that initial wave function on the integration in the path integral in (115.11). In addition, we consider 4-geometries characterized by a parameter $\langle \dot{u}_f \rangle$ which we take to be the rms relative acceleration on the space-like hypersurface at $t = t_f$. Thus, we can rewrite (115.11) for our purposes as

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int_{-\infty}^{\infty} A(\langle \dot{u}_f \rangle) e^{iI(\langle \dot{u}_f \rangle)/\hbar} \, \mathrm{d}\langle \dot{u}_f \rangle \ , \tag{115.12}$$

where $A(\langle \dot{u}_f \rangle)$ is a slowly varying function of $\langle \dot{u}_f \rangle$, $I(\langle \dot{u}_f \rangle)$ is the action, and since for classical spacetimes, the acceleration is a known function of cosmological time, we can consider the action I to depend on the rms acceleration at any cosmological time we choose, say t_f, which we designate as $\langle \dot{u}_f \rangle$, where $\langle \dot{u}_f \rangle^2 \equiv \overline{\dot{u}_f^2}$, where the average is a spatial average over the volume within the past light cone.

The integral in (115.12) is still a path integral. In this case, each value of $\langle \dot{u}_f \rangle$ specifies one "path", in that it specifies one 4-geometry, and that specifies one sequence of 3-geometries. The space of "paths" in this case is often referred to as a mini-superspace because it is restricted to a much smaller space of 4-geometries. The parameter $\langle \dot{u}_f \rangle$, classically determined by initial conditions on the 4-geometry, represents an independent degree-of-freedom of the gravitational field.

Either a stationary-phase path or a steepest-descent path could be used when making the saddlepoint approximation [134, 333, 219], but here, we use a stationary-phase path. Halliwell [218] gives an example of a more detailed path-integral calculation of quantum gravity.

115.8 Background cosmology

We start with the formula for the Hubble parameter neglecting vorticity, shear, and acceleration

$$\frac{1}{a}\frac{da}{dt} = H(a) = H_0\sqrt{\Omega_\Lambda + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2}} = \sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{a^2}} \ , \tag{115.13}$$

where Λ is the cosmological constant, ρ is density, t is global time, $H_0 = 67.66$ km s^{-1} Mpc^{-1} = 6.92×10^{-11} yr^{-1} $\approx 2.193\times10^{-18}$ s^{-1} [409] is the present value of the Hubble parameter, $a = 1/(z+1)$ is the cosmological scale factor, whose present value is 1, z is the redshift factor, and $\Omega_\Lambda = 0.6889$ [409] is the dark energy density divided by the critical density today. Using $z_{\text{eq}} = 3387$ [409] for the redshift at radiation/matter equality with $\Omega_m = 0.311$ [409] for the matter density today divided by the critical density gives $\Omega_r = 9.181 \times 10^{-5} \approx 9 \times 10^{-5}$ for the radiation energy density divided by the critical density today.[6] $\Omega_k = 0.0$ within measurement error [409]. Equation (115.13) is not valid during inflation, but that era is considered later.

For an equation of state, we take

$$p = w\rho \ , \tag{115.14}$$

where $w = 1/3$ in the radiation-dominated era, and $w = 0$ in the matter-dominated era. The variation of density ρ with cosmological scale factor a is given by [342, Table 6.1]

$$\rho = \rho_{eq}(a/a_{eq})^{-3(1+w)} \ , \tag{115.15}$$

where ρ_{eq} is the value of ρ at the boundary between the radiation era and the matter era where $a = a_{eq}$.

[6] The values are from [409, Table 2], and include lensing and baryon acoustic oscillations.

From (115.13), we have

$$\rho = \frac{3H_0^2}{8\pi}\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right) , \tag{115.16}$$

which gives a smooth transition between the radiation era and the matter era, instead of the abrupt transition given by (115.15). We can have a smooth transition for pressure, also, by taking w in (115.14) to be given by

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m}a\right)^{-n} , \tag{115.17}$$

where n is a positive integer. The larger n is, the sharper will be the transition. However, for calculating the action, the result does not depend strongly on how smooth or sharp is the transition. Therefore, to keep the calculations simple, we take $n = 1$ to give

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m}a\right)^{-1} . \tag{115.18}$$

Putting (115.18) and (115.16) in (115.14) gives

$$p = \frac{1}{3}\frac{3H_0^2}{8\pi}\frac{\Omega_r}{a^4} . \tag{115.19}$$

Putting (115.19) and (115.16) into (115.35) gives

$$\tilde{L} = \alpha_1 p + \alpha_2\rho + \alpha_3\Lambda = \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] , \tag{115.20}$$

with $w = 1/3$. We can approximate (115.20) in different eras.

$$\begin{aligned}
\tilde{L} &= \frac{3H_0^2}{8\pi}\left[\alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] \text{ for } a \le a_m \approx 10^{-2} \\
\tilde{L} &= \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3}\right] \text{ for } a \ge a_m \approx 10^{-2}
\end{aligned} \tag{115.21}$$

We can convert (115.13) into an integral to get

$$t = \frac{1}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_\Lambda a^2 + \Omega_m/a + \Omega_r/a^2}} . \tag{115.22}$$

Equation (115.22) is a well-defined integral to give the global time t as a function of the cosmological scale factor a. Although it is not easy to calculate in closed form, there is no region where more than two terms in the radical are significant. That allows a very good approximate evaluation of the integral in closed form. We have

$$t = \frac{2}{3H_0}\frac{\Omega_r^{3/2}}{\Omega_m^2}\left[2 - \left(2 - \frac{\Omega_m}{\Omega_r}a\right)\sqrt{1 + \frac{\Omega_m}{\Omega_r}a}\right] \text{ for } a \le a_m \approx 10^{-2},$$

and

$$t = \frac{1}{3H_0\sqrt{\Omega_\Lambda}}\ln\frac{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda}a^{-3}} + 1}{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda}a^{-3}} - 1} \text{ for } a \ge a_m, \tag{115.23}$$

where

$$a_{\mathrm{eq}} = \frac{\Omega_r}{\Omega_m} \approx 3 \times 10^{-4} \ll a_m \ll \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \approx 0.76 . \tag{115.24}$$

Using (115.13) and (115.23) allows us to make a table that gives H as a function of global time t. When the cosmological scale factor a is very small, we can make some approximations. These approximations are valid in the very early universe.

$$a^2 \approx 2H_0\sqrt{\Omega_r}t, \text{ that is, } t \approx 2.4 \times 10^{19} a^2 \text{ seconds} \quad \text{for } a \ll \Omega_r/\Omega_m. \tag{115.25}$$

$$H \approx \frac{1}{2t} \text{ for } a \ll \Omega_r/\Omega_m. \tag{115.26}$$

For the inflation era (from 10^{-36} seconds to 10^{-34} seconds), we choose a constant value for the Hubble parameter H that will give 50, 55, or 60 e-foldings. Reference [409] estimates that there were about 50 to 60 e-foldings during inflation.

115.9 Action for a perfect fluid

We can take the action in (115.12) to be

$$I = \int (-g^{(4)})^{1/2} L \mathrm{d}^4 x + \frac{1}{8\pi} \int (g^{(3)})^{1/2} K \mathrm{d}^3 x \ , \tag{115.27}$$

where

$$L = L_{\text{geom}} + L_{\text{matter}} \tag{115.28}$$

is the Lagrangian, and the surface term is necessary to insure consistency if the action integral is broken into parts [183, 123]. The quantity

$$K = g^{(3)ij} K_{ij} = -\frac{1}{2} g^{(3)ij} \frac{\partial g^{(3)}_{ij}}{\partial t} \tag{115.29}$$

is the trace of the extrinsic curvature, where $g^{(3)}_{ij}$ is the 3-metric. In this example, we take the Lagrangian for the geometry as

$$L_{\text{geom}} = \frac{R^{(4)} - 2\Lambda}{16\pi} \ , \tag{115.30}$$

where $R^{(4)}$ is the four-dimensional scalar curvature and Λ is the cosmological constant.

For a perfect fluid, the energy momentum tensor is

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \ , \tag{115.31}$$

where p is the pressure, ρ is the density, and u is the 4-velocity. For solutions to Einstein's field equations

$$R^{\mu\nu} = 8\pi (T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T) + \Lambda g^{\mu\nu} \tag{115.32}$$

for a perfect fluid, (115.30) becomes

$$L_{\text{geom}} = \frac{1}{2}\rho - \frac{3}{2} p + \frac{\Lambda}{8\pi} \ , \tag{115.33}$$

and we can take the Lagrangian for the matter as

$$L_{\text{matter}} = \rho + \alpha(p - \rho) \ , \tag{115.34}$$

where α is a constant, and we can take $\alpha = 0$ [161], $\alpha = 1$ [334, 162], or $\alpha = \frac{3}{2}$ (from combining (115.31) with [20, eq. 21.33a]).

For some cosmological models, it is possible to represent K as a time-integral of some quantity. In those cases, the surface term in the formula for action could be included in an effective Lagrangian. Thus

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda \tag{115.35}$$

can be considered to be an effective Lagrangian, where α_1, α_2, and α_3 are dimensionless constants of order unity. In that case, the surface term in the action could be eliminated, and the action could be written as only an integral over the spacetime of the effective Lagrangian.

$$I = \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^4 x = \int_{t_i}^{t_f} \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^3 x \mathrm{d}t \ , \tag{115.36}$$

where t_i is the initial time (which we take to be the beginning of the inflation era), and t_f is the final time, which we choose arbitrarily to calculate the action at any specified cosmic time.

Equation (115.20) gives the effective Lagrangian for the background case, where there is no rotation or acceleration of the matter relative to the local inertial frame. When there is rotation or acceleration, the scale factor ℓ along lines of cosmic flow differ from the cosmological scale factor a. In that case, the effective Lagrangian is

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda = \frac{3H_0^2}{8\pi} \left[8\pi \alpha_3 \Omega_\Lambda + \alpha_2 \frac{\Omega_m}{\ell^3} + (\alpha_1 w + \alpha_2) \frac{\Omega_r}{\ell^4} \right] \ , \tag{115.37}$$

where ℓ is not normal to surfaces of constant global time. However, since we are considering the case where the relative acceleration is small, we can expand (115.37) to second order in a parameter ϵ that is proportional to mean square acceleration in a specified time slice. That gives

$$\tilde{L}(\ell) \approx 3H_0^2 \Omega_\Lambda \left(\alpha_3 - \frac{\alpha_1 w + \alpha_2}{8\pi} \right) + H_f^2 F(a) \left[1 + f_L(a) \epsilon + f_{LL}(a) \epsilon^2 \right] \ , \tag{115.38}$$

where $H_f = H(a_f)$, ϵ is defined in (115.43),

$$\begin{aligned} F(a) &= \frac{3}{8\pi} (\alpha_1 w + \alpha_2) \frac{H_0^2 \Omega_r}{H_f^2 a^4} \text{ for } a \leq a_N \\ F(a) &= \frac{3}{8\pi} (\alpha_1 w + \alpha_2) \frac{H(a)^2}{H_f^2} \text{ for } a \geq a_N \ , \end{aligned} \tag{115.39}$$

$$\begin{aligned} f_L(a) &= -4 f_\ell(a) \text{ for } a \leq a_N \\ f_L(a) &= -\frac{H_0^2}{H(a)^2} \left[3\frac{\Omega_m}{a^3} + 4\frac{\Omega_r}{a^4} \right] f_\ell(a) \text{ for } a \geq a_N \ , \end{aligned} \tag{115.40}$$

$$\begin{aligned} f_{LL}(a) &= \frac{a^4}{\Omega_r} \left[\left(6\frac{\Omega_m}{a^3} + 10\frac{\Omega_r}{a^4} \right) f_\ell(a)^2 - \left(3\frac{\Omega_m}{a^3} + 4\frac{\Omega_r}{a^4} \right) f_{\ell\ell}(a) \right] \text{ for } a \leq a_N \\ f_{LL}(a) &= \frac{H_0^2}{H(a)^2} \left[\left(6\frac{\Omega_m}{a^3} + 10\frac{\Omega_r}{a^4} \right) f_\ell(a)^2 - \left(3\frac{\Omega_m}{a^3} + 4\frac{\Omega_r}{a^4} \right) f_{\ell\ell}(a) \right] \\ &\text{for } a \geq a_N \ , \end{aligned} \tag{115.41}$$

and $f_\ell(a)$ and $f_{\ell\ell}(a)$ are defined in (115.64) and (115.65).

115.10 Relative acceleration

The time-integral of the acceleration gives a velocity, which when divided by the speed of light gives a dimensionless number

$$\theta = \int_0^t \dot{u}\mathrm{d}t = \int_{a_i}^a \frac{\dot{u}}{\dot{a}}\mathrm{d}a \approx \int_{a_i}^a \frac{\dot{u}}{aH(a)}\left[1 + f_H(a)\epsilon + f_{HH}(a)\epsilon^2\right]\mathrm{d}a \ , \tag{115.42}$$

where we have used (115.76) for $1/\dot{a}$, and where

$$\epsilon \equiv \frac{1}{6}\left(\frac{\dot{u}_f}{H_f}\right)^2 \tag{115.43}$$

is a dimensionless small number, $\dot{u}_f$ is the relative acceleration (a function of position) on the final time slice (which we choose arbitrarily) and $H_f = H(a_f)$ is the value of the Hubble parameter on the final time slice.

Table 115.5 shows the possibilities for $\dot{u}$ in the various eras. Using (115.63) in the formulas in table 115.5 then gives

$$\dot{u} \approx \dot{u}_f F_a(a)\left[1 + g_a(a)\epsilon + g_{aa}(a)\epsilon^2\right] \ , \tag{115.44}$$

$$\begin{aligned} F_a(a) &= \left(\frac{a_f}{a}\right)^{m_r} \text{ for } a \leq a_{eq} \text{ and } a_f \leq a_{eq} \\ F_a(a) &= \left(\frac{a_f}{a_{eq}}\right)^{m_m}\left(\frac{a_{eq}}{a}\right)^{m_r} \text{ for } a \leq a_{eq} \text{ and } a_f \geq a_{eq} \\ F_a(a) &= \left(\frac{a_f}{a}\right)^{m_m} \text{ for } a \geq a_{eq} \text{ and } a_f \geq a_{eq} \ , \end{aligned} \tag{115.45}$$

$$\begin{aligned} g_a(a) &= m_r\left[f_\ell(a_f) - f_\ell(a)\right] \text{ for } a \leq a_{eq} \text{ and } a_f \leq a_{eq} \\ g_a(a) &= m_m f_\ell(a_f) + (m_r - m_m)f_\ell(a_{eq}) - m_r f_\ell(a) \text{ for } a \leq a_{eq} \text{ and } a_f \geq a_{eq} \\ g_a(a) &= m_m\left[f_\ell(a_f) - f_\ell(a)\right] \text{ for } a \geq a_{eq} \text{ and } a_f \geq a_{eq} \ , \end{aligned} \tag{115.46}$$

and

$$\begin{aligned} & g_{aa}(a) = m_r\left[f_{\ell\ell}(a_f) - f_{\ell\ell}(a) + \frac{m_r - 1}{2}f_\ell(a_f)^2 + \frac{m_r + 1}{2}f_\ell(a)^2 - m_r f_\ell(a_f)f_\ell(a)\right] \\ & \text{for } a \leq a_{eq} \text{ and } a_f \leq a_{eq} \\ & g_{aa}(a) = m_m f_{\ell\ell}(a_f) + (m_r - m_m)f_{\ell\ell}(a_{eq}) - m_r f_{\ell\ell}(a) \\ & + \frac{m_m(m_m - 1)}{2}f_\ell(a_f)^2 + \frac{(m_r - m_m)(m_r - m_m - 1)}{2}f_\ell(a_{eq})^2 + \frac{m_r(m_r + 1)}{2}f_\ell(a)^2 \\ & + m_m(m_r - m_m)f_\ell(a_f)f_\ell(a_{eq}) - m_m m_r f_\ell(a_f)f_\ell(a) - m_r(m_r - m_m)f_\ell(a_{eq})f_\ell(a) \\ & \text{for } a \leq a_{eq} \text{ and } a_f \geq a_{eq} \\ & g_{aa}(a) = m_m\left[f_{\ell\ell}(a_f) - f_{\ell\ell}(a) + \frac{m_m - 1}{2}f_\ell(a_f)^2 + \frac{m_m + 1}{2}f_\ell(a)^2 - m_m f_\ell(a_f)f_\ell(a)\right] \\ & \text{for } a \geq a_{eq} \text{ and } a_f \geq a_{eq} \ . \end{aligned} \tag{115.47}$$

Using (115.44) into (115.42) gives

$$\theta = \sqrt{2\epsilon}\left[\sqrt{f_\theta(a)} - \frac{1}{2}f_{\theta\theta}(a)\epsilon\right] \ , \tag{115.48}$$

where

$$\sqrt{f_\theta(a)} \equiv \sqrt{3}\int_{a_i}^{a}\frac{H_f}{H(a)}\frac{F_a(a)}{a}\mathrm{d}a\ , \tag{115.49}$$

and

$$f_{\theta\theta}(a) \equiv -2\sqrt{3}\int_{a_i}^{a}\frac{H_f}{H(a)}\frac{F_a(a)}{a}\left[g_a(a)+f_H(a)\right]\mathrm{d}a\ . \tag{115.50}$$

Although it is possible to make very accurate approximations to the integral in (115.49) by keeping two terms in $H(a)$, such closed-form solutions involve either the hypergeometric function or require separate solutions for each value of m_r. Because such high accuracy in calculating the action adds very little to the calculations here, but adds a great deal of complexity, it is more useful to calculate the integral using the approximation of keeping only the dominant term in $H(a)$ in each era.

In that approximation, $H(a)$ will transition at a_{eq} from $H_0\sqrt{\Omega_r/a_{eq}^4}$ to its equal value of $H_0\sqrt{\Omega_m/a_{eq}^3}$. However, the correct value of $H(a_{eq})$ is $H_0\sqrt{\Omega_m/a_{eq}^3+\Omega_r/a_{eq}^4} = \sqrt{2}H_0\sqrt{\Omega_r/a_{eq}^4} = \sqrt{2}H_0\sqrt{\Omega_m/a_{eq}^3}$. Therefore, to be consistent with appendix 115.8, I shall use $H_0\sqrt{\Omega_r/a_{eq}^4} = H_0\sqrt{\Omega_m/a_{eq}^3} = H(a_{eq})/\sqrt{2}$.

With that approximation, we can use (115.45) in (115.49) to give

$$\begin{aligned}
&f_\theta(a) = 3\left[\frac{H_f}{H_i}\frac{e^{Nm_r}}{m_r}\left(\frac{a_f}{a_N}\right)^{m_r}\right]^2\left[1-\left(\frac{a_N}{a}\right)^{m_r}e^{-Nm_r}\right]^2\\
&\text{for } a_i \le a \le a_N \text{ and } a_f \le a_{eq}\\
&f_\theta(a) = 3\left[\frac{H_f}{H_i}\frac{e^{Nm_r}}{m_r}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\right]^2\left[1-\left(\frac{a_N}{a}\right)^{m_r}e^{-Nm_r}\right]^2\\
&\text{for } a_i \le a \le a_N \text{ and } a_f \ge a_{eq}\\
&f_\theta(a) = 3\left[\frac{H_f}{H_i}\left(\frac{a_f}{a_N}\right)^{m_r}\frac{e^{Nm_r}-1}{m_r}+\frac{H_f}{H_N}\left(\frac{a_f}{a_N}\right)^{m_r}\frac{1-\left(\frac{a_N}{a}\right)^{m_r-2}}{m_r-2}\right]^2\\
&\text{for } a_N \le a \le a_{eq} \text{ and } a_f \le a_{eq}\\
&f_\theta(a) =\\
&3\left[\frac{H_f}{H_i}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{e^{Nm_r}-1}{m_r}+\frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{1-\left(\frac{a_N}{a}\right)^{m_r-2}}{m_r-2}\right]^2\\
&\text{for } a_N \le a \le a_{eq} \text{ and } a_f \ge a_{eq}\\
&f_\theta(a) = 3\left[c_1-c_2\left(\frac{a_{eq}}{a}\right)^{m_m-3/2}\right]^2 \text{ for } a_{eq} \le a \le a_\Lambda = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \text{ and } a_f \ge a_{eq}\\
&f_\theta(a) = 3\left[c_4-c_3\left(\frac{a_\Lambda}{a}\right)^{m_m}\right]^2 \text{ for } a \ge a_\Lambda = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \text{ and } a_f \ge a_{eq}
\end{aligned} \tag{115.51}$$

where

$$H_i = H(a_i) = \frac{N}{t_N-t_i} \approx \frac{N}{t_N}\ , \tag{115.52}$$

$$H_N = H(a_N) = \frac{H_0}{a_N^2}\sqrt{\Omega_r} \approx \frac{1}{2t_N}\ , \tag{115.53}$$

$$H_{eq} = H(a_{eq}) = H_0\sqrt{\Omega_m/a_{eq}^3+\Omega_r/a_{eq}^4} = \sqrt{2}H_0\sqrt{\Omega_r/a_{eq}^4} = \sqrt{2}H_0\sqrt{\Omega_m/a_{eq}^3}\ , \tag{115.54}$$

$$H_\Lambda = H(a_\Lambda) = H_0\sqrt{\Omega_\Lambda+\Omega_m/a_\Lambda^3} = \sqrt{2}H_0\sqrt{\Omega_\Lambda} = \sqrt{2}H_0\sqrt{\Omega_m/a_\Lambda^3}\ , \tag{115.55}$$

$$c_1 = \sqrt{\frac{f_\theta(a_{eq})}{3}} + c_2 \ , \tag{115.56}$$

$$c_2 = \frac{\sqrt{2}}{m_m - 3/2} \frac{H_f}{H_{eq}} \left(\frac{a_f}{a_{eq}}\right)^{m_m} \ , \tag{115.57}$$

and

$$c_3 = \frac{\sqrt{2}}{m_m} \frac{H_f}{H_\Lambda} \left(\frac{a_f}{a_\Lambda}\right)^{m_m} \ , \tag{115.58}$$

$$c_4 = c_1 - c_2 \left(\frac{a_{eq}}{a_\Lambda}\right)^{m_m - 3/2} + c_3 \ , \tag{115.59}$$

and

$$\begin{aligned}
&f_\theta(a_{eq}) = \\
&3\left[\frac{H_f}{H_i}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{e^{Nm_r}-1}{m_r} + \frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{1-\left(\frac{a_N}{a_{eq}}\right)^{m_r-2}}{m_r-2}\right]^2 \\
&\text{for } m_r \neq 2 \ , \\
&f_\theta(a_{eq}) = 3\left[\frac{H_f}{H_i}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{e^{Nm_r}-1}{m_r} + \frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\ln\frac{a_{eq}}{a_N}\right]^2 \\
&\text{for } m_r = 2 \ , \\
&f_\theta(a_{eq}) = 3\left[\frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{1-\left(\frac{a_N}{a_{eq}}\right)^{m_r-2}}{m_r-2}\right]^2 \\
&\text{for } m_r \neq 2 \text{ neglecting inflation,} \\
&f_\theta(a_{eq}) = 3\left[\frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\ln\frac{a_{eq}}{a_N}\right]^2 \text{ for } m_r = 2 \text{ neglecting inflation.}
\end{aligned} \tag{115.60}$$

Because inflation dominates the calculation, we can approximate (115.51) by

$$\begin{aligned}
f_\theta(a) &= 3\left[\frac{H_f}{H_i}\left(\frac{a_f}{a_N}\right)^{m_r}\frac{e^{Nm_r}}{m_r}\right]^2 \text{ for } a_f \leq a_{eq} \\
f_\theta(a) &= \frac{3}{4}\frac{e^{2Nm_r}}{N^2 m_r^2} \text{ for } a_f = a_N \\
f_\theta(a) &= 3\left[\frac{H_f}{H_i}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{e^{Nm_r}}{m_r}\right]^2 \text{ for } a_f \geq a_{eq}
\end{aligned} \tag{115.61}$$

It is useful to also calculate $f_\theta(a)$ neglecting inflation. This gives

$$\begin{aligned}
&f_\theta(a) = 3\left[\frac{H_f}{H_N}\left(\frac{a_f}{a_N}\right)^{m_r}\frac{1-\left(\frac{a_N}{a}\right)^{m_r-2}}{m_r-2}\right]^2 \\
&\text{for } a_N \leq a \leq a_{eq} \text{ and } a_f \leq a_{eq} \text{ and } m_r \neq 2 \text{ and no inflation} \\
&f_\theta(a) = 3\left[\frac{H_f}{H_N}\left(\frac{a_f}{a_N}\right)^{m_r}\ln\frac{a}{a_N}\right]^2 \\
&\text{for } a_N \leq a \leq a_{eq} \text{ and } a_f \leq a_{eq} \text{ and } m_r = 2 \text{ and no inflation}
\end{aligned}$$

$$f_\theta(a) = 3\left[\frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\frac{1-\left(\frac{a_N}{a}\right)^{m_r-2}}{m_r-2}\right]^2$$

for $a_N \le a \le a_{eq}$ and $a_f \ge a_{eq}$ and $m_r \ne 2$ and no inflation

$$f_\theta(a) = 3\left[\frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\ln\frac{a}{a_N}\right]^2$$

for $a_N \le a \le a_{eq}$ and $a_f \ge a_{eq}$ and $m_r = 2$ and no inflation

$$f_\theta(a) = 3\left[c_1 - c_2\left(\frac{a_{eq}}{a}\right)^{m_m-3/2}\right]^2$$

for $a_{eq} \le a \le a_\Lambda = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}$ and $a_f \ge a_{eq}$ and no inflation

$$f_\theta(a) = 3\left[c_4 - c_3\left(\frac{a_\Lambda}{a}\right)^{m_m}\right]^2$$

for $a \ge a_\Lambda = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}$ and $a_f \ge a_{eq}$ and no inflation (115.62)

115.11 Approximate global cosmological scale factor

With acceleration, just as with rotation, the scale factor ℓ along lines of cosmic flow differs from the global cosmological scale factor a. Thus

$$\ell \equiv \int \frac{\mathrm{d}a}{\cosh\theta} \approx a[1 + f_\ell(a)\epsilon + f_{\ell\ell}(a)\epsilon^2] \ , \tag{115.63}$$

where

$$f_\ell(a) \equiv -\frac{1}{a}\int_{a_i}^{a} f_\theta(a)\mathrm{d}a \ , \tag{115.64}$$

and

$$f_{\ell\ell}(a) \equiv \frac{1}{a}\int_{a_i}^{a}\left[\frac{5}{6}f_\theta(a)^2 + \sqrt{f_\theta(a)}f_{\theta\theta}(a)\right]\mathrm{d}a \ , \tag{115.65}$$

Using (115.61) in (115.64) gives

$$f_\ell(a) \approx -3\left[\frac{H_f}{H_i}\frac{e^{Nm_r}}{m_r}\left(\frac{a_f}{a_N}\right)^{m_r}\right]^2 \text{ for } a_f \le a_{eq}$$

$$f_\ell(a) \approx -3\left[\frac{H_f}{H_i}\frac{e^{Nm_r}}{m_r}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\right]^2 \text{ for } a_f \ge a_{eq} \ , \tag{115.66}$$

Neglecting inflation, we substitute (115.62) in (115.64) to give

$$f_\ell(a) \approx$$

$$-3\left[\frac{H_f}{H_N}\frac{1}{m_r-2}\left(\frac{a_f}{a_N}\right)^{m_r}\right]^2\left[1 - \frac{a_N}{a}\left(1 - 2\frac{\left(\frac{a_N}{a}\right)^{m_r-3}-1}{m_r-3} + \frac{\left(\frac{a_N}{a}\right)^{2m_r-5}-1}{2m_r-5}\right)\right]$$

for $a_N \le a \le a_{eq}$ and $a_f \le a_{eq}$ and $m_r \ne 2$ and $m_r \ne 3$ and no inflation

$$f_\ell(a) \approx -3\left[\frac{H_f}{H_N}\left(\frac{a_f}{a_N}\right)^{m_r}\right]^2\left[\left(\ln\frac{a}{a_N}\right)^2 - 2\ln\frac{a}{a_N} + 2 - 2\frac{a_N}{a}\right]$$

for $a_N \le a \le a_{eq}$ and $a_f \le a_{eq}$ and $m_r = 2$ and no inflation

$$f_\ell(a) \approx -3\left[\frac{H_f}{H_N}\frac{1}{m_r-2}\left(\frac{a_f}{a_N}\right)^{m_r}\right]^2\left[1-\frac{a_N}{a}\left(1-2\ln\frac{a_N}{a}+\frac{\left(\frac{a_N}{a}\right)^{2m_r-5}-1}{2m_r-5}\right)\right]$$
for $a_N \le a \le a_{eq}$ and $a_f \le a_{eq}$ and $m_r = 3$ and no inflation
$$f_\ell(a) \approx -3\left[\frac{H_f}{H_N}\frac{1}{m_r-2}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\right]^2 \times$$
$$\left[1-\frac{a_N}{a}\left(1-2\frac{\left(\frac{a_N}{a}\right)^{m_r-3}-1}{m_r-3}+\frac{\left(\frac{a_N}{a}\right)^{2m_r-5}-1}{2m_r-5}\right)\right]$$
for $a_N \le a \le a_{eq}$ and $a_f \ge a_{eq}$ and $m_r \ne 2$ and $m_r \ne 3$ and no inflation
$$f_\ell(a) \approx -3\left[\frac{H_f}{H_N}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\right]^2\left[\left(\ln\frac{a}{a_N}\right)^2-2\ln\frac{a}{a_N}+2-2\frac{a_N}{a}\right]$$
for $a_N \le a \le a_{eq}$ and $a_f \ge a_{eq}$ and $m_r = 2$ and no inflation
$$f_\ell(a) \approx -3\left[\frac{H_f}{H_N}\frac{1}{m_r-2}\left(\frac{a_{eq}}{a_N}\right)^{m_r}\left(\frac{a_f}{a_{eq}}\right)^{m_m}\right]^2 \times$$
$$\left[1-\frac{a_N}{a}\left(1-2\ln\frac{a_N}{a}+\frac{\left(\frac{a_N}{a}\right)^{2m_r-5}-1}{2m_r-5}\right)\right]$$
for $a_N \le a \le a_{eq}$ and $a_f \ge a_{eq}$ and $m_r = 3$ and no inflation
$$f_\ell(a) \approx \frac{a_{eq}}{a}f_\ell(a_{eq})-3\left[c_1^2-\frac{a_{eq}}{a}\left(c_1^2-2c_1c_2\frac{\left(\frac{a_{eq}}{a}\right)^{m_m-5/2}-1}{m_m-5/2}+c_2^2\frac{\left(\frac{a_{eq}}{a}\right)^{2m_m-4}-1}{2m_m-4}\right)\right]$$
for $a_{eq} \le a \le a_\Lambda$ and $m_m \ne 2$ and no inflation
$$f_\ell(a) \approx \frac{a_{eq}}{a}f_\ell(a_{eq})-3\left[c_1^2-\frac{a_{eq}}{a}\left(c_1^2-2c_1c_2\frac{\left(\frac{a_{eq}}{a}\right)^{m_m-5/2}-1}{m_m-5/2}+c_2^2\ln\frac{a_{eq}}{a}\right)\right]$$
for $a_{eq} \le a \le a_\Lambda$ and $m_m = 2$ and no inflation
$$f_\ell(a) \approx \frac{a_\Lambda}{a}f_\ell(a_\Lambda)-3\left[c_4^2-\frac{a_\Lambda}{a}\left(c_4^2-2c_3c_4\frac{\left(\frac{a_\Lambda}{a}\right)^{m_m-1}-1}{m_m-1}+c_3^2\frac{\left(\frac{a_\Lambda}{a}\right)^{2m_m-1}-1}{2m_m-1}\right)\right]$$
for $a \ge a_\Lambda$ and no inflation . (115.67)

115.12 Approximate Generalized Friedmann equation for small acceleration

If there were no acceleration, then we could use the Friedmann equation to calculate $\dot{a}$ in (115.80). However, with acceleration, the Friedmann equation, generalized to include vorticity, shear, and acceleration can be calculated from the Raychoudhury equation to give [386, Appendix F]

$$\dot{\ell} = \ell\sqrt{H(a)^2 + H_\omega^2 + H_\sigma^2 + H_a^2}\ , \tag{115.68}$$

where ℓ is a scale factor along lines of cosmic flow. In the presence of acceleration, a and ℓ differ. $H(a)$ [given by (115.13) in Appendix 115.8] is the Hubble parameter without vorticity, shear, or acceleration,

$$H_\omega^2 \equiv \frac{4}{3\ell^2}\int \ell\omega^2\,\mathrm{d}\ell \tag{115.69}$$

is the vorticity term, ω is vorticity,

$$H_\sigma^2 \equiv -\frac{4}{3\ell^2}\int \ell\sigma^2\,\mathrm{d}\ell \tag{115.70}$$

is the shear term, σ is shear, and

$$H_a^2 \equiv \frac{2}{3\ell^2}\int \ell \dot{u}^a{}_{;a}\,\mathrm{d}\ell = \frac{2}{3\ell^2}\int \ell\ {}^{(3)}\nabla_a \dot{u}^a\,\mathrm{d}\ell + \frac{2}{3\ell^2}\int \ell \dot{u}_a \dot{u}^a\,\mathrm{d}\ell \to \frac{2}{3\ell^2}\int \ell \dot{u}_a \dot{u}^a\,\mathrm{d}\ell \qquad (115.71)$$

is the acceleration term, where ${}^{(3)}\nabla_a$ is a 3-dimensional spatial gradient.

We can take acceleration to depend on the distance along flow lines ℓ as

$$\dot{u} \propto \ell^{-m}\ , \qquad (115.72)$$

where the value of m depends on the assumptions we make.

Substituting the appropriate formula for $\dot{u}$ from table 115.5 in (115.71) gives the formulas for H_a^2 in table 115.6.

Using (115.63) in the formulas in table 115.6 then gives

$$H_a^2 \approx H_f^2\left[f_a(a)\epsilon + f_{aa}(a)\epsilon^2\right]\ , \qquad (115.73)$$

where

$$\begin{aligned}
f_a(a) &= 2\left(\frac{a_f}{a}\right)^2\left[2\ln\frac{a_f}{a} + \alpha_4 - 1\right] \text{ for } a \le a_{eq} \text{ and } a_f \le a_{eq} \text{ and } m_r = 1\\
f_a(a) &= 2\left(\frac{a_f}{a}\right)^2\left[\alpha_4 - \frac{1}{m_r - 1}\left(\frac{a_f}{a}\right)^{2(m_r-1)}\right]\\
&\text{for } a \le a_{eq} \text{ and } a_f \le a_{eq} \text{ and } m_r \ne 1\\
f_a(a) &= 2\left(\frac{a_f}{a_{eq}}\right)^{2(m_m-m_r)}\left(\frac{a_f}{a}\right)^2\left[2\ln\frac{a_f}{a} + \alpha_4 - 1\right]\\
&\text{for } a \le a_{eq} \text{ and } a_f \ge a_{eq} \text{ and } m_r = 1\\
f_a(a) &= 2\left(\frac{a_f}{a_{eq}}\right)^{2(m_m-m_r)}\left(\frac{a_f}{a}\right)^2\left[\alpha_4 - \frac{1}{m_r - 1}\left(\frac{a_f}{a}\right)^{2(m_r-1)}\right]\\
&\text{for } a \le a_{eq} \text{ and } a_f \ge a_{eq} \text{ and } m_r \ne 1\\
f_a(a) &= 2\left(\frac{a_f}{a}\right)^2\left[\alpha_4 - \frac{1}{m_m - 1}\left(\frac{a_f}{a}\right)^{2(m_m-1)}\right]\\
&\text{for } a \ge a_{eq} \text{ and } a_f \ge a_{eq} \text{ and } m_m \ne 1\ ,
\end{aligned} \qquad (115.74)$$

and

$$\begin{aligned}
f_{aa}(a) &= 4\left(\frac{a_f}{a}\right)^2\left[2\ln\frac{a_f}{a} + \alpha_4\right]\left[f_\ell(a_f) - f_\ell(a)\right]\\
&\text{for } a \le a_{eq} \text{ and } a_f \le a_{eq} \text{ and } m_r = 1\\
f_{aa}(a) &= 4\left(\frac{a_f}{a}\right)^2\left[\alpha_4 - \frac{m_r}{m_r - 1}\left(\frac{a_f}{a}\right)^{2(m_r-1)}\right]\left[f_\ell(a_f) - f_\ell(a)\right]\\
&\text{for } a \le a_{eq} \text{ and } a_f \le a_{eq} \text{ and } m_r \ne 1\\
f_{aa}(a) &= 4\left(\frac{a_f}{a_{eq}}\right)^{2(m_m-m_r)}\left(\frac{a_f}{a}\right)^2\left[2\ln\frac{a_f}{a} + \alpha_4 + m_m - m_r\right]\left[f_\ell(a_f) - f_\ell(a)\right]\\
&\text{for } a \le a_{eq} \text{ and } a_f \ge a_{eq} \text{ and } m_r = 1\\
f_{aa}(a) &= 4\left(\frac{a_f}{a_{eq}}\right)^{2(m_m-m_r)}\left(\frac{a_f}{a}\right)^2 \times
\end{aligned}$$

$$\left[\alpha_4 - \frac{m_r}{m_r - 1}\left(\frac{a_f}{a}\right)^{2(m_r-1)} + m_m - m_r\right][f_\ell(a_f) - f_\ell(a)]$$
for $a \le a_{eq}$ and $a_f \ge a_{eq}$ and $m_r \ne 1$
$$f_{aa}(a) = 4\left(\frac{a_f}{a}\right)^2\left[\alpha_4 - \frac{m_m}{m_m - 1}\left(\frac{a_f}{a}\right)^{2(m_m-1)}\right][f_\ell(a_f) - f_\ell(a)]$$
for $a \ge a_{eq}$ and $a_f \ge a_{eq}$ and $m_r \ne 1$, (115.75)

Using (115.73) and (115.63) in (115.68) gives

$$\frac{1}{\dot{a}} \approx \frac{1}{aH(a)}\left[1 + f_H(a)\epsilon + f_{HH}(a)\epsilon^2\right] , \tag{115.76}$$

where

$$f_H(a) = -\frac{1}{2}f_L(a) - f_\ell(a) - f_\theta(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2}f_a(a) \tag{115.77}$$

and

$$\begin{aligned} f_{HH}(a) = &-\frac{1}{2}f_{LL}(a) + f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) + \frac{3}{8}f_L(a)^2 \\ &+\frac{1}{2}f_L(a)f_\ell(a) + \frac{1}{2}f_L(a)f_\theta(a) + f_\ell(a)f_\theta(a) \\ &+\left(\frac{3}{4}f_L(a)f_a(a) + \frac{1}{2}f_\ell(a)f_a(a) + \frac{1}{2}f_\theta(a)f_a(a) - \frac{1}{2}f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2}f_a(a)^2\right) \\ &\times\frac{H_f^2}{H(a)^2} . \end{aligned} \tag{115.78}$$

115.13 Approximate action for small acceleration

We can convert the time integral in (115.36) to an integral over the cosmological scale factor a

$$I = \int_{a_i}^{a_f}\int \frac{(-g^{(4)})^{1/2}\tilde{L}\mathrm{d}^3x\,\mathrm{d}\,a}{\dot{a}} , \tag{115.79}$$

where $\dot{a} = \mathrm{d}a/\mathrm{d}t$, a_i is the value of the cosmological scale factor at $t = t_i$, and a_f is the value of the cosmological scale factor at $t = t_f$.

We can express the volume integral in (115.79) as a product of the spatial volume and the spatial average.

$$I = \int_{a_i}^{a_f} V(a)\overline{\left(\frac{\tilde{L}}{\dot{a}}\right)}\mathrm{d}\,a , \tag{115.80}$$

where we have used the knowledge that our universe is spatially flat, an overbar indicates a spatial average,

$$V(a) = \frac{4}{3}\pi a^3 r_0^3 \tag{115.81}$$

is the approximate spatial volume,[7] and r_0 is the present radius of the cosmological horizon.

Using (115.38) for $\tilde{L}$, (115.76) for $1/\dot{a}$, and (115.81) for $V(a)$ in (115.80) gives

$$I \approx I_0 + I_1\overline{\epsilon} + I_2\overline{\epsilon^2} , \tag{115.82}$$

[7]Since our universe is approximately spatially flat. Although there are arguments that V should be infinite because this is an open cosmology [351], considering causality leads to restricting the spatial part of the action to the past light cone.

where I_0 is the action for zero vorticity, zero shear, and zero acceleration,

$$\begin{aligned} I_1 &= 4\pi r_0^3 H_0^2 \Omega_\Lambda \left(\alpha_3 - \frac{\alpha_1 w + \alpha_2}{8\pi}\right) \int_{a_i}^{a_f} \frac{a^2 f_H(a)}{H(a)} \,\mathrm{d}\, a \\ &+ \frac{4}{3}\pi r_0^3 H_f^2 \int_{a_i}^{a_f} \frac{a^2 F(a)}{H(a)} \left[f_L(a) + f_H(a)\right] \mathrm{d}\, a \;, \end{aligned} \tag{115.83}$$

$$\begin{aligned} I_2 &= 4\pi r_0^3 H_0^2 \Omega_\Lambda \left(\alpha_3 - \frac{\alpha_1 w + \alpha_2}{8\pi}\right) \int_{a_i}^{a_f} \frac{a^2 f_{HH}(a)}{H(a)} \,\mathrm{d}\, a \\ &+ \frac{4}{3}\pi r_0^3 H_f^2 \int_{a_i}^{a_f} \frac{a^2 F(a)}{H(a)} \left[f_L(a) f_H(a) + f_{LL}(a) + f_{HH}(a)\right] \mathrm{d}\, a \;, \end{aligned} \tag{115.84}$$

$$\bar{\epsilon} = \frac{1}{6}\overline{\left(\frac{\dot{u}_f}{H_f}\right)^2} = \frac{1}{6}\left(\frac{\langle \dot{u}_f \rangle}{H_f}\right)^2 \tag{115.85}$$

is the spatial average of ϵ given by (115.43),

$$\overline{\epsilon^2} = \bar{\epsilon}^2 + \frac{1}{36}\frac{\sigma_a^2}{H_f^4} \tag{115.86}$$

is the spatial average of the square of ϵ, σ_a^2 is the variance of $\dot{u}_f^2$, and we have neglected shear and vorticity, keeping only acceleration.

We can write (115.82) as

$$I \approx I_0 + \hbar \left(\frac{\langle \dot{u}_f \rangle}{\dot{u}_m}\right)^2 \left[C_I(a_f) + \frac{\langle \dot{u}_f \rangle^2 + \sigma_a^2 / \langle \dot{u}_f \rangle^2}{H_f^2} C_{II}(a_f)\right] \;, \tag{115.87}$$

where

$$\dot{u}_m = \left(\frac{\hbar H_f}{r_0^3 a_f^3}\right)^{1/2} = \frac{L^*}{r_f}\sqrt{\frac{H_f}{r_f}} = L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \;, \tag{115.88}$$

H_f is the value of the Hubble parameter at $t = t_f$, r_f is the radius of the universe at $t = t_f$ (which we approximate by the inverse of the Hubble parameter at $t = t_f$), L^* is the Planck length,

$$\begin{aligned} C_I(a_f) &= \frac{H_0^2 \Omega_\Lambda}{H_f a_f^3} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right) \int_{a_i}^{a_f} \frac{a^2 f_H(a)}{H(a)} \,\mathrm{d}\, a \\ &+ \frac{2\pi H_f}{9 a_f^3} \int_{a_i}^{a_f} \frac{a^2 F(a)}{H(a)} \left[f_L(a) + f_H(a)\right] \mathrm{d}\, a \;, \end{aligned} \tag{115.89}$$

and

$$\begin{aligned} C_{II}(a_f) &= \frac{H_0^2 \Omega_\Lambda}{6 H_f a_f^3} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right) \int_{a_i}^{a_f} \frac{a^2 f_{HH}(a)}{H(a)} \,\mathrm{d}\, a \\ &+ \frac{\pi H_f}{27 a_f^3} \int_{a_i}^{a_f} \frac{a^2 F(a)}{H(a)} \left[f_L(a) f_H(a) + f_{LL}(a) + f_{HH}(a)\right] \mathrm{d}\, a \;. \end{aligned} \tag{115.90}$$

We can use (115.39) for $F(a)$, and (115.77) for $f_H(a)$ in (115.89) to give

$$C_I(a_f) = \frac{H_0^2 \Omega_\Lambda}{H_f a_f^3} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right)$$

$$
\begin{aligned}
&\int_{a_i}^{a_f} \frac{a^2}{H(a)} \left[-\frac{1}{2} f_L(a) - f_\ell(a) - f_\theta(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2} f_a(a)\right] \mathrm{d}\, a \\
&+\frac{\alpha_1 w + \alpha_2}{12 H_f a_f^3} \int_{a_i}^{a_f} \frac{a^2 H_0^2 \Omega_r}{H(a) a^4} \left[\frac{1}{2} f_L(a) - f_\ell(a) - f_\theta(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2} f_a(a)\right] \mathrm{d}\, a \\
&\quad \text{for } a_i \le a_f \le a_N \ , \\
&C_I(a_f) = \frac{H_0^2 \Omega_\Lambda}{H_f a_f^3} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right) \\
&\int_{a_i}^{a_f} \frac{a^2}{H(a)} \left[-\frac{1}{2} f_L(a) - f_\ell(a) - f_\theta(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2} f_a(a)\right] \mathrm{d}\, a \\
&+\frac{\alpha_1 w + \alpha_2}{12 H_f a_f^3} \int_{a_i}^{a_f} a^2 H(a) \left[\frac{1}{2} f_L(a) - f_\ell(a) - f_\theta(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2} f_a(a)\right] \mathrm{d}\, a \\
&\quad \text{for } a_f \ge a_N \ .
\end{aligned} \tag{115.91}
$$

We can use (115.39) for $F(a)$, (115.77) for $f_H(a)$, and (115.78) for $f_{HH}(a)$ in (115.90) to give

$$
\begin{aligned}
&C_{II}(a_f) = \frac{H_0^2 \Omega_\Lambda}{6 H_f a_f^3} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right) \times \\
&\int_{a_i}^{a_f} \frac{a^2}{H(a)} \left[-\frac{1}{2} f_{LL}(a) + f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6} f_\theta^2(a)\right. \\
&+\sqrt{f_\theta(a)} f_{\theta\theta}(a) + \frac{3}{8} f_L(a)^2 + \frac{1}{2} f_L(a) f_\ell(a) + \frac{1}{2} f_L(a) f_\theta(a) + f_\ell(a) f_\theta(a) \\
&+\left(\frac{3}{4} f_L(a) f_a(a) + \frac{1}{2} f_\ell(a) f_a(a) + \frac{1}{2} f_\theta(a) f_a(a) - \frac{1}{2} f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2} f_a(a)^2\right) \\
&\left.\times \frac{H_f^2}{H(a)^2}\right] \mathrm{d}\, a \\
&+\frac{\alpha_1 w + \alpha_2}{72 H_f a_f^3} \times \\
&\int_{a_i}^{a_f} \frac{H_0^2 \Omega_r}{H(a) a^2} \left[\frac{1}{2} f_{LL}(a) + f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6} f_\theta^2(a) + \sqrt{f_\theta(a)} f_{\theta\theta}(a) - \frac{1}{8} f_L(a)^2\right. \\
&-\frac{1}{2} f_L(a) f_\ell(a) - \frac{1}{2} f_L(a) f_\theta(a) + f_\ell(a) f_\theta(a) + \left(\frac{1}{4} f_L(a) f_a(a) + \frac{1}{2} f_\ell(a) f_a(a)\right. \\
&\left.\left.+\frac{1}{2} f_\theta(a) f_a(a) - \frac{1}{2} f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2} f_a(a)^2\right) \frac{H_f^2}{H(a)^2}\right] \mathrm{d}\, a \qquad \text{for } a_i \le a_f \le a_N \ , \\
&C_{II}(a_f) = \frac{H_0^2 \Omega_\Lambda}{6 H_f a_f^3} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right) \times \\
&\int_{a_i}^{a_f} \frac{a^2}{H(a)} \left[-\frac{1}{2} f_{LL}(a) + f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6} f_\theta^2(a)\right. \\
&+\sqrt{f_\theta(a)} f_{\theta\theta}(a) + \frac{3}{8} f_L(a)^2 + \frac{1}{2} f_L(a) f_\ell(a) + \frac{1}{2} f_L(a) f_\theta(a) + f_\ell(a) f_\theta(a) \\
&+\left(\frac{3}{4} f_L(a) f_a(a) + \frac{1}{2} f_\ell(a) f_a(a) + \frac{1}{2} f_\theta(a) f_a(a) - \frac{1}{2} f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2} f_a(a)^2\right) \\
&\left.\times \frac{H_f^2}{H(a)^2}\right] \mathrm{d}\, a
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha_1 w+\alpha_2}{72 H_f a_f^3}\times \\
&\int_{a_i}^{a_f} a^2 H(a)\left[\frac{1}{2} f_{LL}(a)+f_\ell^2(a)-f_{\ell\ell}(a)+\frac{5}{6} f_\theta^2(a)+\sqrt{f_\theta(a)}\, f_{\theta\theta}(a)-\frac{1}{8} f_L(a)^2\right. \\
&-\frac{1}{2} f_L(a) f_\ell(a)-\frac{1}{2} f_L(a) f_\theta(a)+f_\ell(a) f_\theta(a)+\left(\frac{1}{4} f_L(a) f_a(a)+\frac{1}{2} f_\ell(a) f_a(a)\right. \\
&\left.\left.+\frac{1}{2} f_\theta(a) f_a(a)-\frac{1}{2} f_{aa}(a)+\frac{3}{8} \frac{H_f^2}{H(a)^2} f_a(a)^2\right) \frac{H_f^2}{H(a)^2}\right] \mathrm{d}\, a \qquad \text{for } a_f \geq a_N\ . \qquad (115.92)
\end{aligned}
$$

Although relative acceleration is associated with a scalar mode perturbation [387, Chapter 29], which leads to $m = 4$, we include calculations for $m = 1$, $m = 2$, and $m = 3$ for completeness and to allow for various quantum gravity models.

115.13.1 Formulas for $C_I(a_f)$

Formulas for $C_I(a_f)$ including inflation

We can use (115.40) for $f_L(a)$, (115.66) for $f_\ell(a)$, (115.61) for $f_\theta(a)$, and (115.74) for $f_a(a)$, in (115.91) to give

$$
\begin{aligned}
&C_I(a_f)=\frac{\Omega_\Lambda a_f^4}{\Omega_r}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_1 w+\alpha_2}{12}\right)\times \\
&\left[-\frac{2}{5} f_\theta(a_f)\left(1-\left(\frac{a_N}{a_f}\right)^5 e^{-5N}\right)-\frac{\alpha_4}{7}\left(1-\left(\frac{a_N}{a_f}\right)^7 e^{-7N}\right)\right. \\
&\left.+\frac{1-\left(\frac{a_N}{a_f}\right)^{9-2m_r} e^{-(9-2m_r)N}}{(m_r-1)(9-2m_r)}\right] \\
&+\frac{\alpha_1 w+\alpha_2}{12}\frac{t_N}{N}\frac{H_0^2\Omega_r}{H_f a_f^4}\times \\
&\left[2 f_\theta(a_f)\left(\frac{a_f}{a_N}e^N-1\right)-\alpha_4\left(1-\frac{a_N}{a_f}e^{-N}\right)+\frac{1-\left(\frac{a_N}{a_f}\right)^{3-2m_r} e^{-(3-2m_r)N}}{(m_r-1)(3-2m_r)}\right] \\
&\text{for } a_i \leq a_f \leq a_N \text{ and } m_r \neq 1 \\
&C_I(a_f)=\frac{\Omega_\Lambda a_f^4}{\Omega_r}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_1 w+\alpha_2}{12}\right)\times \\
&\left[-\frac{2}{5} f_\theta(a_f)\left(1-\left(\frac{a_N}{a_f}\right)^5 e^{-5N}\right)-\frac{\alpha_4-1}{7}\left(1-\left(\frac{a_N}{a_f}\right)^7 e^{-7N}\right)\right. \\
&\left.-\frac{2}{49}-\frac{2}{7}\left(\frac{a_N}{a_f}\right)^7 e^{-7N}\left(\ln\frac{a_N}{a_f}-N-\frac{1}{7}\right)\right] \\
&+\frac{\alpha_1 w+\alpha_2}{12}\frac{t_N}{N}\frac{H_0^2\Omega_r}{H_f a_f^4}\times \\
&\left[2 f_\theta(a_f)\left(\frac{a_f}{a_N}e^N-1\right)-(\alpha_4-1)\left(1-\frac{a_N}{a_f}e^{-N}\right)-2-2\frac{a_N}{a_f}e^{-N}\left(\ln\frac{a_N}{a_f}-N-1\right)\right] \\
&\text{for } a_i \leq a_f \leq a_N \text{ and } m_r = 1\ ,
\end{aligned}
$$

$$
\begin{aligned}
& C_I(a_f) = C_I(a_N) + \frac{\Omega_\Lambda H_0^2}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right) \times \\
& \left[-\frac{2}{5} f_\theta(a_f)\left(1 - \left(\frac{a_N}{a_f}\right)^5\right) - \frac{\alpha_4}{7}\left(1 - \left(\frac{a_N}{a_f}\right)^7\right) + \frac{1 - \left(\frac{a_N}{a_f}\right)^{9-2m_r}}{(m_r-1)(9-2m_r)}\right] \\
& + \frac{\alpha_1 w + \alpha_2}{12}\left[2 f_\theta(a_f)\left(1 - \frac{a_N}{a_f}\right) - \frac{\alpha_4}{3}\left(1 - \left(\frac{a_N}{a_f}\right)^3\right) + \frac{1 - \left(\frac{a_N}{a_f}\right)^{5-2m_r}}{(m_r-1)(5-2m_r)}\right] \\
& \text{for } a_N \leq a_f \leq a_{eq} \text{ and } m_r \neq 1 \\
& C_I(a_f) = C_I(a_N) + \\
& \frac{\Omega_\Lambda H_0^2}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right)\left[-\frac{2}{5} f_\theta(a_f)\left(1 - \left(\frac{a_N}{a_f}\right)^5\right) - \frac{\alpha_4 - 1}{7}\left(1 - \left(\frac{a_N}{a_f}\right)^7\right)\right. \\
& \left. - \frac{2}{49} - \frac{2}{7}\left(\frac{a_N}{a_f}\right)^7\left(\ln\frac{a_N}{a_f} - \frac{1}{7}\right)\right] \\
& + \frac{\alpha_1 w + \alpha_2}{12} \times \\
& \left[2 f_\theta(a_f)\left(1 - \frac{a_N}{a_f}\right) - \frac{\alpha_4 - 1}{3}\left(1 - \left(\frac{a_N}{a_f}\right)^3\right) - \frac{2}{9} - \frac{2}{3}\left(\frac{a_N}{a_f}\right)^3\left(\ln\frac{a_N}{a_f} - \frac{1}{3}\right)\right] \\
& \text{for } a_N \leq a_f \leq a_{eq} \text{ and } m_r = 1\ , \\
& C_I(a_f) = C_I(a_{eq}) + \frac{\Omega_\Lambda H_0^2}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right) \times \\
& \left[-\frac{1}{3} f_\theta(a_f)\left(1 - \left(\frac{a_{eq}}{a_f}\right)^{9/2}\right) - \frac{2\alpha_4}{11}\left(1 - \left(\frac{a_{eq}}{a_f}\right)^{11/2}\right) + \frac{1 - \left(\frac{a_{eq}}{a_f}\right)^{15/2-2m_m}}{(m_m-1)(15/2-2m_m)}\right] \\
& + \frac{\alpha_1 w + \alpha_2}{12} \times \\
& \left[f_\theta(a_f)\left(1 - \left(\frac{a_{eq}}{a_f}\right)^{3/2}\right) - \frac{2\alpha_4}{5}\left(1 - \left(\frac{a_{eq}}{a_f}\right)^{5/2}\right) + \frac{1 - \left(\frac{a_{eq}}{a_f}\right)^{9/2-2m_m}}{(m_m-1)(9/2-2m_m)}\right] \\
& \text{for } a_{eq} \leq a_f \leq a_\Lambda \text{ and } m_m \neq 1\ , \\
& C_I(a_f) = C_I(a_\Lambda) + \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right)\left[-\frac{3\Omega_m}{2\Omega_\Lambda a_f^3} f_\theta(a_f)\ln\frac{a_f}{a_\Lambda} - \alpha_4\left(1 - \frac{a_\Lambda}{a_f}\right)\right. \\
& \left. + \frac{1 - \left(\frac{a_\Lambda}{a_f}\right)^{3-2m_m}}{(m_m-1)(3-2m_m)}\right] \\
& + \frac{\alpha_1 w + \alpha_2}{12}\left[\frac{3\Omega_m}{2\Omega_\Lambda a_f^3} f_\theta(a_f)\ln\frac{a_f}{a_\Lambda} - \alpha_4\left(1 - \frac{a_\Lambda}{a_f}\right) + \frac{1 - \left(\frac{a_\Lambda}{a_f}\right)^{3-2m_m}}{(m_m-1)(3-2m_m)}\right] \\
& \text{for } a_\Lambda \leq a_f \leq a_0 = 1 \text{ and } m_m \neq 1\ .
\end{aligned}
\tag{115.93}
$$

For $a_f = a_N$, and using (115.61) for $f_\theta(a_f)$ in (115.93), some terms dominate over others, so that we have

$$C_I(a_N) \approx \frac{\alpha_1 w + \alpha_2}{16} \frac{e^{(2m_r+1)N}}{N^3 m_r^2} \ . \tag{115.94}$$

For $a_f \gg a_N$, and using (115.61) for $f_\theta(a_f)$ in (115.93), some terms dominate over others, so that we have

$$\begin{aligned}
&C_I(a_f) \approx C_I(a_N) + \frac{\alpha_1 w + \alpha_2}{6} f_\theta(a_f) \approx \frac{\alpha_1 w + \alpha_2}{16} \frac{e^{2Nm_r}}{N^2 m_r^2} \left[\frac{e^N}{N} + \left(\frac{a_f}{a_N}\right)^{2m_r - 4} \right] \\
&\text{for } a_N \ll a_f \leq a_{eq} \ , \\
&C_I(a_f) \approx C_I(a_{eq}) + f_\theta(a_f) \times \\
&\left[-\frac{\Omega_\Lambda H_0^2}{3H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12} \right) \left(1 - \left(\frac{a_{eq}}{a_f}\right)^{9/2} \right) + \frac{\alpha_1 w + \alpha_2}{12} \left(1 - \left(\frac{a_{eq}}{a_f}\right)^{3/2} \right) \right] \\
&\approx \frac{e^{2Nm_r}}{(2Nm_r)^2} \left\{ \frac{\alpha_1 w + \alpha_2}{4} \left[\frac{e^N}{N} + \left(\frac{a_{eq}}{a_N}\right)^{2m_r - 4} \right] + 3 \frac{\Omega_m}{\Omega_r} \frac{a_N^4}{a_f^3} \left(\frac{a_{eq}}{a_N}\right)^{2m_r} \left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times \right. \\
&\left. \left[-\frac{\Omega_\Lambda H_0^2}{3H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12} \right) \left(1 - \left(\frac{a_{eq}}{a_f}\right)^{9/2} \right) + \frac{\alpha_1 w + \alpha_2}{12} \left(1 - \left(\frac{a_{eq}}{a_f}\right)^{3/2} \right) \right] \right\} \\
&\text{for } a_{eq} \leq a_f \leq a_\Lambda, \\
&C_I(a_f) \approx C_I(a_\Lambda) - f_\theta(a_f) \frac{\pi \alpha_3 \Omega_m}{\Omega_\Lambda a_f^3} \ln \frac{a_f}{a_\Lambda} \\
&\approx \frac{e^{2Nm_r}}{(2Nm_r)^2} \left\{ \frac{\alpha_1 w + \alpha_2}{4} \left[\frac{e^N}{N} + \left(\frac{a_{eq}}{a_N}\right)^{2m_r - 4} \right] + 3 \frac{\Omega_m}{\Omega_r} \frac{a_N^4}{a_\Lambda^3} \left(\frac{a_{eq}}{a_N}\right)^{2m_r} \left(\frac{a_\Lambda}{a_{eq}}\right)^{2m_m} \times \right. \\
&\left[-\frac{1}{3} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12} \right) \left(1 - \left(\frac{a_{eq}}{a_\Lambda}\right)^{9/2} \right) \right. \\
&\left. \left. + \frac{\alpha_1 w + \alpha_2}{12} \left(1 - \left(\frac{a_{eq}}{a_\Lambda}\right)^{3/2} \right) - \pi \alpha_3 \left(\frac{a_f}{a_\Lambda}\right)^{2m_m - 3} \ln \frac{a_f}{a_\Lambda} \right] \right\} \\
&\text{for } a_\Lambda \leq a_f \leq a_0 = 1 \ .
\end{aligned} \tag{115.95}$$

From (115.95), we have

$$\begin{aligned}
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{16} \frac{e^{2Nm_r}}{N^2 m_r^2} \frac{e^N}{N} \text{ for } m_r = 1 \text{ or } m_r = 2 \ , \\
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{16} \frac{e^{2Nm_r}}{N^2 m_r^2} \left(\frac{a_{eq}}{a_N}\right)^{2m_r - 4} \text{ for } m_r = 3 \text{ or } m_r = 4 \ .
\end{aligned} \tag{115.96}$$

Formulas for $C_I(a_f)$ neglecting inflation

For $a_N \leq a_f \leq a_\Lambda$, we can make some approximations in (115.91) to give, neglecting inflation

$$\begin{aligned}
&C_I(a_f) = \frac{H_0^2 \Omega_\Lambda}{H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12} \right) \times \\
&\left[\frac{1}{a_f^5} \int_{a_N}^{a_f} a^4 f_\ell(a) \, \mathrm{d}\, a - \frac{1}{a_f^5} \int_{a_N}^{a_f} a^4 f_\theta(a) \, \mathrm{d}\, a - \frac{1}{2a_f^9} \int_{a_N}^{a_f} a^8 f_a(a) \, \mathrm{d}\, a \right] \\
&+ \frac{\alpha_1 w + \alpha_2}{12} \left[-\frac{3}{a_f} \int_{a_N}^{a_f} f_\ell(a) \, \mathrm{d}\, a - \frac{1}{a_f} \int_{a_N}^{a_f} f_\theta(a) \, \mathrm{d}\, a - \frac{1}{2a_f^5} \int_{a_N}^{a_f} a^4 f_a(a) \, \mathrm{d}\, a \right]
\end{aligned}$$

$$
\begin{aligned}
&\text{for } a_N \le a_f \le a_{eq} \ , \\
&C_I(a_f) = C_I(a_{eq}) + \frac{H_0^2 \Omega_\Lambda}{H_f^2} \left(\frac{2\pi}{3} \alpha_3 - \frac{\alpha_2}{12} \right) \times \\
&\left[\frac{1}{2 a_f^{9/2}} \int_{a_{eq}}^{a_f} a^{7/2} f_\ell(a) \, \mathrm{d}\, a - \frac{1}{a_f^{9/2}} \int_{a_{eq}}^{a_f} a^{7/2} f_\theta(a) \, \mathrm{d}\, a - \frac{1}{2 a_f^{15/2}} \int_{a_{eq}}^{a_f} a^{13/2} f_a(a) \, \mathrm{d}\, a \right] \\
&+ \frac{\alpha_2}{12} \times \\
&\left[- \frac{5}{2 a_f^{3/2}} \int_{a_{eq}}^{a_f} a^{1/2} f_\ell(a) \, \mathrm{d}\, a - \frac{1}{a_f^{3/2}} \int_{a_{eq}}^{a_f} a^{1/2} f_\theta(a) \, \mathrm{d}\, a - \frac{1}{2 a_f^{9/2}} \int_{a_{eq}}^{a_f} a^{7/2} f_a(a) \, \mathrm{d}\, a \right] \\
&\text{for } a_{eq} \le a_f \le a_\Lambda \ , \\
&C_I(a_f) \approx C_I(a_\Lambda) \\
&+ \frac{2\pi H_0}{3 H_f} \alpha_3 \left[\frac{3\Omega_m - 2}{2 a_f^3} \int_{a_\Lambda}^{a_f} a^2 f_\ell(a) \, \mathrm{d}\, a - \frac{1}{a_f^3} \int_{a_\Lambda}^{a_f} a^2 f_\theta(a) \, \mathrm{d}\, a - \frac{1}{2 a_f^3} \int_{a_\Lambda}^{a_f} a^2 f_a(a) \, \mathrm{d}\, a \right] \\
&- \frac{\alpha_2}{12} \frac{H_0}{H_f} \left[\frac{3\Omega_m}{a_f^3} \int_{a_\Lambda}^{a_f} a^2 f_\ell(a) \, \mathrm{d}\, a \right] \\
&\text{for } a_\Lambda \le a_f \le a_0 = 1 \ ,
\end{aligned}
\qquad (115.97)
$$

where

$$
\begin{aligned}
&\frac{1}{a_f^5} \int_{a_N}^{a_f} a^4 f_\ell(a) \, \mathrm{d}\, a = - \frac{3}{(m_r - 2)^2} \left(\frac{a_f}{a_N} \right)^{2m_r - 4} \left[\frac{1}{5} \left(1 - \left(\frac{a_N}{a_f} \right)^5 \right) \right. \\
&- \frac{1}{4} \left(1 + \frac{2}{m_r - 3} - \frac{1}{2m_r - 5} \right) \left(\frac{a_N}{a_f} - \left(\frac{a_N}{a_f} \right)^5 \right) \\
&\left. + 2 \frac{\left(\frac{a_N}{a_f} \right)^{m_r - 2} - \left(\frac{a_N}{a_f} \right)^5}{(m_r - 3)(7 - m_r)} - \frac{\left(\frac{a_N}{a_f} \right)^{2m_r - 4} - \left(\frac{a_N}{a_f} \right)^5}{(2m_r - 5)(9 - 2m_r)} \right] \\
&\text{for } a_N \le a_f \le a_{eq} \text{ and } m_r \neq 2 \text{ and } m_r \neq 3 \text{ and no inflation,} \\
&\frac{1}{a_f^5} \int_{a_N}^{a_f} a^4 f_\ell(a) \, \mathrm{d}\, a = - \frac{3}{5} \left(\ln \frac{a_f}{a_N} \right)^2 + \frac{36}{25} \ln \frac{a_f}{a_N} - \frac{372}{250} + \frac{3}{2} \frac{a_N}{a_f} - \frac{3}{250} \left(\frac{a_N}{a_f} \right)^5 \\
&\text{for } a_N \le a_f \le a_{eq} \text{ and } m_r = 2 \text{ and no inflation,} \\
&\frac{1}{a_f^5} \int_{a_N}^{a_f} a^4 f_\ell(a) \, \mathrm{d}\, a = - \frac{3}{5} \left(\frac{a_f}{a_N} \right)^2 + \frac{3}{2} \left(\ln \frac{a_f}{a_N} - \frac{1}{4} \right) \frac{a_f}{a_N} + 1 - \frac{1}{40} \left(\frac{a_N}{a_f} \right)^3 \\
&\text{for } a_N \le a_f \le a_{eq} \text{ and } m_r = 3 \text{ and no inflation,}
\end{aligned}
\qquad (115.98)
$$

$$
\begin{aligned}
&\frac{1}{a_f^5} \int_{a_N}^{a_f} a^4 f_\theta(a) \, \mathrm{d}\, a = \frac{3}{5} \left(\ln \frac{a_f}{a_N} \right)^2 - \frac{6}{25} \ln \frac{a_f}{a_N} + \frac{6}{125} - \frac{6}{125} \left(\frac{a_N}{a_f} \right)^5 \\
&\text{for } a_N \le a_f \le a_{eq} \text{ and } m_r = 2 \text{ and no inflation,} \\
&\frac{1}{a_f^5} \int_{a_N}^{a_f} a^4 f_\theta(a) \, \mathrm{d}\, a = \frac{3}{(m_r - 2)^2} \left(\frac{a_f}{a_N} \right)^{2m_r - 4} \left[\frac{1}{5} + \frac{2}{7 - m_r} - \frac{1}{9 - 2m_r} \right.
\end{aligned}
$$

$$
-\frac{2}{7-m_r}\left(\frac{a_N}{a_f}\right)^{m_r-2}+\frac{1}{9-2m_r}\left(\frac{a_N}{a_f}\right)^{2m_r-4}-\frac{1}{5}\left(\frac{a_N}{a_f}\right)^{5}\Bigg]
$$

for $a_N \le a_f \le a_{eq}$ and $m_r \neq 2$ and no inflation, (115.99)

$$
\frac{1}{2a_f^9}\int_{a_N}^{a_f} a^8 f_a(a)\,\mathrm{d}\,a = -\frac{2}{7}\ln\frac{a_f}{a_N}
$$

for $a_N \le a_f \le a_{eq}$ and $m_r = 1$ and no inflation,

$$
\frac{1}{2a_f^9}\int_{a_N}^{a_f} a^8 f_a(a)\,\mathrm{d}\,a = \frac{\alpha_4}{7}\left(1-\left(\frac{a_N}{a_f}\right)^7\right)-\frac{1-\left(\frac{a_N}{a_f}\right)^{9-2m_r}}{(m_r-1)(9-2m_r)}
$$

for $a_N \le a_f \le a_{eq}$ and $m_r \neq 1$ and no inflation, (115.100)

$$
\frac{3}{a_f}\int_{a_N}^{a_f} f_\ell(a)\,\mathrm{d}\,a = -\frac{9}{(m_r-2)^2}\left(\frac{a_f}{a_N}\right)^{2m_r-5}\Bigg[\frac{a_f}{a_N}-1
$$

$$
-\left(1+\frac{2}{m_r-3}-\frac{1}{2m_r-5}\right)\ln\frac{a_f}{a_N}+2\frac{1-\left(\frac{a_N}{a_f}\right)^{m_r-3}}{(m_r-3)^2}-\frac{1-\left(\frac{a_N}{a_f}\right)^{2m_r-5}}{(2m_r-5)^2}\Bigg]
$$

for $a_N \le a_f \le a_{eq}$ and $m_r \neq 2$ and $m_r \neq 3$ and no inflation,

$$
\frac{3}{a_f}\int_{a_N}^{a_f} f_\ell(a)\,\mathrm{d}\,a = -9\left(\ln\frac{a_f}{a_N}\right)^2+36\ln\frac{a_f}{a_N}-54+54\frac{a_N}{a_f}+18\frac{a_N}{a_f}\ln\frac{a_f}{a_N}
$$

for $a_N \le a_f \le a_{eq}$ and $m_r = 2$ and no inflation,

$$
\frac{3}{a_f}\int_{a_N}^{a_f} f_\ell(a)\,\mathrm{d}\,a = -9\left(\frac{a_f}{a_N}\right)^2+9\left(\ln\frac{a_f}{a_N}+2\right)\frac{a_f}{a_N}\ln\frac{a_f}{a_N}
$$

for $a_N \le a_f \le a_{eq}$ and $m_r = 3$ and no inflation, (115.101)

$$
\frac{1}{a_f}\int_{a_N}^{a_f} f_\theta(a)\,\mathrm{d}\,a = 3\left(\ln\frac{a_f}{a_N}\right)^2-6\ln\frac{a_f}{a_N}+6-6\frac{a_N}{a_f}
$$

for $a_N \le a_f \le a_{eq}$ and $m_r = 2$ and no inflation,

$$
\frac{1}{a_f}\int_{a_N}^{a_f} f_\theta(a)\,\mathrm{d}\,a = 3\frac{a_f}{a_N}\left[\frac{a_f}{a_N}-2\frac{a_N}{a_f}\ln\frac{a_f}{a_N}-\frac{a_N}{a_f}\right]
$$

for $a_N \le a_f \le a_{eq}$ and $m_r = 3$ and no inflation,

$$
\frac{1}{a_f}\int_{a_N}^{a_f} f_\theta(a)\,\mathrm{d}\,a = \frac{3}{(m_r-2)^2}\left(\frac{a_f}{a_N}\right)^{2m_r-5}\times
$$

$$
\left[\frac{a_f}{a_N}-1-2\frac{1-\left(\frac{a_N}{a_f}\right)^{m_r-3}}{(m_r-3)}+\frac{1-\left(\frac{a_N}{a_f}\right)^{2m_r-5}}{(2m_r-5)}\right]
$$

for $a_N \le a_f \le a_{eq}$ and $m_r \neq 2$ and $m_r \neq 3$ and no inflation, (115.102)

$$
\frac{1}{2a_f^5}\int_{a_N}^{a_f} a^4 f_a(a)\,\mathrm{d}\,a = -\frac{1}{9}+\frac{1}{3}\left(\frac{a_N}{a_f}\right)^3\left(\ln\frac{a_f}{a_N}+\frac{1}{3}\right)+\frac{\alpha_4-1}{3}\left(1-\left(\frac{a_N}{a_f}\right)^3\right)
$$

for $a_N \le a_f \le a_{eq}$ and $m_r = 1$ and no inflation,

$$\frac{1}{2a_f^5}\int_{a_N}^{a_f} a^4 f_a(a)\,\mathrm{d}\,a = \frac{\alpha_4}{3}\left(1-\left(\frac{a_N}{a_f}\right)^3\right) + \frac{1-\left(\frac{a_N}{a_f}\right)^{5-2m_r}}{(m_r-1)(2m_r-5)}$$

for $a_N \le a_f \le a_{eq}$ and $m_r \neq 1$ and no inflation, (115.103)

$$\frac{1}{2a_f^{9/2}}\int_{a_{eq}}^{a_f} a^{7/2} f_\ell(a)\,\mathrm{d}\,a =$$

$$\frac{1}{7}\left(f_\ell(a_{eq}) + 3c_1^2 + \frac{6c_1c_2}{m_m-5/2} - \frac{3c_2^2}{2m_m-4}\right)\left(\frac{a_{eq}}{a_f} - \left(\frac{a_{eq}}{a_f}\right)^{9/2}\right)$$

$$-\frac{c_1^2}{3}\left(1-\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right) - 3c_1c_2\frac{\left(\frac{a_{eq}}{a_f}\right)^{m_m-3/2} - \left(\frac{a_{eq}}{a_f}\right)^{9/2}}{(m_m-5/2)(6-m_m)}$$

$$+\frac{3}{2}c_2^2\frac{\left(\frac{a_{eq}}{a_f}\right)^{2m_m-3} - \left(\frac{a_{eq}}{a_f}\right)^{9/2}}{(2m_m-4)(15/2-2m_m)}$$

for $a_{eq} \le a_f \le a_\Lambda$ and $m_m \neq 2$ and no inflation,

$$\frac{1}{2a_f^{9/2}}\int_{a_{eq}}^{a_f} a^{7/2} f_\ell(a)\,\mathrm{d}\,a = \frac{1}{7}\left(f_\ell(a_{eq}) + 3c_1^2 + \frac{6c_1c_2}{m_m-5/2}\right)\left(\frac{a_{eq}}{a_f} - \left(\frac{a_{eq}}{a_f}\right)^{9/2}\right)$$

$$-\frac{c_1^2}{3}\left(1-\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right) - 3c_1c_2\frac{\left(\frac{a_{eq}}{a_f}\right)^{m_m-3/2} - \left(\frac{a_{eq}}{a_f}\right)^{9/2}}{(m_m-5/2)(6-m_m)}$$

$$-\frac{3}{7}c_2^2\frac{a_{eq}}{a_f}\left(\ln\frac{a_f}{a_{eq}} - \frac{2}{7} + \frac{2}{7}\left(\frac{a_{eq}}{a_f}\right)^{7/2}\right)$$

for $a_{eq} \le a_f \le a_\Lambda$ and $m_m = 2$ and no inflation, (115.104)

$$-\frac{1}{a_f^{9/2}}\int_{a_{eq}}^{a_f} a^{7/2} f_\theta(a)\,\mathrm{d}\,a = -\frac{2}{3}c_1^2 + \frac{2}{3}c_1^2\left(\frac{a_{eq}}{a_f}\right)^{9/2} + \frac{6c_1c_2}{6-m_m}\left(\frac{a_{eq}}{a_f}\right)^{m_m-3/2}$$

$$-\frac{6c_1c_2}{6-m_m}\left(\frac{a_{eq}}{a_f}\right)^{9/2} - \frac{3c_2^2}{15/2-2m_m}\left(\frac{a_{eq}}{a_f}\right)^{2m_m-3} + \frac{3c_2^2}{15/2-2m_m}\left(\frac{a_{eq}}{a_f}\right)^{9/2}$$

for $a_{eq} \le a_f \le a_\Lambda$ and no inflation, (115.105)

$$-\frac{1}{2a_f^{15/2}}\int_{a_{eq}}^{a_f} a^{13/2} f_a(a)\,\mathrm{d}\,a = -\frac{2\alpha_4}{11}\left(1-\left(\frac{a_{eq}}{a_f}\right)^{11/2}\right) + \frac{1-\left(\frac{a_{eq}}{a_f}\right)^{15/2-2m_m}}{(m_m-1)(15/2-2m_m)}$$

for $a_{eq} \le a_f \le a_\Lambda$ and $m_m \neq 1$ and no inflation, (115.106)

$$-\frac{5}{2a_f^{3/2}}\int_{a_{eq}}^{a_f} a^{1/2} f_\ell(a)\,\mathrm{d}\,a =$$

$$-5\left(f_\ell(a_{eq})+3c_1^2+\frac{6c_1c_2}{m_m-5/2}-\frac{3c_2^2}{2m_m-4}\right)\left(\frac{a_{eq}}{a_f}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)$$
$$+5c_1^2\left(1-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)+15c_1c_2\frac{\left(\frac{a_{eq}}{a_f}\right)^{m_m-3/2}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}}{(m_m-5/2)(3-m_m)}$$
$$-\frac{15}{2}c_2^2\frac{\left(\frac{a_{eq}}{a_f}\right)^{2m_m-3}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}}{(2m_m-4)(9/2-2m_m)}$$
for $a_{eq}\le a_f\le a_\Lambda$ and $m_m\ne 2$ and $m_m\ne 3$ and no inflation,
$$-\frac{5}{2a_f^{3/2}}\int_{a_{eq}}^{a_f}a^{1/2}f_\ell(a)\,\mathrm{d}\,a=-5\left(f_\ell(a_{eq})+3c_1^2+\frac{6c_1c_2}{m_m-5/2}\right)\left(\frac{a_{eq}}{a_f}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)$$
$$+5c_1^2\left(1-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)-15c_2^2\frac{a_{eq}}{a_f}\left(\ln\frac{a_f}{a_{eq}}-2+2\left(\frac{a_{eq}}{a_f}\right)^{1/2}\right)$$
for $a_{eq}\le a_f\le a_\Lambda$ and $m_m=2$ and no inflation,
$$-\frac{5}{2a_f^{3/2}}\int_{a_{eq}}^{a_f}a^{1/2}f_\ell(a)\,\mathrm{d}\,a=$$
$$-5\left(f_\ell(a_{eq})+3c_1^2+\frac{6c_1c_2}{m_m-5/2}-\frac{3c_2^2}{2m_m-4}\right)\left(\frac{a_{eq}}{a_f}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)$$
$$+5c_1^2\left(1-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)+30c_1c_2\left(\frac{a_{eq}}{a_f}\right)^{3/2}\ln\frac{a_f}{a_{eq}}-\frac{15}{2}c_2^2\frac{\left(\frac{a_{eq}}{a_f}\right)^{2m_m-3}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}}{(2m_m-4)(9/2-2m_m)}$$
for $a_{eq}\le a_f\le a_\Lambda$ and $m_m=3$ and no inflation,

(115.107)

$$-\frac{1}{a_f^{3/2}}\int_{a_{eq}}^{a_f}a^{1/2}f_\theta(a)\,\mathrm{d}\,a=$$
$$-2c_1^2\left(1-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)+6c_1c_2\frac{\left(\frac{a_{eq}}{a_f}\right)^{m_m-3/2}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}}{(3-m_m)}-3c_2^2\frac{\left(\frac{a_{eq}}{a_f}\right)^{2m_m-3}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}}{(9/2-2m_m)}$$
for $a_{eq}\le a_f\le a_\Lambda$ and $m_m\ne 3$ and no inflation,
$$-\frac{1}{a_f^{3/2}}\int_{a_{eq}}^{a_f}a^{1/2}f_\theta(a)\,\mathrm{d}\,a=$$
$$-2c_1^2\left(1-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right)+6c_1c_2\left(\frac{a_{eq}}{a_f}\right)^{3/2}\ln\frac{a_f}{a_{eq}}-3c_2^2\frac{\left(\frac{a_{eq}}{a_f}\right)^{2m_m-3}-\left(\frac{a_{eq}}{a_f}\right)^{3/2}}{(9/2-2m_m)}$$
for $a_{eq}\le a_f\le a_\Lambda$ and $m_m=3$ and no inflation,

(115.108)

$$-\frac{1}{2a_f^{9/2}}\int_{a_{eq}}^{a_f}a^{7/2}f_a(a)\,\mathrm{d}\,a=-\frac{2\alpha_4}{5}\left(1-\left(\frac{a_{eq}}{a_f}\right)^{5/2}\right)+\frac{1-\left(\frac{a_{eq}}{a_f}\right)^{9/2-2m_m}}{(m_m-1)(9/2-2m_m)}$$

for $a_{eq} \leq a_f \leq a_\Lambda$ and $m_m \neq 1$ and no inflation. (115.109)

$$\frac{1}{a_f^3}\int_{a_\Lambda}^{a_f} a^2 f_\ell(a)\,\mathrm{d}\,a = \frac{1}{2}\left(f_\ell(a_\Lambda) + 3c_4^2 + \frac{6c_3c_4}{m_m-1} - \frac{3c_3^2}{2m_m-1}\right)\left(\frac{a_\Lambda}{a_f} - \left(\frac{a_\Lambda}{a_f}\right)^3\right)$$
$$-c_4^2\left(1-\left(\frac{a_\Lambda}{a_f}\right)^3\right) - 6c_3c_4\frac{\left(\frac{a_\Lambda}{a_f}\right)^{m_m} - \left(\frac{a_\Lambda}{a_f}\right)^3}{(m_m-1)(3-m_m)} + 3c_3^2\frac{\left(\frac{a_\Lambda}{a_f}\right)^{2m_m} - \left(\frac{a_\Lambda}{a_f}\right)^3}{(2m_m-1)(3-2m_m)}$$
for $a_\Lambda \leq a_f \leq a_0 = 1$ and $m_m \neq 3$ and no inflation,
$$\frac{1}{a_f^3}\int_{a_\Lambda}^{a_f} a^2 f_\ell(a)\,\mathrm{d}\,a = \frac{1}{2}\left(f_\ell(a_\Lambda) + 3c_4^2 + \frac{6c_3c_4}{m_m-1} - \frac{3c_3^2}{2m_m-1}\right)\left(\frac{a_\Lambda}{a_f} - \left(\frac{a_\Lambda}{a_f}\right)^3\right)$$
$$-c_4^2\left(1-\left(\frac{a_\Lambda}{a_f}\right)^3\right) - \frac{6c_3c_4}{m_m-1}\left(\frac{a_\Lambda}{a_f}\right)^3 \ln\frac{a_f}{a_\Lambda} + 3c_3^2\frac{\left(\frac{a_\Lambda}{a_f}\right)^{2m_m} - \left(\frac{a_\Lambda}{a_f}\right)^3}{(2m_m-1)(3-2m_m)}$$
for $a_\Lambda \leq a_f \leq a_0 = 1$ and $m_m = 3$ and no inflation,

(115.110)

$$-\frac{1}{a_f^3}\int_{a_\Lambda}^{a_f} a^2 f_\theta(a)\,\mathrm{d}\,a =$$
$$-c_4^2\left(1-\left(\frac{a_\Lambda}{a_f}\right)^3\right) + 6c_3c_4\frac{\left(\frac{a_\Lambda}{a_f}\right)^{m_m} - \left(\frac{a_\Lambda}{a_f}\right)^3}{(3-m_m)} - 3c_3^2\frac{\left(\frac{a_\Lambda}{a_f}\right)^{2m_m} - \left(\frac{a_\Lambda}{a_f}\right)^3}{(3-2m_m)}$$
for $a_\Lambda \leq a_f \leq a_0 = 1$ and $m_m \neq 3$ and no inflation,
$$-\frac{1}{a_f^3}\int_{a_\Lambda}^{a_f} a^2 f_\theta(a)\,\mathrm{d}\,a =$$
$$-c_4^2\left(1-\left(\frac{a_\Lambda}{a_f}\right)^3\right) + 6c_3c_4\left(\frac{a_\Lambda}{a_f}\right)^3 \ln\frac{a_f}{a_\Lambda} - 3c_3^2\frac{\left(\frac{a_\Lambda}{a_f}\right)^{2m_m} - \left(\frac{a_\Lambda}{a_f}\right)^3}{(3-2m_m)}$$
for $a_\Lambda \leq a_f \leq a_0 = 1$ and $m_m = 3$ and no inflation,

(115.111)

and

$$-\frac{1}{2a_f^3}\int_{a_\Lambda}^{a_f} a^2 f_a(a)\,\mathrm{d}\,a = -\alpha_4\left(1-\frac{a_\Lambda}{a_f}\right) + \frac{\left(\frac{a_f}{a_\Lambda}\right)^{2m_m-3} - 1}{(m_m-1)(2m_m-3)}$$
for $a_\Lambda \leq a_f \leq a_0 = 1$ and $m_m \neq 1$ and no inflation. (115.112)

For $a_f \gg a_N$, keeping only the most significant terms in (115.97) gives

$$C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{27} \quad \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 1 \text{ and for no inflation,}$$
$$C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{2}\left(\ln\frac{a_f}{a_N}\right)^2$$
for $a_N \ll a_f \leq a_{eq}$ and for $m_r = 2$ and for no inflation,

$$C_I(a_f) \approx (\alpha_1 w + \alpha_2)\left[\frac{1}{2}\left(\frac{a_f}{a_N}\right)^2 - \frac{3}{4}\frac{a_f}{a_N}\left(\ln\frac{a_f}{a_N}\right)^2\right]$$

for $a_N \ll a_f \le a_{eq}$ and for $m_r = 3$ and for no inflation,

$$C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{8}\left(\frac{a_f}{a_N}\right)^4$$

for $a_N \ll a_f \le a_{eq}$ and for $m_r = 4$ and for no inflation,

$$C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2\Omega_r}\left\{\frac{H_0^2\Omega_\Lambda}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\times\right.$$
$$\left[-9 + 63\left(\frac{a_{eq}}{a_f}\right)^{1/2} - \frac{1}{7}\left(67 - \frac{3}{7} + 12\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - \frac{1}{7}\left(311 + \frac{3}{7}\right)\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right]$$
$$\left. + \frac{\alpha_2}{12}\left[27 + 36\left(\frac{a_{eq}}{a_f}\right)^{1/2} + \left(326 - 60\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - 389\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}$$

for $a_{eq} \le a_f \le a_\Lambda$ and for $m_r = 1$ and for $m_m = 2$ and for no inflation.

$$C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2\Omega_r}\left\{\frac{H_0^2\Omega_\Lambda}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\times\right.$$
$$\left[-\left(\ln\frac{a_{eq}}{a_N}\right)^2 - 4\ln\frac{a_{eq}}{a_N} - 4 + 6\left(\ln\frac{a_{eq}}{a_N} + 2\right)\left(\frac{a_{eq}}{a_f}\right)^{1/2}\right.$$
$$\left. - \frac{1}{7}\left(12\ln\frac{a_{eq}}{a_N} + 57 - \frac{3}{7} + 12\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} + \left(\left(\ln\frac{a_{eq}}{a_N}\right)^2 - \frac{2}{7}\ln\frac{a_{eq}}{a_N} + \frac{4}{49}\right)\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right]$$
$$+ \frac{\alpha_2}{12}\left[3\left(\ln\frac{a_{eq}}{a_N}\right)^2 + 12\ln\frac{a_{eq}}{a_N} + 12 + 12\ln\frac{a_{eq}}{a_N}\left(\frac{a_{eq}}{a_f}\right)^{1/2}\right.$$
$$\left.\left. + \left(-12\ln\frac{a_{eq}}{a_N} + 84 - 60\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - \left(3\left(\ln\frac{a_{eq}}{a_N}\right)^2 + 12\ln\frac{a_{eq}}{a_N} + 96\right)\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}$$

for $a_{eq} \le a_f \le a_\Lambda$ and for $m_r = 2$ and for $m_m = 2$ and for no inflation.

$$C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_N^4}{H_0^2\Omega_r(m_r-2)^2}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a_f}{a_{eq}}\right)^{2m_m}\times$$
$$\left\{-\frac{H_0^2\Omega_\Lambda}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1 - \left(\frac{a_{eq}}{a_f}\right)^{9/2}\right] + \frac{\alpha_2}{4}\left[1 - \left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}$$

for $a_{eq} \le a_f \le a_\Lambda$ and for $m_r > 2$ and for $m_m > 2$ and for no inflation.

$$C_I(a_f) \approx C_I(a_\Lambda) - 3c_1^2\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1 - \left(\frac{a_\Lambda}{a_f}\right)^3\right]$$

for $a_\Lambda \le a_f \le a_0 = 1$ and for no inflation, specifically,

$$C_I(a_f) \approx C_I(a_\Lambda) - 27\frac{H_f^2 a_f^4}{H_0^2\Omega_r}\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1 - \left(\frac{a_\Lambda}{a_f}\right)^3\right]$$

for $a_\Lambda \le a_f \le a_0 = 1$ and for $m_r = 1$ and for $m_m = 2$ and for no inflation,

$$C_I(a_f) \approx C_I(a_\Lambda) - 3\frac{H_f^2 a_f^4}{H_0^2\Omega_r}\left[\ln\frac{a_{eq}}{a_N} + 2\right]^2\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1 - \left(\frac{a_\Lambda}{a_f}\right)^3\right]$$

$$
\begin{aligned}
&\text{for } a_\Lambda \leq a_f \leq a_0 = 1 \text{ and for } m_r = 2 \text{ and for } m_m = 2 \text{ and for no inflation,}\\
&C_I(a_f) \approx C_I(a_\Lambda) - 3\frac{H_f^2 a_N^4}{H_0^2 \Omega_r (m_r - 2)^2} \left(\frac{a_{eq}}{a_N}\right)^{2m_r} \left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times\\
&\Omega_m \frac{H_0}{H_f} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) \left[1 - \left(\frac{a_\Lambda}{a_f}\right)^3\right]\\
&\text{for } a_\Lambda \leq a_f \leq a_0 = 1 \text{ and for } m_r > 2 \text{ and for } m_m > 2 \text{ and for no inflation,}
\end{aligned}
\tag{115.113}
$$

where

$$
\begin{aligned}
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{27} \text{ for } m_r = 1 \text{ and for no inflation,}\\
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{2} \left(\ln\frac{a_{eq}}{a_N}\right)^2 \text{ for } m_r = 2 \text{ and for no inflation,}\\
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{2} \left(\frac{a_{eq}}{a_N}\right)^2 \text{ for } m_r = 3 \text{ and for no inflation,}\\
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{8} \left(\frac{a_{eq}}{a_N}\right)^4 \text{ for } m_r = 4 \text{ and for no inflation,}
\end{aligned}
\tag{115.114}
$$

and

$$
\begin{aligned}
&C_I(a_\Lambda) \approx C_I(a_{eq}) + c_1^2 \left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) + \frac{\alpha_2}{4}\right\}\\
&\quad \text{for no inflation, specifically,}\\
&C_I(a_\Lambda) \approx C_I(a_{eq}) + 9\frac{H_\Lambda^2 a_\Lambda^4}{H_0^2\Omega_r} \left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) + \frac{\alpha_2}{4}\right\}\\
&\text{for } m_r = 1 \text{ and for } m_m = 2 \text{ and for no inflation,}\\
&C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_\Lambda^4}{H_0^2\Omega_r} \left[\ln\frac{a_{eq}}{a_N} + 2\right]^2 \left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) + \frac{\alpha_2}{4}\right\}\\
&\text{for } m_r = 2 \text{ and for } m_m = 2 \text{ and for no inflation,}\\
&C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_N^4}{H_0^2\Omega_r (m_r - 2)^2} \left(\frac{a_{eq}}{a_N}\right)^{2m_r} \left(\frac{a_\Lambda}{a_{eq}}\right)^{2m_m} \times\\
&\left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) + \frac{\alpha_2}{4}\right\}\\
&\text{for } m_r > 2 \text{ and for } m_m > 2 \text{ and for no inflation.}
\end{aligned}
\tag{115.115}
$$

115.13.2 Formulas for $C_{II}(a_f)$

We can use (115.40) for $f_L(a)$ and (115.41) for $f_{LL}(a)$ in (115.92) to give

$$
\begin{aligned}
&C_{II}(a_f) = \frac{H_0^2\Omega_\Lambda}{6H_f a_f^3}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w+\alpha_2}{12}\right)\times \\
&\int_{a_i}^{a_f}\frac{a^2}{H(a)}\left[-3\frac{\Omega_m}{\Omega_r}af_\ell(a)^2 + \frac{3}{2}\frac{\Omega_m}{\Omega_r}af_{\ell\ell}(a) + f_{\ell\ell}(a)\right. \\
&+\frac{5}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) - f_\theta(a)f_\ell(a) \\
&\left.-\frac{5}{2}\frac{H_f^2}{H(a)^2}f_\ell(a)f_a(a) + \frac{1}{2}\frac{H_f^2}{H(a)^2}f_\theta(a)f_a(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2}f_{aa}(a) + \frac{3}{8}\frac{H_f^4}{H(a)^4}f_a(a)^2\right]\mathrm{d}\,a \\
&+\frac{\alpha_1 w+\alpha_2}{72H_f a_f^3}\times \\
&\int_{a_i}^{a_f}\frac{H_0^2\Omega_r}{H(a)a^2}\left[3\frac{\Omega_m}{\Omega_r}af_\ell(a)^2 + 6f_\ell(a)^2 - \frac{3}{2}\frac{\Omega_m}{\Omega_r}af_{\ell\ell}(a) - 3f_{\ell\ell}(a)\right. \\
&+\frac{5}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) + 3f_\theta(a)f_\ell(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2}f_\ell(a)f_a(a) \\
&\left.+\frac{1}{2}\frac{H_f^2}{H(a)^2}f_\theta(a)f_a(a) - \frac{1}{2}\frac{H_f^2}{H(a)^2}f_{aa}(a) + \frac{3}{8}\frac{H_f^4}{H(a)^4}f_a(a)^2\right]\mathrm{d}\,a \qquad \text{for } a_i \le a_f \le a_N\ , \\
&C_{II}(a_f) = \frac{H_0^2\Omega_\Lambda}{6H_f a_f^3}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w+\alpha_2}{12}\right)\times \\
&\left\{\int_{a_I}^{a_N}\frac{a^2}{H(a)}\left[-\frac{1}{2}\frac{a^4}{\Omega_r}\left(6\frac{\Omega_m}{a^3}+10\frac{\Omega_r}{a^4}\right)f_\ell(a)^2 + \frac{1}{2}\frac{a^4}{\Omega_r}\left(3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right)f_{\ell\ell}(a)\right.\right. \\
&+f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) + 4f_\ell(a)^2 - f_\theta(a)f_\ell(a) \\
&\left.+\left(-\frac{5}{2}f_\ell(a)f_a(a) + \frac{1}{2}f_\theta(a)f_a(a) - \frac{1}{2}f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2}f_a(a)^2\right)\times\frac{H_f^2}{H(a)^2}\right]\mathrm{d}\,a + \\
&\int_{a_N}^{a_f}\frac{a^2}{H(a)}\left[-\frac{1}{2}\frac{H_0^2}{H(a)^2}\left(6\frac{\Omega_m}{a^3}+10\frac{\Omega_r}{a^4}\right)f_\ell(a)^2 + \frac{1}{2}\frac{H_0^2}{H(a)^2}\left(3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right)f_{\ell\ell}(a)\right. \\
&+f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) + \frac{3}{8}\frac{H_0^4}{H(a)^4}\left[3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right]^2 f_\ell(a)^2 \\
&-\frac{1}{2}\frac{H_0^2}{H(a)^2}\left[3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right]\Big(f_\ell(a)^2 + f_\theta(a)f_\ell(a)\Big) + f_\ell(a)f_\theta(a) \\
&+\left(-\frac{3}{4}\frac{H_0^2}{H(a)^2}\left[3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right]f_\ell(a)f_a(a)\right. \\
&\left.+\frac{1}{2}f_\ell(a)f_a(a) + \frac{1}{2}f_\theta(a)f_a(a) - \frac{1}{2}f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2}f_a(a)^2\right) \\
&\left.\left.\times\frac{H_f^2}{H(a)^2}\right]\mathrm{d}\,a\right\} + \frac{\alpha_1 w+\alpha_2}{72H_f a_f^3}\times \\
&\left\{\int_{a_i}^{a_N}a^2H(a)\left[\frac{1}{2}\frac{a^4}{\Omega_r}\left(6\frac{\Omega_m}{a^3}+10\frac{\Omega_r}{a^4}\right)f_\ell(a)^2 - \frac{1}{2}\frac{a^4}{\Omega_r}\left(3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right)f_{\ell\ell}(a)\right.\right. \\
&+f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) + 3f_\theta(a)f_\ell(a) + \Big(-\frac{1}{2}f_\ell(a)f_a(a)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}f_\theta(a)f_a(a) - \frac{1}{2}f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2}f_a(a)^2\Big)\frac{H_f^2}{H(a)^2}\Bigg]\,\mathrm{d}\,a + \\
&\int_{a_N}^{a_f} a^2 H(a)\left[\frac{1}{2}\frac{H_0^2}{H(a)^2}\left(6\frac{\Omega_m}{a^3}+10\frac{\Omega_r}{a^4}\right)f_\ell(a)^2 - \frac{1}{2}\frac{H_0^2}{H(a)^2}\left(3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right)f_{\ell\ell}(a)\right. \\
&+f_\ell^2(a) - f_{\ell\ell}(a) + \frac{5}{6}f_\theta^2(a) + \sqrt{f_\theta(a)}f_{\theta\theta}(a) - \frac{1}{8}\frac{H_0^4}{H(a)^4}\left[3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right]^2 f_\ell(a)^2 \\
&+\frac{1}{2}\frac{H_0^2}{H(a)^2}\left[3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right]\Big(f_\ell(a)^2 + f_\theta(a)f_\ell(a)\Big) + f_\ell(a)f_\theta(a) \\
&+\Bigg(-\frac{1}{4}\frac{H_0^2}{H(a)^2}\left[3\frac{\Omega_m}{a^3}+4\frac{\Omega_r}{a^4}\right]f_\ell(a)f_a(a) + \frac{1}{2}f_\ell(a)f_a(a) \\
&\left.+\frac{1}{2}f_\theta(a)f_a(a) - \frac{1}{2}f_{aa}(a) + \frac{3}{8}\frac{H_f^2}{H(a)^2}f_a(a)^2\Bigg)\frac{H_f^2}{H(a)^2}\right]\mathrm{d}\,a\Bigg\} \qquad \text{for } a_f \geq a_N\ . \qquad (115.116)
\end{aligned}
$$

Formulas for $C_{II}(a_f)$ including inflation

We can use (115.66) for $f_\ell(a)$ and (115.61) for $f_\theta(a)$ in (115.116) to give

Formulas for $C_{II}(a_f)$ neglecting inflation

We can use (115.67) for $f_\ell(a)$ and (115.62) for $f_\theta(a)$ in (115.116) to give

Table 115.1: Cosmological scale factor a_f, Hubble parameter H_f, and maximum rms acceleration $\langle \dot{u}_f \rangle \mathrm{max}$ as a function of global time t_f, m_r, and m_m for an inflation era with 50 e-foldings. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. Everything but the last 3 columns is calculated using the formulas in 115.8. The last column is calculated using equations (115.7), (115.94), and (115.95). $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds. $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length.

t_f		a_f	H_f s^{-1}	$L^* \sqrt{\frac{H_f}{(r_0 a_f)^3}}$ $\mathrm{m\ s}^{-2}$	m_r	m_m	$\langle \dot{u}_f \rangle \mathrm{max}$ $\mathrm{m\ s}^{-2}$
beginning of inflation 10^{-36} s		$a_i =$ 3.9×10^{-42}					
inflation era			5×10^{35}				
end of inflation 10^{-34} s		$a_N =$ 2×10^{-20}	5×10^{33}	2.3×10^{-16}	1 2 3 4		9×10^{-46} 3×10^{-67} 9×10^{-89} 2×10^{-110}
electroweak transition 2.4×10^{-11} s		10^{-15}	2×10^{10}	4.0×10^{-35}	1 2 3 4		1×10^{-64} 6×10^{-86} 2×10^{-107} 3×10^{-129}
radiation era 0.24 s		10^{-10}	2	1.3×10^{-47}	1 2 3 4		4×10^{-77} 2×10^{-102} 2×10^{-124} 1×10^{-155}
matter/ radiation equality 5×10^4 yr		$a_{\mathrm{eq}} =$ 3×10^{-4}	3×10^{-13}	9.5×10^{-63}	1 2 3 4		4×10^{-92} 1×10^{-117} 1×10^{-145} 1×10^{-183}
recombi- nation 3.8×10^5 yr		$.9 \times 10^{-3}$	5×10^{-14}	7.5×10^{-63}	1 2 3 4	2 2 3 4	3×10^{-92} 1×10^{-113} $\approx 10^{-141}$ $\approx 10^{-179}$
matter/ dark energy equality 10×10^9 yr		$a_\Lambda = 0.76$	2.5×10^{-18}	2.1×10^{-71}	1 2 3 4	2 2 3 4	8×10^{-101} 3×10^{-122} $\approx 10^{-155}$ $\approx 10^{-196}$
today 13.8×10^9 yr		$a_0 = 1.0$	2.2×10^{-18}	1.3×10^{-71}	1 2 3 4	2 2 3 4	5×10^{-101} 2×10^{-122} $\approx 10^{-155}$ $\approx 10^{-196}$

Table 115.2: Cosmological scale factor a_f, Hubble parameter H_f, and maximum rms acceleration $\langle \dot{u}_f \rangle \mathrm{max}$ as a function of global time t_f, m_r, and m_m for an inflation era with 55 e-foldings. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. Everything but the last 3 columns is calculated using the formulas in 115.8. The last column is calculated using equations (115.7), (115.94), and (115.95). $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds. $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length.

t_f		a_f	H_f s^{-1}	$L^*\sqrt{\frac{H_f}{(r_0 a_f)^3}}$ m s^{-2}	m_r	m_m	$\langle \dot{u}_f \rangle \mathrm{max}$ m s^{-2}
beginning of inflation 10^{-36} s		$a_i =$ 3.9×10^{-42}					
inflation era			5×10^{35}				
end of inflation 10^{-34} s		$a_N =$ 2×10^{-20}	5×10^{33}	2.3×10^{-16}	1 2 3 4		6×10^{-49} 1×10^{-72} 3×10^{-96} 6×10^{-120}
electroweak transition 2.4×10^{-11} s		10^{-15}	2×10^{10}	4.0×10^{-35}	1 2 3 4		9×10^{-68} 2×10^{-91} 6×10^{-115} 9×10^{-139}
radiation era 0.24 s		10^{-10}	2	1.3×10^{-47}	1 2 3 4		1×10^{-83} 8×10^{-108} 4×10^{-132} 3×10^{-164}
matter/ radiation equality 5×10^4 yr		$a_{\mathrm{eq}} =$ 3×10^{-4}	3×10^{-13}	9.5×10^{-63}	1 2 3 4		8×10^{-99} 5×10^{-123} 3×10^{-152} 2×10^{-192}
recombi- nation 3.8×10^5 yr		$.9 \times 10^{-3}$	5×10^{-14}	7.5×10^{-63}	1 2 3 4	2 2 3 4	2×10^{-97} 5×10^{-119} $\approx 10^{-147}$ $\approx 10^{-188}$
matter/ dark energy equality 10×10^9 yr		$a_\Lambda = 0.76$	2.5×10^{-18}	2.1×10^{-71}	1 2 3 4	2 2 3 4	5×10^{-106} 1×10^{-127} $\approx 10^{-162}$ $\approx 10^{-205}$
today 13.8×10^9 yr		$a_0 = 1.0$	2.2×10^{-18}	1.3×10^{-71}	1 2 3 4	2 2 3 4	3×10^{-106} 8×10^{-128} $\approx 10^{-162}$ $\approx 10^{-205}$

Table 115.3: Cosmological scale factor a_f, Hubble parameter H_f, and maximum rms acceleration $\langle \dot{u}_f \rangle \mathrm{max}$ as a function of global time t_f, m_r, and m_m for an inflation era with 60 e-foldings. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. Everything but the last 3 columns is calculated using the formulas in 115.8. The last column is calculated using equations (115.7), (115.94), and (115.95). $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds. $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length.

t_f		a_f	H_f s^{-1}	$L^* \sqrt{\frac{H_f}{(r_0 a_f)^3}}$ $\mathrm{m\ s}^{-2}$	m_r	m_m	$\langle \dot{u}_f \rangle \mathrm{max}$ $\mathrm{m\ s}^{-2}$
beginning of inflation 10^{-36} s		$a_i =$ 3.9×10^{-42}					
inflation era			5×10^{35}				
end of inflation 10^{-34} s		$a_N =$ 2×10^{-20}	5×10^{33}	2.3×10^{-16}	1 2 3 4		3×10^{-52} 6×10^{-76} 9×10^{-104} 9×10^{-130}
electroweak transition 2.4×10^{-11} s		10^{-15}	2×10^{10}	4.0×10^{-35}	1 2 3 4		6×10^{-71} 9×10^{-97} 1×10^{-122} 2×10^{-148}
radiation era 0.24 s		10^{-10}	2	1.3×10^{-47}	1 2 3 4		4×10^{-87} 2×10^{-113} 1×10^{-139} 5×10^{-173}
matter/ radiation equality 5×10^4 yr		$a_{\mathrm{eq}} =$ 3×10^{-4}	3×10^{-13}	9.5×10^{-63}	1 2 3 4		4×10^{-114} 2×10^{-140} 1×10^{-158} 4×10^{-201}
recombi- nation 3.8×10^5 yr		$.9 \times 10^{-3}$	5×10^{-14}	7.5×10^{-63}	1 2 3 4	2 2 3 4	1×10^{-98} 2×10^{-154} $\approx 10^{-141}$ $\approx 10^{-197}$
matter/ dark energy equality 10×10^9 yr		$a_\Lambda = 0.76$	2.5×10^{-18}	2.1×10^{-71}	1 2 3 4	2 2 3 4	3×10^{-107} 6×10^{-133} $\approx 10^{-168}$ $\approx 10^{-214}$
today 13.8×10^9 yr		$a_0 = 1.0$	2.2×10^{-18}	1.3×10^{-71}	1 2 3 4	2 2 3 4	2×10^{-107} 3×10^{-133} $\approx 10^{-168}$ $\approx 10^{-214}$

Table 115.4: Cosmological scale factor a_f, Hubble parameter H_f, and maximum rms acceleration $\langle \dot{u}_f \rangle \mathrm{max}$ as a function of global time t_f, m_r, and m_m neglecting inflation. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. Everything but the last 3 columns is calculated using the formulas in 115.8. The last column is calculated using equations (115.7) and (115.113). $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds. $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length.

t_f		a_f	H_f s^{-1}	$L^* \sqrt{\frac{H_f}{(r_0 a_f)^3}}$ $\mathrm{m\ s}^{-2}$	m_r	m_m	$\langle \dot{u}_f \rangle \mathrm{max}$ $\mathrm{m\ s}^{-2}$
end of inflation 10^{-34} s		$a_N =$ 2×10^{-20}	5×10^{33}	2.3×10^{-16}			
electroweak transition 2.4×10^{-11} s		10^{-15}	2×10^{10}	4.0×10^{-35}	1 2 3 4		2×10^{-34} 6×10^{-36} 1×10^{-39} 3×10^{-44}
radiation era 0.24 s		10^{-10}	2	1.3×10^{-47}	1 2 3 4		7×10^{-47} 8×10^{-49} 4×10^{-57} 1×10^{-66}
matter/ radiation equality 5×10^4 yr		$a_{\mathrm{eq}} =$ 3×10^{-4}	3×10^{-13}	9.5×10^{-63}	1 2 3 4		5×10^{-62} 4×10^{-64} 9×10^{-79} 1×10^{-94}
recombi- nation 3.8×10^5 yr		$.9 \times 10^{-3}$	5×10^{-14}	7.5×10^{-63}	1 2 3 4	2 2 3 4	2×10^{-63} 3×10^{-64} 2×10^{-79} 8×10^{-96}
matter/ dark energy equality 10×10^9 yr		$a_\Lambda = 0.76$	2.5×10^{-18}	2.1×10^{-71}	1 2 3 4	2 2 3 4	1×10^{-73} 8×10^{-75} 8×10^{-93} 2×10^{-112}
today 13.8×10^9 yr		$a_0 = 1.0$	2.2×10^{-18}	1.3×10^{-71}	1 2 3 4	2 2 3 4	7×10^{-74} 5×10^{-75} 5×10^{-93} 1×10^{-112}

Table 115.5: Acceleration propagation. ℓ is a scale factor along lines of cosmic flow. [387, Chapter 29] gives $m_r = m_m = 4$. Some of the quantities in this table are a function of position.

Era	a	ℓ	a_f	$\dot{u} \propto$	$\dot{u} =$
	a_N	ℓ_N			$\dot{u}_N$
Radiation			$< a_{eq}$ $> a_{eq}$	ℓ^{-m_r} ℓ^{-m_r}	$\dot{u}_f \left(\frac{\ell_f}{\ell}\right)^{m_r}$ $\dot{u}_f \left(\frac{\ell_f}{\ell_{eq}}\right)^{m_m} \left(\frac{\ell_{eq}}{\ell}\right)^{m_r}$
	a_{eq}	ℓ_{eq}			$\dot{u}_{eq}$
Matter			$> a_{eq}$	ℓ^{-m_m}	$\dot{u}_f \left(\frac{\ell_f}{\ell}\right)^{m_m}$
Now	a_0	ℓ_0			$\dot{u}_0$

Table 115.6: Acceleration contribution to the Hubble parameter. $H_a^2 = H_f^2 F_a(\ell)\epsilon$, where $\epsilon = \frac{1}{6}\left(\frac{\dot{u}_f}{H_f}\right)^2$ and ℓ is a scale factor along lines of cosmic flow. [387, Chapter 29] gives $m_r = m_m = 4$. Some of the quantities in this table are a function of position.

Era	a	ℓ	a_f	$F_a(\ell)$
	a_i	ℓ_i		
Inflation			$< a_{eq}$	$2\left(\frac{\ell_f}{\ell}\right)^2 \left[2\ln\left(\frac{\ell_f}{\ell}\right) + \alpha_4 - 1\right]$, $m_r = 1$ $2\left(\frac{\ell_f}{\ell}\right)^2 \left[\alpha_4 - \frac{1}{m_r-1}\left(\frac{\ell_f}{\ell}\right)^{2m_r-2}\right]$, $m_r \neq 1$
Inflation			$> a_{eq}$	$2\left(\frac{\ell_f}{\ell_{eq}}\right)^{2(m_m-m_r)} (\frac{\ell_f}{\ell})^2 \left[2\ln\left(\frac{\ell_f}{\ell}\right) + \alpha_4 - 1\right]$, $m_r = 1$ $2\left(\frac{\ell_f}{\ell_{eq}}\right)^{2(m_m-m_r)} \left(\frac{\ell_f}{\ell}\right)^2 \left[\alpha_4 - \frac{1}{m_r-1}\left(\frac{\ell_f}{\ell}\right)^{2m_r-2}\right]$, $m_r \neq 1$
	a_N	ℓ_N		
Radiation			$< a_{eq}$	$2\left(\frac{\ell_f}{\ell}\right)^2 \left[2\ln\left(\frac{\ell_f}{\ell}\right) + \alpha_4 - 1\right]$, $m_r = 1$ $2\left(\frac{\ell_f}{\ell}\right)^2 \left[\alpha_4 - \frac{1}{m_r-1}\left(\frac{\ell_f}{\ell}\right)^{2m_r-2}\right]$, $m_r \neq 1$
Radiation			$> a_{eq}$	$2\left(\frac{\ell_f}{\ell_{eq}}\right)^{2(m_m-m_r)} (\frac{\ell_f}{\ell})^2 \left[2\ln\left(\frac{\ell_f}{\ell}\right) + \alpha_4 - 1\right]$, $m_r = 1$ $2\left(\frac{\ell_f}{\ell_{eq}}\right)^{2(m_m-m_r)} \left(\frac{\ell_f}{\ell}\right)^2 \left[\alpha_4 - \frac{1}{m_r-1}\left(\frac{\ell_f}{\ell}\right)^{2m_r-2}\right]$, $m_r \neq 1$
	a_{eq}	ℓ_{eq}		
Matter			$> a_{eq}$	$2\left(\frac{\ell_f}{\ell}\right)^2 \left[\alpha_4 - \frac{1}{m_m-1}\left(\frac{\ell_f}{\ell}\right)^{2m_m-2}\right]$, $m_m \neq 1$
	$a_{\Lambda m}$	$\ell_{\Lambda m}$		
Λ			$> a_{\Lambda m}$	$2\left(\frac{\ell_f}{\ell}\right)^2 \left[\alpha_4 - \frac{1}{m_m-1}\left(\frac{\ell_f}{\ell}\right)^{2m_m-2}\right]$, $m_m \neq 1$
Now	a_0	ℓ_0	$= a_0$	$2\left(\frac{\ell_0}{\ell}\right)^2 \left[\alpha_4 - \frac{1}{m_m-1}\left(\frac{\ell_0}{\ell}\right)^{2m_m-2}\right]$, $m_m \neq 1$

Chapter 116

Relative acceleration of matter and inertial frames[1]

abstract

Any reasonable form of quantum gravity can explain (by phase interference) why, on a large scale, inertial frames would not have significant acceleration relative to the average matter distribution without the need for absolute space, finely tuned initial conditions, or giving up on independent degrees of freedom for the gravitational field. A simple saddlepoint approximation to a path-integral calculation is used to estimate the limits on the relative acceleration as a function of cosmic time for inflation rates of 50, 55, and 60 e-foldings, and for values of the dependence of relative acceleration on cosmological scale factor a as a^{-m} for various values of m. Inflation dominates the calculation and gives a limit of relative acceleration of about 10^{-64} m s^{-2} even as early as 10^{-11} s for $m = 1$ and for 50 e-foldings, with tighter restrictions for larger m and for 55 and 60 e-foldings, and even tighter restrictions today. However, even if there were no inflation, relative acceleration would be restricted to be less than about 10^{-34} m s^{-2} at 10^{-11} s for $m = 1$, with tighter restrictions for larger m, and even tighter restrictions today. Although the calculations are based on solutions to Einstein's field equations, the results are valid for a more general dependence of the Lagrangian on the pressure, density, and cosmological constant.

116.1 Introduction

We do not observe large-scale relative rotation between matter and inertial frames [164, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312]. Nor do we observe relative acceleration between matter and inertial frames on very large scales. We would expect evidence of relative acceleration as anisotropy in the Cosmic Microwave Background. However, the lack of observed anisotropy in the CMB [390, 369] suggests that there is no significant evidence for relative acceleration. We are not surprised at these results.

However, we should be surprised because in General Relativity, gravitation (including inertia) (as expressed by the metric tensor) is determined not only by the distribution of matter (in terms of the stress-energy tensor), but also by initial and boundary conditions. There are many solutions of Einstein's field equations for General Relativity that have large-scale relative rotation of matter and inertial frames, e.g. [347, 348, 163, 370, 371], or large-scale relative acceleration of matter and inertial frames, e.g. [396, 397, 398, 399, 400, 401, 402, 403, 404].

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It is difficult to explain the absence of relative rotation and relative acceleration in our universe classically without absolute space (as proposed by Newton) or without assuming very finely tuned initial conditions for the universe.

Ernst Mach [120, 102, 122, 15] suggested that inertia might be determined by distant matter. Various versions of that proposal have come to be known as Mach's principle. Since we now know (from General Relativity) that inertia is a gravitational force, such an implementation of Mach's principle would require that the gravitational field (or at least part of it) be determined only by its sources (matter) rather than having independent degrees of freedom (in terms of initial and boundary conditions).

If the many proposals to implement Mach's principle for General Relativity, e.g. [11, 16, 156, 109, 315, 159] were correct, then gravitation would behave very differently from the electromagnetic interaction, in that electric and magnetic fields are determined not only from sources (charges and currents), but also from initial and boundary conditions.

Although General Relativity provides a partial mechanism for Mach's principle through "frame dragging," it is necessary to explain the very finely tuned initial conditions needed to give the apparent absence of relative rotation and relative acceleration between matter and inertial frames today.

The observed absence of relative rotation has been explained by quantum gravity [386]. Specifically, any reasonable form of quantum gravity can explain (in terms of phase interference) the lack of observed rotation of inertial frames relative to the matter distribution.

Here, we show that any reasonable form of quantum gravity could also explain why, on a large scale, inertial frames would not have significant acceleration relative to the matter distribution. The idea is that in quantum cosmology, we have a quantum superposition of many spacetimes, each with a different rms relative acceleration between the matter distribution and inertial frames, and each with a specific phase that is proportional to the action. Most of these spacetimes cancel each other out by phase interference, but there is constructive interference for those spacetimes within the first Fresnel zone where the action is an extremum for variation of the relative acceleration, which occurs for zero relative acceleration. This calculation shows that relative acceleration would be restricted to small values even within a fraction of a second after the initial singularity. Although inflation dominates the calculation, relative acceleration would have been restricted to very small values even without inflation.

Section 118.2 reviews path integrals in quantum cosmology. Section 118.3 gives the approximate action for small relative acceleration. Section 118.4 calculates a saddlepoint approximation to the path integral. Section 118.5 discusses the results.

118.6 gives the background cosmology. 118.7 gives the approximate generalized Friedmann equation for small acceleration. 118.8 and 118.9 give formulas for the function $C_I(a_f)$ that is used to calculate the action for small relative acceleration. 118.8 includes inflation, while 118.9 neglects inflation. Although the derivation of the formulas in 118.8 and 118.9 is straightforward, and all of the information necessary to derive those formulas is included here, the derivation is long and tedious. For those interested, the details for the derivation of these formulas are in [410].

116.2 Review of path integrals in quantum cosmology

There are strong reasons why a theory of quantum gravity should exist, e.g.[406], and it is generally believed that such a theory exists. There are many difficulties with formulating a theory of quantum gravity, some of which are discussed in [373, 374]. Although we do not have a final theory of quantum gravity, and therefore, no universally accepted theory of quantum cosmology, we have some speculations for theories of quantum gravity, e.g. [62, 63, 64, 19, 375, 376, 377].

However, some calculations (including the present one) can be made without having a full theory of quantum gravity by using a path-integral representation because the action is most likely to dominate over the measure (which we do not know), and the action in the case of acceleration depends only weakly on the exact form of the Lagrangian.

The amplitude for measuring a particular value for some quantity is equal to the amplitude for measuring that value given a particular 4-geometry times the amplitude for that 4-geometry, and then we sum over all 4-geometries.

For example, following [124], the amplitude for the 3-geometry and matter field to be fixed at specified values on two spacelike hypersurfaces is

$$\langle {}^{(3)}\mathcal{G}_f, \phi_f \,|^{(3)}\, \mathcal{G}_i, \phi_i \rangle = \int \exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right) \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi \ , \tag{116.1}$$

where the integral is over all 4-geometries and field configurations that match the given values on the two spacelike hypersurfaces, and $\exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right)$ is the contribution of the 4-geometry ${}^{(4)}\mathcal{G}$ and matter field ϕ on that 4-geometry to the path integral, where $I[{}^{(4)}\mathcal{G}, \phi]$ is the action. The proper time between the two hypersurfaces is not specified. A correct theory of quantum gravity would be necessary to specify the measures $\mathcal{D}^{(4)}\mathcal{G}$ and $\mathcal{D}\phi$, but that will not be necessary for the purposes here. Hartle and Hawking [124] restricted the integration in (116.1) to compact (closed) 4-geometries, but (116.1) can be applied to open 4-geometries if that is done carefully.

Equation (116.1) is a path integral. In this case, the "path" is the sequence of 3-geometries that form the 4-geometry ${}^{(4)}\mathcal{G}$. Thus, each 4-geometry is one "path." This is related to a sum-over-histories approach [316, 165, 317, 318, 319].[2] The space in which these paths exist is often referred to as superspace, e.g. [20]. As pointed out by Hajicek [217], there are two kinds of path integrals: those in which the time is specified at the endpoints, and those in which the time is not specified. The path integral in (116.1) is the latter. References [217] and [221] consider refinements to the path integral in (116.1), but such refinements are not necessary here.

Because of diffeomorphisms, a given 4-geometry can be specified by different metrics that are connected by coordinate transformations. This makes it difficult to avoid duplications when making path integral calculations. We avoid that difficulty here by considering only simple models.

Let $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ be the amplitude that the 3-geometry was ${}^{(3)}\mathcal{G}_i$ on some initial space-like hypersurface and that the matter fields on that 3-geometry were ϕ_i. Let $\psi_f({}^{(3)}\mathcal{G}_f, \phi_f)$ be the amplitude that the 3-geometry is ${}^{(3)}\mathcal{G}_f$ on some final space-like hypersurface and that the matter fields on that 3-geometry are ϕ_f. Then, we have

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int \langle {}^{(3)}\mathcal{G}_f, \phi_f \,|^{(3)}\, \mathcal{G}_i, \phi_i \rangle \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ . \tag{116.2}$$

Substituting (116.1) into (116.2) gives

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int\int \exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right) \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi\, \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ . \tag{116.3}$$

Although in (118.1), the integration is over all possible 4-geometries, not just classical 4-geometries, the main contribution to the integral (in most cases) comes from classical 4-geometries, e.g. [221, 220]. Thus, we shall now restrict (118.1) to be an integration over classical 4-geometries. This is appropriate for our purposes, in any case, since we are trying to explain why we do not measure relative acceleration of matter and inertial frames in what appears to be a classical universe.

Any measurement to determine the inertial frame will give a result that depends on the 4-geometry. If several 4-geometries contribute significantly to an amplitude, such as in (118.1), then any measurement to determine an inertial frame might give the inertial frame corresponding to any

[2] also called decoherent histories [320, 321, 322, 323, 324, 325, 326, 327] or consistent histories [328, 329, 330, 331]

one of those 4-geometries. However, the probability for the result being a particular inertial frame will depend on the contribution of the corresponding 4-geometry to calculations such as that in (118.1).

The condition that there are not finely tuned initial conditions is equivalent to $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ being a broad wave function. That allows us to neglect the effect of that initial wave function on the integration in the path integral in (118.1). In addition, we consider 4-geometries characterized by a parameter $\langle \dot{u}_f \rangle$ which we take to be the rms relative acceleration on the space-like hypersurface at $t = t_f$. Thus, we can rewrite (118.1) for our purposes as

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) \propto \int_{-\infty}^{\infty} A(\langle \dot{u}_f \rangle) e^{iI(\langle \dot{u}_f \rangle)/\hbar} \, \mathrm{d}\langle \dot{u}_f \rangle \ , \tag{116.4}$$

where $A(\langle \dot{u}_f \rangle)$ is a slowly varying function of $\langle \dot{u}_f \rangle$, $I(\langle \dot{u}_f \rangle)$ is the action, and since for classical spacetimes, the acceleration is a known function of cosmological time, we can consider the action I to depend on the rms acceleration at any cosmological time we choose, say t_f, which we designate as $\langle \dot{u}_f \rangle$, where $\langle \dot{u}_f \rangle^2 \equiv \overline{\dot{u}_f^2}$, where the average is a spatial average over the volume within the past light cone.

118.6 discusses the background cosmology, including the inflation, radiation, matter, and dark-energy eras because the Lagrangian would depend differently on the cosmological scale factor in the four eras.

116.3 Approximate action for small acceleration

The action in (118.2) is equal to a volume integral over spacetime plus a surface integral. The surface term is necessary to insure consistency if the action integral is broken into parts [183, 123]. For the present purposes, it is not necessary to consider the surface term.

In addition, for 4-geometries that are solutions of Einstein's field equations, the Lagrangian can be expressed as an effective Lagrangian e.g. [386, Appendix B]

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda \ , \tag{116.5}$$

where p is pressure, ρ is density, Λ is the cosmological constant, and α_1, α_2, and α_3 are dimensionless constants of order unity.

That allows us to write the action as

$$I = \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^4 x = \int_{t_i}^{t_f} \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^3 x \mathrm{d}t \ , \tag{116.6}$$

where t_i is the initial time (which we take to be the beginning of the inflation era), and t_f is the final time, which we choose arbitrarily to calculate the action at any specified cosmic time.

We can convert the time integral in (118.4) to an integral over the cosmological scale factor a

$$I = \int_{a_i}^{a_f} \int \frac{(-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^3 x \, \mathrm{d}\, a}{\dot{a}} \ , \tag{116.7}$$

where $\dot{a} = \mathrm{d}a/\mathrm{d}t$ is given by a generalization of the Friedmann equation that includes acceleration (118.7), a_i is the value of the cosmological scale factor at $t = t_i$, and a_f is the value of the cosmological scale factor at $t = t_f$.

Including relative acceleration in the calculation of the action (even approximately) is tedious, even though straightforward. The details of the calculation of the total action to second order in the mean square of the acceleration at $t = t_f$ is given by [410]. That calculation includes taking into account the fact that if there is relative acceleration of the matter distribution and the inertial frame, then density and pressure will depend on the scale length ℓ along flow lines rather than depend on the cosmological scale factor a. The significance is that the cosmological scale factor a is a function of global time, but scale length ℓ along flow lines is not normal to surfaces of constant global time. This causes the calculation to be much more complicated.

We can express the volume integral in (118.5) as a product of the spatial volume and the spatial average.

$$I = \int_{a_i}^{a_f} V(a) \overline{\left(\frac{\tilde{L}}{\dot{a}}\right)} \mathrm{d}\, a \ , \tag{116.8}$$

where we have used the knowledge that our universe is spatially flat, an overbar indicates a spatial average,

$$V(a) = \frac{4}{3}\pi a^3 r_0^3 \tag{116.9}$$

is the approximate spatial volume[3], and r_0 is the present radius of the cosmological horizon.

The result to lowest order in $\langle \dot{u}_f \rangle / H_f$ is

$$I(\langle \dot{u}_f \rangle) \approx I_0 + \hbar \left(\frac{\langle \dot{u}_f \rangle}{\dot{u}_m}\right)^2 f_I(\langle \dot{u}_f \rangle) \ , \tag{116.10}$$

where I_0 is the action for the standard cosmological model (the Robertson-Walker cosmology),

$$\dot{u}_m = \left(\frac{\hbar H_f}{r_0^3 a_f^3}\right)^{1/2} = L^* \sqrt{\frac{H_f}{r_f^3}} = L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \ , \tag{116.11}$$

$r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds, $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length,

$$\begin{aligned} f_I(\langle \dot{u}_f \rangle) &\approx \left[C_I(a_f) + \tfrac{\langle \dot{u}_f \rangle^2 + \sigma_a^2 / \langle \dot{u}_f \rangle^2}{H_f^2} C_{II}(a_f) \right] \\ &\approx C_I(a_f) \text{ for small } \langle \dot{u}_f \rangle \end{aligned} \tag{116.12}$$

is some slowly varying dimensionless even function of $\langle \dot{u}_f \rangle$, σ_a^2 is the variance of $\dot{u}_f^2$, $C_I(a_f)$ and $C_{II}(a_f)$ are dimensionless functions, and the details for the derivation of these formulas are in [410].

Notice that (118.8) reduces to I_0, the action for the standard cosmological model, when the rms acceleration $\langle \dot{u}_f \rangle$ is zero, and that (118.8) is an even function of $\langle \dot{u}_f \rangle$ if $f_I(\langle \dot{u}_f \rangle)$ is an even function of $\langle \dot{u}_f \rangle$.

116.4 Saddlepoint approximation

Comparing (118.2) and (118.8) shows that (118.2) has a saddlepoint at $\langle \dot{u}_f \rangle = 0$. If that saddlepoint is the only significant saddlepoint (and other criteria are satisfied), then the only significant contributions to the path integral (118.2) comes from values of the rms acceleration at the time t_f of

$$\langle \dot{u}_f \rangle = 0 \pm \dot{u}_m / \sqrt{|\ f_I(0)\ |} \ . \tag{116.13}$$

Either a stationary-phase path or a steepest-descent path could be used when making the saddlepoint approximation [134, 333, 219], but here, we use a stationary-phase path. Halliwell [332] gives an example of a more detailed path-integral calculation of quantum gravity.

[3] Since our universe is approximately spatially flat. Although there are arguments that V should be infinite because this is an open cosmology [351], considering causality leads to restricting the spatial part of the action to the past light cone.

The saddlepoint at $\langle \dot{u}_f \rangle = 0$ in (118.2) is isolated from other saddlepoints and any possible non-analytic points as shown by (118.10). The integral in (118.2) can be approximated by a saddlepoint integration to give

$$
\begin{aligned}
&\psi_f \propto A(0)\dot{u}_m\sqrt{\pi}/\sqrt{f_I(0)}e^{i\pi/4} \\
&\text{for } \langle \dot{u}_f \rangle < \frac{\dot{u}_m}{\sqrt{\mid f_I(0) \mid}} \approx L^*\sqrt{\frac{H_f}{r_0^3 a_f^3}}\frac{1}{\sqrt{\mid f_I(0) \mid}} \approx L^*\sqrt{\frac{H_f}{r_0^3 a_f^3}}\frac{1}{\sqrt{\mid C_I(a_f) \mid}}, \\
&\psi_f \approx 0, \text{ otherwise,}
\end{aligned}
\tag{116.14}
$$

where $r_0 \approx 46.5 \times 10^9$ light years is the present radius of the observable universe and $L^* \approx 2 \times 10^{-35}$ m is the Planck length.

118.8 calculates $C_I(a_f)$ including inflation. This allows calculating estimates for allowed upper limits on the values for the rms relative acceleration $\langle \dot{u}_f \rangle$ from (118.12) given in table 118.1. 118.9 calculates $C_I(a_f)$ neglecting inflation. This allows calculating estimates for allowed upper limits on the values for the rms relative acceleration $\langle \dot{u}_f \rangle$ from (118.12) given in table 118.2.

Although relative acceleration is associated with a scalar mode perturbation [411, Chapter 29], which leads to $m = 4$, we include calculations for $m = 1$, $m = 2$, and $m = 3$ for completeness.

116.5 Discussion

Table 118.1 gives the limits on rms relative acceleration as a function of cosmic time from (118.12) including the effects of inflation with 50, 55, and 60 e-foldings. Calculations are included for $m = 1$ in the radiation era and $m = 2$ in the matter era and also for $m = 2$, 3, and 4 in all eras, where m gives the dependence of relative acceleration on cosmological scale factor a as a^{-m}. As listed in table 118.1, m_r is the value in the radiation era, and m_m is the value in the matter era.

Inflation dominates the calculation and gives a limit of relative acceleration at the electroweak transition[4] ($\approx 2.4 \times 10^{-11}$ s) of 10^{-45}, 10^{-67}, 10^{-88}, and 10^{-110} m s^{-2} for $m = 1$, 2, 3, and 4 for 50 e-foldings, with tighter restrictions for 55 and 60 e-foldings. The corresponding values at the present time are 10^{-100}, 10^{-122}, 10^{-155}, and 10^{-196} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

However, although inflation dominates the calculation, inflation is not necessary to restrict the relative acceleration to small values, as shown in table 118.2, which gives the limits on rms relative acceleration as a function of cosmic time from (118.12) neglecting inflation. Even if there were no inflation, relative acceleration would be restricted to be not much larger than about $L^*\sqrt{H/(r_0 a)^3}$, where $L^* \approx 2 \times 10^{-35}$ m is the Planck length, $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds is the approximate radius of the observable universe today, H is the Hubble parameter, which varies from about 2×10^{10} s^{-1} at the electroweak transition to $\approx 2 \times 10^{-18}$ s^{-1} today, and a is the cosmological scale factor, which varies from about 10^{-15} s^{-1} at the electroweak transition to 1 today.

As shown by table 118.2, even in the absence of inflation, relative acceleration at the electroweak transition would have been limited to about 10^{-34}, 10^{-35}, 10^{-39}, and 10^{-44} m s^{-2} for $m = 1$, 2, 3, and 4. The corresponding values at the present time without inflation are 10^{-73}, 10^{-74}, 10^{-92}, and 10^{-112} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

The calculations used for tables 118.1 and 118.2 take the quantities $\alpha_1 w + \alpha_2$ and α_3 to be of order unity. If desired, other specific values could be used instead.

[4]Whittle [407, page 230] gives $\approx 10^{-15}$ for the value of the cosmological scale factor at the electroweak transition. Using that value in (118.25) gives $\approx 2.4 \times 10^{-11}$ s for the approximate time of the electroweak transition.

Because quantum gravity (specifically, quantum cosmology) explains through phase interference why we should not observe relative rotation or relative acceleration between matter and inertial frames on a large scale, we could say that quantum gravity may provide the mechanism for implementing Mach's principle.

116.6 Background cosmology

We start with the formula for the Hubble parameter neglecting vorticity, shear, and acceleration

$$\frac{1}{a}\frac{da}{dt} = H(a) = H_0\sqrt{\Omega_\Lambda + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2}} = \sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{a^2}}\ , \tag{116.15}$$

where Λ is the cosmological constant, ρ is density, t is global time, $H_0 = 67.74$ km s^{-1} Mpc^{-1} = 6.928×10^{-11} yr^{-1} $\approx 2.195 \times 10^{-18}$ s^{-1} [390] is the present value of the Hubble parameter, $a = 1/(z+1)$ is the cosmological scale factor, whose present value is 1, z is the redshift factor, and $\Omega_\Lambda = 0.6911$ [390] is the dark energy density divided by the critical density today. Using $z_{\rm eq} = 3402$ [412] for the redshift at radiation/matter equality with $\Omega_m = 0.3089$ [390] for the matter density today divided by the critical density gives $\Omega_r = 9.077 \times 10^{-5} \approx 9 \times 10^{-5}$ for the radiation energy density divided by the critical density today. $\Omega_k = 0.0$ within measurement error [390]. Equation (118.13) is not valid during inflation, but that era is considered later.

For an equation of state, we take

$$p = w\rho\ , \tag{116.16}$$

where $w = 1/3$ in the radiation-dominated era, and $w = 0$ in the matter-dominated era. The variation of density ρ with cosmological scale factor a is given by [342, Table 6.1]

$$\rho = \rho_{eq}(a/a_{eq})^{-3(1+w)}\ , \tag{116.17}$$

where ρ_{eq} is the value of ρ at the boundary between the radiation era and the matter era where $a = a_{eq}$.

From (118.13), we have

$$\rho = \frac{3H_0^2}{8\pi}\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right)\ , \tag{116.18}$$

which gives a smooth transition between the radiation era and the matter era, instead of the abrupt transition given by (118.15). We can have a smooth transition for pressure, also, by taking w in (118.14) to be given by

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m}a\right)^{-n}\ , \tag{116.19}$$

where n is a positive integer. The larger n is, the sharper will be the transition. However, for calculating the action, the result does not depend strongly on how smooth or sharp is the transition. Therefore, to keep the calculations simple, we take $n = 1$ to give

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m}a\right)^{-1}\ . \tag{116.20}$$

Putting (118.18) and (118.16) in (118.14) gives

$$p = \frac{1}{3}\frac{3H_0^2}{8\pi}\frac{\Omega_r}{a^4}\ . \tag{116.21}$$

Putting (118.19) and (118.16) into (118.3) gives

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda = \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] , \tag{116.22}$$

with $w = 1/3$. We can approximate (118.20) in different eras.

$$\begin{aligned} \tilde{L} &= \frac{3H_0^2}{8\pi}\left[\alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] \text{ for } a \leq a_m \approx 10^{-2} \\ \tilde{L} &= \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3}\right] \text{ for } a \geq a_m \approx 10^{-2} \end{aligned} \tag{116.23}$$

We can convert (118.13) into an integral to get

$$t = \frac{1}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_\Lambda a^2 + \Omega_m/a + \Omega_r/a^2}} . \tag{116.24}$$

Equation (118.22) is a well-defined integral to give the global time t as a function of the cosmological scale factor a. Although it is not easy to calculate in closed form, there is no region where more than two terms in the radical are significant. That allows a very good approximate evaluation of the integral in closed form. We have

$$\begin{aligned} t &= \frac{2}{3H_0}\frac{\Omega_r^{3/2}}{\Omega_m^2}\left[2 - \left(2 - \frac{\Omega_m}{\Omega_r}a\right)\sqrt{1 + \frac{\Omega_m}{\Omega_r}a}\right] \text{ for } a \leq a_m \approx 10^{-2}, \\ &\text{and} \\ t &= \frac{1}{3H_0\sqrt{\Omega_\Lambda}}\ln\frac{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda}a^{-3}} + 1}{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda}a^{-3}} - 1} \text{ for } a \geq a_m, \end{aligned} \tag{116.25}$$

where

$$a_{\text{eq}} = \frac{\Omega_r}{\Omega_m} \approx 3 \times 10^{-4} \ll a_m \ll \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \approx 0.76 . \tag{116.26}$$

Using (118.13) and (118.23) allows us to make a table that gives H as a function of global time t. When the cosmological scale factor a is very small, we can make some approximations. These approximations are valid in the very early universe.

$$a^2 \approx 2H_0\sqrt{\Omega_r}t, \text{ that is, } t \approx 2.4 \times 10^{19} a^2 \text{ seconds } \text{ for } a \ll \Omega_r/\Omega_m. \tag{116.27}$$

$$H \approx \frac{1}{2t} \text{ for } a \ll \Omega_r/\Omega_m. \tag{116.28}$$

For the inflation era (from 10^{-36} seconds to 10^{-34} seconds), we choose a constant value for the Hubble parameter H that will give 50, 55, or 60 e-foldings. [390] estimates that there were about 50 to 60 e-foldings during inflation.

Table 118.2 gives the results of the calculations neglecting inflation for selected values of global time corresponding to values of the cosmological scale factor a. Table 118.1 gives the results of these calculations including inflation with 50, 55, or 60 e-foldings for the same values of global time and values of the cosmological scale factor a as in table 118.2.

116.7 Approximate Generalized Friedmann equation for small acceleration

If there were no acceleration, then we could use the Friedmann equation to calculate $\dot{a}$ in (118.6). However, with acceleration, the Friedmann equation, generalized to include vorticity, shear, and acceleration can be calculated from the Raychoudhury equation to give [386, Appendix F]

$$\dot{\ell} = \ell\sqrt{H(a)^2 + H_\omega^2 + H_\sigma^2 + H_a^2} \ , \tag{116.29}$$

where ℓ is a scale factor along lines of cosmic flow. In the presence of acceleration, a and ℓ differ. $H(a)$ [given by (118.13) in 118.6] is the Hubble parameter without vorticity, shear, or acceleration,

$$H_\omega^2 \equiv \frac{4}{3\ell^2}\int \ell\omega^2 \, \mathrm{d}\ell \tag{116.30}$$

is the vorticity term, ω is vorticity,

$$H_\sigma^2 \equiv -\frac{4}{3\ell^2}\int \ell\sigma^2 \, \mathrm{d}\ell \tag{116.31}$$

is the shear term, σ is shear, and

$$H_a^2 \equiv \frac{2}{3\ell^2}\int \ell\dot{u}^a{}_{;a} \, \mathrm{d}\ell = \frac{2}{3\ell^2}\int \ell \ {}^{(3)}\nabla_a \dot{u}^a \, \mathrm{d}\ell + \frac{2}{3\ell^2}\int \ell\dot{u}_a\dot{u}^a \, \mathrm{d}\ell \to \frac{2}{3\ell^2}\int \ell\dot{u}_a\dot{u}^a \, \mathrm{d}\ell \tag{116.32}$$

is the acceleration term, where ${}^{(3)}\nabla_a$ is a 3-dimensional spatial gradient.

We can take acceleration to depend on the distance along flow lines ℓ as

$$\dot{u} \propto \ell^{-m} \ , \tag{116.33}$$

where the value of m depends on the assumptions we make.

116.8 Formulas for $C_I(a_f)$ including inflation

The details for the derivation of these formulas are in [410].

$$C_I(a_N) \approx \frac{\alpha_1 w + \alpha_2}{16}\frac{e^{(2m_r+1)N}}{N^3 m_r^2} \ . \tag{116.34}$$

For $a_f \gg a_N$, some terms dominate over others, so that we have

$$C_I(a_f) \approx f(a_f)\frac{e^{2Nm_r}}{4N^2 m_r^2} \ , \tag{116.35}$$

where

$$\begin{aligned} f(a_f) &= f_1(a_f) \text{ for } a_N \ll a_f \le a_{eq} \ , \\ f(a_f) &= f(a_{eq}) + f_2(a_f) f_3(a_f) \text{ for } a_{eq} \le a_f \le a_\Lambda, \\ f(a_f) &= f(a_\Lambda) - f_2(a_f)\pi\alpha_3 \left(\frac{a_f}{a_\Lambda}\right)^{2m_m-3} \ln\frac{a_f}{a_\Lambda} \text{ for } a_\Lambda \le a_f \le a_0 = 1 \ , \end{aligned} \tag{116.36}$$

$$f_1(a) \equiv \frac{\alpha_1 w + \alpha_2}{4}\left[\frac{e^N}{N} + \left(\frac{a}{a_N}\right)^{2m_r-4}\right] \ , \tag{116.37}$$

$$f_2(a) \equiv 3\frac{\Omega_m}{\Omega_r}\frac{a_N^4}{a^3}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a}{a_{eq}}\right)^{2m_m} \ , \tag{116.38}$$

and

$$f_3(a) \equiv -\frac{\Omega_\Lambda H_0^2}{3H(a)^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right)\left[1 - \left(\frac{a_{eq}}{a}\right)^{9/2}\right] + \frac{\alpha_1 w + \alpha_2}{12}\left[1 - \left(\frac{a_{eq}}{a}\right)^{3/2}\right] \ . \tag{116.39}$$

116.9 Formulas for $C_I(a_f)$ neglecting inflation

The details for the derivation of these formulas are in [410].

For $a_f \gg a_N$, keeping only the most significant terms gives

$$
\begin{aligned}
&C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{27} \text{ for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 1 \text{ and for no inflation,}\\
&C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{2} \left(\ln \frac{a_f}{a_N}\right)^2\\
&\quad \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 2 \text{ and for no inflation,}\\
&C_I(a_f) \approx (\alpha_1 w + \alpha_2) \left[\frac{1}{2}\left(\frac{a_f}{a_N}\right)^2 - \frac{3}{4}\frac{a_f}{a_N}\left(\ln \frac{a_f}{a_N}\right)^2\right]\\
&\quad \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 3 \text{ and for no inflation,}\\
&C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{8} \left(\frac{a_f}{a_N}\right)^4\\
&\quad \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 4 \text{ and for no inflation,}\\
&C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2 \Omega_r} \left\{ \frac{H_0^2 \Omega_\Lambda}{H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) \times \right.\\
&\left[-9 + 63\left(\frac{a_{eq}}{a_f}\right)^{1/2} - \frac{1}{7}\left(67 - \frac{3}{7} + 12 \ln \frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - \frac{1}{7}\left(311 + \frac{3}{7}\right)\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right]\\
&\left. + \frac{\alpha_2}{12}\left[27 + 36\left(\frac{a_{eq}}{a_f}\right)^{1/2} + \left(326 - 60 \ln \frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - 389\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}\\
&\quad \text{for } a_{eq} \leq a_f \leq a_\Lambda \text{ and for } m_r = 1 \text{ and for } m_m = 2 \text{ and for no inflation.}\\
&C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2 \Omega_r} \left\{ \frac{H_0^2 \Omega_\Lambda}{H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) \times \right.\\
&\left[-\left(\ln \frac{a_{eq}}{a_N}\right)^2 - 4 \ln \frac{a_{eq}}{a_N} - 4 + 6\left(\ln \frac{a_{eq}}{a_N} + 2\right)\left(\frac{a_{eq}}{a_f}\right)^{1/2}\right.\\
&\left. -\frac{1}{7}\left(12 \ln \frac{a_{eq}}{a_N} + 57 - \frac{3}{7} + 12 \ln \frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} + \left(\left(\ln \frac{a_{eq}}{a_N}\right)^2 - \frac{2}{7}\ln \frac{a_{eq}}{a_N} + \frac{4}{49}\right)\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right]\\
&+\frac{\alpha_2}{12}\left[3\left(\ln \frac{a_{eq}}{a_N}\right)^2 + 12 \ln \frac{a_{eq}}{a_N} + 12 + 12 \ln \frac{a_{eq}}{a_N}\left(\frac{a_{eq}}{a_f}\right)^{1/2}\right.\\
&\left.\left. + \left(-12 \ln \frac{a_{eq}}{a_N} + 84 - 60 \ln \frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - \left(3\left(\ln \frac{a_{eq}}{a_N}\right)^2 + 12 \ln \frac{a_{eq}}{a_N} + 96\right)\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}\\
&\quad \text{for } a_{eq} \leq a_f \leq a_\Lambda \text{ and for } m_r = 2 \text{ and for } m_m = 2 \text{ and for no inflation.}\\
&C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_N^4}{H_0^2 \Omega_r (m_r - 2)^2}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times\\
&\left\{-\frac{H_0^2 \Omega_\Lambda}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1 - \left(\frac{a_{eq}}{a_f}\right)^{9/2}\right] + \frac{\alpha_2}{4}\left[1 - \left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}\\
&\quad \text{for } a_{eq} \leq a_f \leq a_\Lambda \text{ and for } m_r > 2 \text{ and for } m_m > 2 \text{ and for no inflation.}
\end{aligned}
$$

$$
\begin{aligned}
&C_I(a_f) \approx C_I(a_\Lambda) - 27\frac{H_f^2 a_f^4}{H_0^2\Omega_r}\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1-\left(\frac{a_\Lambda}{a_f}\right)^3\right] \\
&\quad \text{for } a_\Lambda \le a_f \le a_0 = 1 \text{ and for } m_r = 1 \text{ and for } m_m = 2 \text{ and for no inflation,} \\
&C_I(a_f) \approx C_I(a_\Lambda) - 3\frac{H_f^2 a_f^4}{H_0^2\Omega_r}\left[\ln\frac{a_{eq}}{a_N}+2\right]^2\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1-\left(\frac{a_\Lambda}{a_f}\right)^3\right] \\
&\quad \text{for } a_\Lambda \le a_f \le a_0 = 1 \text{ and for } m_r = 2 \text{ and for } m_m = 2 \text{ and for no inflation,} \\
&C_I(a_f) \approx C_I(a_\Lambda) - 3\frac{H_f^2 a_N^4}{H_0^2\Omega_r(m_r-2)^2}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times \\
&\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\left[1-\left(\frac{a_\Lambda}{a_f}\right)^3\right] \\
&\quad \text{for } a_\Lambda \le a_f \le a_0 = 1 \text{ and for } m_r > 2 \text{ and for } m_m > 2 \text{ and for no inflation,}
\end{aligned}
\tag{116.40}
$$

where

$$
\begin{aligned}
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{27} \quad \text{for } m_r = 1 \text{ and for no inflation,} \\
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{2}\left(\ln\frac{a_{eq}}{a_N}\right)^2 \quad \text{for } m_r = 2 \text{ and for no inflation,} \\
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{2}\left(\frac{a_{eq}}{a_N}\right)^2 \quad \text{for } m_r = 3 \text{ and for no inflation,} \\
C_I(a_{eq}) &\approx \frac{\alpha_1 w + \alpha_2}{8}\left(\frac{a_{eq}}{a_N}\right)^4 \quad \text{for } m_r = 4 \text{ and for no inflation,}
\end{aligned}
\tag{116.41}
$$

and

$$
\begin{aligned}
&C_I(a_\Lambda) \approx C_I(a_{eq}) + 9\frac{H_\Lambda^2 a_\Lambda^4}{H_0^2\Omega_r}\left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)+\frac{\alpha_2}{4}\right\} \\
&\quad \text{for } m_r = 1 \text{ and for } m_m = 2 \text{ and for no inflation,} \\
&C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_\Lambda^4}{H_0^2\Omega_r}\left[\ln\frac{a_{eq}}{a_N}+2\right]^2\left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)+\frac{\alpha_2}{4}\right\} \\
&\quad \text{for } m_r = 2 \text{ and for } m_m = 2 \text{ and for no inflation,} \\
&C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_N^4}{H_0^2\Omega_r(m_r-2)^2}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a_\Lambda}{a_{eq}}\right)^{2m_m} \times \\
&\left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)+\frac{\alpha_2}{4}\right\} \\
&\quad \text{for } m_r > 2 \text{ and for } m_m > 2 \text{ and for no inflation.}
\end{aligned}
\tag{116.42}
$$

Table 116.1: Maximum rms acceleration $\langle \dot{u}_f \rangle_{\max}$ as a function of global time t_f, m_r, and m_m for an inflation era with 50, 55, and 60 e-foldings. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. The last 3 columns are calculated using equations (118.12), (118.32), and (118.33).

t_f	m_r	m_m	$\langle \dot{u}_f \rangle$max for 50 e-foldings m s^{-2}	$\langle \dot{u}_f \rangle$max for 55 e-foldings m s^{-2}	$\langle \dot{u}_f \rangle$max for 60 e-foldings m s^{-2}
beginning of inflation 10^{-36} s					
end of inflation 10^{-34} s	1		9×10^{-46}	6×10^{-49}	3×10^{-52}
	2		3×10^{-67}	1×10^{-72}	6×10^{-76}
	3		9×10^{-89}	3×10^{-96}	9×10^{-104}
	4		2×10^{-110}	6×10^{-120}	9×10^{-130}
electroweak transition 2.4×10^{-11} s	1		1×10^{-64}	9×10^{-68}	6×10^{-71}
	2		6×10^{-86}	2×10^{-91}	9×10^{-97}
	3		2×10^{-107}	6×10^{-115}	1×10^{-122}
	4		3×10^{-129}	9×10^{-139}	2×10^{-148}
radiation era 0.24 s	1		4×10^{-77}	1×10^{-83}	4×10^{-87}
	2		2×10^{-102}	8×10^{-108}	2×10^{-113}
	3		2×10^{-124}	4×10^{-132}	1×10^{-139}
	4		1×10^{-155}	3×10^{-164}	5×10^{-173}
matter/ radiation equality 5×10^4 yr	1		4×10^{-92}	8×10^{-99}	4×10^{-114}
	2		1×10^{-117}	5×10^{-123}	2×10^{-140}
	3		1×10^{-145}	3×10^{-152}	1×10^{-158}
	4		1×10^{-183}	2×10^{-192}	4×10^{-201}
recombi- nation 3.8×10^5 yr	1	2	3×10^{-92}	2×10^{-97}	1×10^{-98}
	2	2	1×10^{-113}	5×10^{-119}	2×10^{-154}
	3	3	$\approx 10^{-141}$	$\approx 10^{-147}$	$\approx 10^{-141}$
	4	4	$\approx 10^{-179}$	$\approx 10^{-188}$	$\approx 10^{-197}$
matter/ dark energy equality 10×10^9 yr	1	2	8×10^{-101}	5×10^{-106}	3×10^{-107}
	2	2	3×10^{-122}	1×10^{-127}	6×10^{-133}
	3	3	$\approx 10^{-155}$	$\approx 10^{-162}$	$\approx 10^{-168}$
	4	4	$\approx 10^{-196}$	$\approx 10^{-205}$	$\approx 10^{-214}$
today 13.8×10^9 yr	1	2	5×10^{-101}	3×10^{-106}	2×10^{-107}
	2	2	2×10^{-122}	8×10^{-128}	3×10^{-133}
	3	3	$\approx 10^{-155}$	$\approx 10^{-162}$	$\approx 10^{-168}$
	4	4	$\approx 10^{-196}$	$\approx 10^{-205}$	$\approx 10^{-214}$

Table 116.2: Cosmological scale factor a_f, Hubble parameter H_f, and maximum rms acceleration $\langle \dot{u}_f \rangle$max as a function of global time t_f, m_r, and m_m neglecting inflation. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. Everything but the last 3 columns is calculated using the formulas in 118.6. The last column is calculated using equations (118.12) and (118.38). $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds. $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length.

t_f		a_f	H_f s^{-1}	$L^* \sqrt{\frac{H_f}{(r_0 a_f)^3}}$ m s^{-2}	m_r	m_m	$\langle \dot{u}_f \rangle$max m s^{-2}
end of inflation 10^{-34} s		$a_N =$ 2×10^{-20}	5×10^{33}	2.3×10^{-16}			
electroweak transition 2.4×10^{-11} s		10^{-15}	2×10^{10}	4.0×10^{-35}	1 2 3 4		2×10^{-34} 6×10^{-36} 1×10^{-39} 3×10^{-44}
radiation era 0.24 s		10^{-10}	2	1.3×10^{-47}	1 2 3 4		7×10^{-47} 8×10^{-49} 4×10^{-57} 1×10^{-66}
matter/ radiation equality 5×10^4 yr		$a_{\text{eq}} =$ 3×10^{-4}	3×10^{-13}	9.5×10^{-63}	1 2 3 4		5×10^{-62} 4×10^{-64} 9×10^{-79} 1×10^{-94}
recombi- nation 3.8×10^5 yr		$.9 \times 10^{-3}$	5×10^{-14}	7.5×10^{-63}	1 2 3 4	2 2 3 4	2×10^{-63} 3×10^{-64} 2×10^{-79} 8×10^{-96}
matter/ dark energy equality 10×10^9 yr		$a_\Lambda = 0.76$	2.5×10^{-18}	2.1×10^{-71}	1 2 3 4	2 2 3 4	1×10^{-73} 8×10^{-75} 8×10^{-93} 2×10^{-112}
today 13.8×10^9 yr		$a_0 = 1.0$	2.2×10^{-18}	1.3×10^{-71}	1 2 3 4	2 2 3 4	7×10^{-74} 5×10^{-75} 5×10^{-93} 1×10^{-112}

Chapter 117

The fluctuating field interpretation of quantum mechanics[1]

abstract

Combining the fluctuating field interpretation of quantum mechanics with the transactional interpretation shows that the wave function represents neither information nor a physical field but instead represents the average behavior of the actual physical field, which fluctuates randomly.

117.1 Introduction

As pointed out by Cramer [413], an interpretation of quantum mechanics tries to explain what cannot be measured. Therefore, it is not possible to validate an interpretation by measurements. To be acceptable, an interpretation should:

1. be consistent
2. give insight into the true nature of quantum physics
3. be able to explain several of the difficulties of quantum theory
4. give direction in developing new theories, such as quantum gravity

The transactional interpretation of quantum mechanics [413] is a mathematical interpretation of the solutions to the wave equation in terms of advanced and retarded solutions. It solves some of the problems with the Copenhagen interpretation of quantum mechanics, but it is not a physical interpretation, and does not explain how the source of the offer wave chooses which confirmation wave to accept.

117.2 Fluctuating fields

The fluctuating field interpretation explains the following:

1. It answers the question about whether the wave function is a physical field or whether it represents knowledge (it is neither).
2. It explains collapse of the wave packet.

[1] 9 January 2022

3. It explains the measurement problem.

The wave function represents the actual average behavior of the corresponding physical field. The actual physical field fluctuates very quickly. It can move faster than light because it does not carry energy or information. It is not possible to send signals faster than the speed of light using it. It is analogous to a phase velocity in that sense.

"Collapse of the wave packet" is explained in the following way: The actual field is always fluctuating, so in a sense, it is always collapsing.

Our present equations tell us about the wave function, which represents the average behavior of the physical field. In the future, it might be possible to find equations that describe the actual physical field. Such equations will most likely be nonlinear, and most likely have chaotic solutions, thus explaining the apparently random fluctuations.

In the present theory, a "measurement" is an interaction that is not included in the interactions in Schrödinger's or Dirac's equation. The as-yet-to-be-discovered equations describing the actual physical field should include those interactions now classified as "measurements."

The fluctuating field interpretation of quantum mechanics is consistent with the transactional interpretation of quantum mechanics [413], but supplements it by explaining that the wave function (or state vector, as it is referred to by Cramer [413]) is not the physical field, but only represents the average of the fluctuating physical field.

117.3 Entanglement

Entanglement is more difficult to deal with, but it can be done. Consider a standard EPR experiment, in which two particles in a spin-zero state head off in opposite directions. The spins must be opposite, but not determined. The system is in a superposition of spin-up left particle with spin-down right particle and spin-down left particle with spin-up right particle. That is the wave function, which represents the average behavior. The actual physical fields have the particles fluctuating between those two cases.

Now we let one of the particles go through an inhomogeneous magnetic field. That splits the particle onto two paths, depending on the spin. We still have a quantum superposition, that now includes a quantum superposition of one of the particles on two paths. The actual physical field fluctuates between those two paths for that particle.

Now suppose the particle that is split hits a screen (it gets detected). That collapses the wave function. It does not matter if anyone observes it. At that point, the spin of both particles is determined, and we no longer have a quantum superposition. It is not the relativity of simultaneity of special relativity that explains the collapse of the wave function; it is that the fluctuating field is not constrained by light speed.

117.4 Summary

As far as I can tell, the fluctuating field interpretation of quantum mechanics can explain all quantum phenomena.

Chapter 118

Relative acceleration of matter and inertial frames[1]

abstract

Any reasonable form of quantum gravity can explain (by phase interference) why we do not observe large-scale average acceleration of inertial frames relative to the average matter distribution, without the need for absolute space or finely tuned initial conditions, while still keeping independent degrees of freedom for the gravitational field. A simple saddle-point approximation to a path-integral calculation is used to estimate the limits on the large-scale relative acceleration of matter and inertial frames as a function of cosmic time for inflation rates of 50, 55, and 60 e-foldings, and for values of the dependence of relative acceleration on cosmological scale factor a as a^{-m} for various values of m. Even if there were no inflation, relative acceleration would be restricted to be less than about 10^{-34} meters/sec^2 at 10^{-11} sec for $m = 1$, with tighter restrictions for larger m, tighter restrictions for inflation, and even tighter restrictions today. Although these calculations are based on solutions to Einstein's field equations, the formulas are valid for a more general dependence of the Lagrangian on the pressure, density, and cosmological constant.

118.1 Introduction

Ernst Mach [120, 102, 122, 15] argued over a hundred years ago that absolute space (as proposed by Newton) was unobservable. Therefore, Newton's second law, written in terms of acceleration relative to absolute space made no sense. He proposed that physical law should be written in terms of relative coordinates, for example, the coordinates of a local body relative to all of the rest of the matter in the universe. This would explain why we do not observe large-scale rotation of inertial frames relative to the average distribution of matter [164, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312], nor why we do not observe large-scale relative acceleration between matter and inertial frames.

Einstein tried to incorporate Mach's ideas (which he called "Mach's Principle" [152]) in General Relativity. However, he was only partially successful because solutions to Einstein's field equations depend not only on the distribution of matter (in terms of the stress-energy tensor), but depend also on initial conditions and boundary conditions. There are many solutions of Einstein's field equations for General Relativity that have large-scale relative rotation of matter and inertial frames, e.g. [347, 348, 163, 370, 371], and there are many solutions of Einstein's field equations for General Relativity that have large-scale relative acceleration of matter and inertial frames, e.g. [396, 397,

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398, 399, 400, 401, 402, 403, 404]. It would require very finely tuned initial conditions to give us the universe we observe, in which we do not observe large-scale rotation or acceleration relative to the matter distribution. In response, Sciama and others [11, 16, 156, 109, 315, 159] have proposed using Mach's Principle as a boundary condition to restrict solutions to Einstein's equations. This would mean that the gravitational field would be completely determined by its sources and would have no independent degrees of freedom in terms of initial conditions and boundary conditions. Thus, the gravitational field would behave very differently from the electromagnetic field, for example.

The problem is to get the effect of Mach's Principle without actually requiring it. That is, can we find a mechanism for implementing Mach's principle? As it turns out, at least for rotation, we can. In a path-integral representation of quantum cosmology, those cosmologies that have significant relative rotation would cancel each other out by phase interference because the action has an extremum at zero relative rotation and the region of constructive interference around zero relative rotation is very small. [386]

We use a similar calculation here to explain why we do not observe large-scale relative acceleration of matter and inertial frames. We would expect evidence of relative acceleration to appear as anisotropy in the Cosmic Microwave Background. From the CMB temperature anisotropy "dipole," we know that the Earth has a peculiar velocity deviating from the "cosmic flow." There are assignable causes for this such as the rotation of our galaxy, attraction to other galaxies such as Andromeda, and attraction to aggregate large clusters of galaxies. Apart from these knowable causes, we do not observe any other relative acceleration between matter and inertial frames on very large scales. [390, 369].

Here, we calculate the size of the region of constructive interference as a function of cosmic time. We show not only that phase interference leads to a very small relative acceleration, but even though inflation dominates the calculation, the allowed relative acceleration neglecting inflation would already have been very small a fraction of a second after the initial singularity.

As is well-known, we do not have a generally accepted theory of quantum gravity or quantum cosmology. There are many difficulties in developing a theory of quantum gravity e.g. [373, 374], but it is generally believed that a theory of quantum gravity should exist e.g.[406]. Some ideas for developing a theory of quantum gravity are discussed in [62, 63, 64, 19, 375, 376, 377]. Fortunately, some calculations (including the present one) can be made without a final theory of quantum gravity.

Section 118.2 reviews path integrals in quantum cosmology. Section 118.3 gives the approximate action for small relative acceleration. Section 118.4 calculates a saddlepoint approximation to the path integral. Section 118.5 discusses the results.

Appendix 118.6 gives the background cosmology. Appendix 118.7 gives the approximate generalized Friedmann equation for small acceleration. Appendix 118.8 and appendix 118.9 give formulas for the function $C_I(a_f)$ that is used to calculate the action for small relative acceleration. Appendix 118.8 includes inflation, while appendix 118.9 neglects inflation. Although the derivation of the formulas in appendix 118.8 and appendix 118.9 is straightforward, and all of the information necessary to derive those formulas is included here, the derivation is long and tedious. For those interested, the details for the derivation of these formulas are in [414].

The speed of light c and Newton's gravitational constant G are set to 1 throughout except when converting to conventional units.

118.2 Review of path integrals in quantum cosmology

Following Hartle and Hawking [124] and Jones [386, Appendix A], a path-integral calculation gives

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int\int \exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right) \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi\, \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ . \tag{118.1}$$

where $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ is the amplitude that the 3-geometry was ${}^{(3)}\mathcal{G}_i$ on some initial space-like hypersurface and the matter fields on that 3-geometry were ϕ_i, $\psi_f({}^{(3)}\mathcal{G}_f, \phi_f)$ is the amplitude that the 3-geometry is ${}^{(3)}\mathcal{G}_f$ on some final space-like hypersurface and that the matter fields on that 3-geometry are ϕ_f, $\exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right)$ is the contribution of the 4-geometry ${}^{(4)}\mathcal{G}$ and matter field ϕ on that 4-geometry to the path integral, where $I[{}^{(4)}\mathcal{G}, \phi]$ is the action. A correct theory of quantum gravity would be necessary to specify the measures $\mathcal{D}^{(4)}\mathcal{G}$ and $\mathcal{D}\phi$.

Although in (118.1), the integration is over all possible 4-geometries, not just classical 4-geometries, the main contribution to the integral (in most cases) comes from classical 4-geometries, e.g. [221, 220]. That allows us to restrict (118.1) to be an integration over classical 4-geometries.

The condition that there are not finely tuned initial conditions is equivalent to $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ being a broad wave function. That allows us to neglect the effect of that initial wave function on the integration in the path integral in (118.1). In addition, we consider 4-geometries characterized by a parameter $\langle \dot{u}_f \rangle$ which we take to be the rms relative acceleration on the space-like hypersurface at $t = t_f$. Thus, we can rewrite (118.1) for our purposes as

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) \propto \int_{-\infty}^{\infty} A(\langle \dot{u}_f \rangle) e^{iI(\langle \dot{u}_f \rangle)/\hbar} \, \mathrm{d}\langle \dot{u}_f \rangle \ , \tag{118.2}$$

where $A(\langle \dot{u}_f \rangle)$ is a slowly varying function of $\langle \dot{u}_f \rangle$, $I(\langle \dot{u}_f \rangle)$ is the action, and since for classical spacetimes, the acceleration is a known function of cosmological time, we can consider the action I to depend on the rms acceleration at any cosmological time we choose, say t_f, which we designate as $\langle \dot{u}_f \rangle$, where $\langle \dot{u}_f \rangle^2 \equiv \overline{\dot{u}_f^2}$, where the average is a spatial average over the volume within the past light cone.

Appendix 118.6 discusses the background cosmology, including the inflation, radiation, matter, and dark-energy eras because the Lagrangian would depend differently on the cosmological scale factor in the four eras.

118.3 Approximate action for small acceleration

The action in (118.2) is equal to a volume integral over spacetime plus a surface integral. The surface term is necessary to insure consistency if the action integral is broken into parts [183, 123]. For the present purposes, it is not necessary to consider the surface term.

In addition, for 4-geometries that are solutions of Einstein's field equations, the Lagrangian can be expressed as an effective Lagrangian e.g. [386, Appendix B]

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda \ , \tag{118.3}$$

where p is pressure, ρ is density, Λ is the cosmological constant, and α_1, α_2, and α_3 are dimensionless constants of order unity.

That allows us to write the action as

$$I = \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^4 x = \int_{t_i}^{t_f} \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^3 x \mathrm{d}t \ , \tag{118.4}$$

where t_i is the initial time (which we take to be the beginning of the inflation era), and t_f is the final time, which we choose arbitrarily to calculate the action at any specified cosmic time.

We can convert the time integral in (118.4) to an integral over the cosmological scale factor a

$$I = \int_{a_i}^{a_f} \int \frac{(-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^3 x \, \mathrm{d}\, a}{\dot{a}} \ , \tag{118.5}$$

where $\dot{a} = \mathrm{d}a/\mathrm{d}t$ is given by a generalization of the Friedmann equation that includes acceleration (appendix 118.7), a_i is the value of the cosmological scale factor at $t = t_i$, and a_f is the value of the cosmological scale factor at $t = t_f$.

Including relative acceleration in the calculation of the action (even approximately) is tedious, even though straightforward. The details of the calculation of the total action to second order in the mean square of the relative acceleration at $t = t_f$ is given by [414]. That calculation includes taking into account the fact that if there is relative acceleration of the matter distribution and the inertial frame, then density and pressure will depend on the scale length ℓ along flow lines rather than depend on the cosmological scale factor a. The significance is that the cosmological scale factor a is a function of global time, but scale length ℓ along flow lines is not normal to surfaces of constant global time. We have $\ell = \int \mathrm{d}\,a/\cosh\theta$, where $\theta = \int \dot{u}\,\mathrm{d}\,t$. The formulas are then expanded to second order in $\dot{u}$ and θ. This causes the calculation to be much more complicated.

We can express the volume integral in (118.5) as a product of the spatial volume and the spatial average.

$$I = \int_{a_i}^{a_f} V(a)\overline{\left(\frac{\tilde{L}}{\dot{a}}\right)}\,\mathrm{d}\,a\ , \tag{118.6}$$

where we have used the knowledge that our universe is spatially flat, an overbar indicates a spatial average,

$$V(a) = \frac{4}{3}\pi a^3 r_0^3 \tag{118.7}$$

is the approximate spatial volume[2], and r_0 is the present radius of the cosmological horizon.

The result to lowest order in $\langle \dot{u}_f \rangle / H_f$ is

$$I(\langle \dot{u}_f \rangle) \approx I_0 + \hbar \left(\frac{\langle \dot{u}_f \rangle}{\dot{u}_m}\right)^2 f_I(\langle \dot{u}_f \rangle)\ , \tag{118.8}$$

where I_0 is the action for the standard cosmological model (the Robertson-Walker cosmology),

$$\dot{u}_m = \left(\frac{\hbar H_f}{r_0^3 a_f^3}\right)^{1/2} = L^* \sqrt{\frac{H_f}{r_f^3}} = L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}}\ , \tag{118.9}$$

$r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds, $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length,

$$\begin{aligned} f_I(\langle \dot{u}_f \rangle) &\approx \left[C_I(a_f) + \tfrac{\langle \dot{u}_f \rangle^2 + \sigma_a^2/\langle \dot{u}_f \rangle^2}{H_f^2} C_{II}(a_f)\right] \\ &\approx C_I(a_f) \text{ for small } \langle \dot{u}_f \rangle \end{aligned} \tag{118.10}$$

is some slowly varying dimensionless even function of $\langle \dot{u}_f \rangle$, σ_a^2 is the variance of $\dot{u}_f^2$, $C_I(a_f)$ and $C_{II}(a_f)$ are dimensionless functions, and the details for the derivation of these formulas are in [414].

Notice that (118.8) reduces to I_0, the action for the standard cosmological model, when the rms acceleration $\langle \dot{u}_f \rangle$ is zero, and that (118.8) is an even function of $\langle \dot{u}_f \rangle$ if $f_I(\langle \dot{u}_f \rangle)$ is an even function of $\langle \dot{u}_f \rangle$.

118.4 Saddlepoint approximation

Comparing (118.2) and (118.8) shows that (118.2) has a saddlepoint at $\langle \dot{u}_f \rangle = 0$. If that saddlepoint is the only significant saddlepoint (and other criteria are satisfied), then the only significant

[2] Since our universe is approximately spatially flat. Although there are arguments that V should be infinite because this is an open cosmology [351], considering causality leads to restricting the spatial part of the action to the past light cone.

contribution to the path integral (118.2) comes from values of the rms acceleration at the time t_f of

$$\langle \dot{u}_f \rangle = 0 \pm \dot{u}_m / \sqrt{|\, f_I(0)\,|}\ . \tag{118.11}$$

Either a stationary-phase path or a steepest-descent path could be used when making the saddlepoint approximation [134, 333, 219], but here, we use a stationary-phase path. Halliwell [332] gives an example of a more detailed path-integral calculation of quantum gravity.

The saddlepoint at $\langle \dot{u}_f \rangle = 0$ in (118.2) is isolated from other saddlepoints and any possible non-analytic points as shown by (118.10). The integral in (118.2) can be approximated by a saddlepoint integration to give

$$\begin{aligned}
&\psi_f \propto A(0)\dot{u}_m\sqrt{\pi}/\sqrt{f_I(0)}e^{i\pi/4} \\
&\text{for } \langle \dot{u}_f \rangle < \frac{\dot{u}_m}{\sqrt{|\, f_I(0)\,|}} \approx L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \frac{1}{\sqrt{|\, f_I(0)\,|}} \approx L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \frac{1}{\sqrt{|\, C_I(a_f)\,|}}, \\
&\psi_f \approx 0, \text{ otherwise,}
\end{aligned} \tag{118.12}$$

where $r_0 \approx 46.5 \times 10^9$ light years is the present radius of the observable universe and $L^* \approx 2 \times 10^{-35}$ m is the Planck length.

Appendix 118.8 calculates $C_I(a_f)$ including inflation. This allows calculating estimates for allowed upper limits on the values for the rms relative acceleration $\langle \dot{u}_f \rangle$ from (118.12) given in table 118.1. Appendix 118.9 calculates $C_I(a_f)$ neglecting inflation. This allows calculating estimates for allowed upper limits on the values for the rms relative acceleration $\langle \dot{u}_f \rangle$ from (118.12) given in table 118.2.

Although relative acceleration is associated with a scalar mode perturbation [411, Chapter 29], which leads to $m = 4$, we include also calculations for $m = 1$, $m = 2$, and $m = 3$ for completeness.

118.5 Discussion

Table 118.1 gives the limits on rms relative acceleration as a function of cosmic time from (118.12) including the effects of inflation with 50, 55, and 60 e-foldings. Calculations are included for $m = 1$ in the radiation era and $m = 2$ in the matter era and also for $m = 2$, 3, and 4 in all eras, where m gives the dependence of relative acceleration on cosmological scale factor a as a^{-m}. As listed in table 118.1, m_r is the value in the radiation era, and m_m is the value in the matter era.

Inflation dominates the calculation and gives a limit of relative acceleration at the electroweak transition[3] ($\approx 2.4 \times 10^{-11}$ s) of 10^{-64}, 10^{-85}, 10^{-107}, and 10^{-129} m s^{-2} for $m = 1$, 2, 3, and 4 for 50 e-foldings, with tighter restrictions for 55 and 60 e-foldings. The corresponding values at the present time are 10^{-100}, 10^{-122}, 10^{-155}, and 10^{-196} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

However, although inflation dominates the calculation, inflation is not necessary to restrict the relative acceleration to small values, as shown in table 118.2, which gives the limits on rms relative acceleration as a function of cosmic time from (118.12) neglecting inflation. Even if there were no inflation, relative acceleration would be restricted to be not much larger than about $L^* \sqrt{H/(r_0 a)^3}$, where $L^* \approx 2 \times 10^{-35}$ m is the Planck length, $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds is the approximate radius of the observable universe today, H is the Hubble parameter, which varies from about 2×10^{10} s^{-1} at the electroweak transition to $\approx 2 \times 10^{-18}$ s^{-1} today, and a is the cosmological scale factor, which varies from about 10^{-15} at the electroweak transition to 1 today.

[3] Whittle [407, page 230] gives $\approx 10^{-15}$ for the value of the cosmological scale factor at the electroweak transition. Using that value in (118.25) gives $\approx 2.4 \times 10^{-11}$ s for the approximate time of the electroweak transition.

As shown by table 118.2, even in the absence of inflation, relative acceleration at the electroweak transition would have been limited to about 10^{-34}, 10^{-35}, 10^{-39}, and 10^{-44} m s^{-2} for $m = 1$, 2, 3, and 4. The corresponding values at the present time without inflation are 10^{-73}, 10^{-74}, 10^{-92}, and 10^{-112} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

The calculations used for tables 118.1 and 118.2 take the quantities $\alpha_1 w + \alpha_2$ and α_3 to be of order unity. If desired, other specific values could be used instead.

Because quantum gravity (specifically, quantum cosmology) explains through phase interference why we should not observe relative rotation or relative acceleration between matter and inertial frames on a large scale, we could say that quantum gravity may provide the mechanism for implementing Mach's principle.

118.6 Background cosmology

We start with the formula for the Hubble parameter neglecting vorticity, shear, and acceleration

$$\frac{1}{a}\frac{da}{dt} = H(a) = H_0\sqrt{\Omega_\Lambda + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2}} = \sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{a^2}} \; , \tag{118.13}$$

where Λ is the cosmological constant, ρ is density, t is global time, $H_0 = 67.74$ km s^{-1} Mpc^{-1} = 6.928×10^{-11} yr^{-1} $\approx 2.195 \times 10^{-18}$ s^{-1} [390] is the present value of the Hubble parameter, $a = 1/(z+1)$ is the cosmological scale factor, whose present value is 1, z is the redshift factor, and $\Omega_\Lambda = 0.6911$ [390] is the dark energy density divided by the critical density today. Using $z_{\rm eq} = 3402$ [412] for the redshift at radiation/matter equality with $\Omega_m = 0.3089$ [390] for the matter density today divided by the critical density gives $\Omega_r = 9.077 \times 10^{-5} \approx 9 \times 10^{-5}$ for the radiation energy density divided by the critical density today. $\Omega_k = 0.0$ within measurement error [390]. Equation (118.13) is not valid during inflation, but that era is considered later.

For an equation of state, we take

$$p = w\rho \; , \tag{118.14}$$

where $w = 1/3$ in the radiation-dominated era, and $w = 0$ in the matter-dominated era. The variation of density ρ with cosmological scale factor a is given by [342, Table 6.1]

$$\rho = \rho_{eq}(a/a_{eq})^{-3(1+w)} \; , \tag{118.15}$$

where ρ_{eq} is the value of ρ at the boundary between the radiation era and the matter era where $a = a_{eq}$.

From (118.13), we have

$$\rho = \frac{3H_0^2}{8\pi}\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right) \; , \tag{118.16}$$

which gives a smooth transition between the radiation era and the matter era, instead of the abrupt transition given by (118.15). We can have a smooth transition for pressure, also, by taking w in (118.14) to be given by

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m}a\right)^{-n} \; , \tag{118.17}$$

where n is a positive integer. The larger n is, the sharper will be the transition. However, for calculating the action, the result does not depend strongly on how smooth or sharp is the transition. Therefore, to keep the calculations simple, we take $n = 1$ to give

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m}a\right)^{-1} \; . \tag{118.18}$$

Putting (118.18) and (118.16) in (118.14) gives

$$p = \frac{1}{3}\frac{3H_0^2}{8\pi}\frac{\Omega_r}{a^4} \ . \tag{118.19}$$

Putting (118.19) and (118.16) into (118.3) gives

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda = \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] \ , \tag{118.20}$$

with $w = 1/3$. We can approximate (118.20) in different eras.

$$\begin{aligned}
\tilde{L} &= \frac{3H_0^2}{8\pi}\left[\alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] \ \text{for } a \leq a_m \approx 10^{-2} \\
\tilde{L} &= \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3}\right] \ \text{for } a \geq a_m \approx 10^{-2}
\end{aligned} \tag{118.21}$$

We can convert (118.13) into an integral to get

$$t = \frac{1}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_\Lambda a^2 + \Omega_m/a + \Omega_r/a^2}} \ . \tag{118.22}$$

Equation (118.22) is a well-defined integral to give the global time t as a function of the cosmological scale factor a. Although it is not easy to calculate in closed form, there is no region where more than two terms in the radical are significant. That allows a very good approximate evaluation of the integral in closed form. We have

$$t = \frac{2}{3H_0}\frac{\Omega_r^{3/2}}{\Omega_m^2}\left[2 - \left(2 - \frac{\Omega_m}{\Omega_r}a\right)\sqrt{1 + \frac{\Omega_m}{\Omega_r}a}\right] \ \text{for } a \leq a_m \approx 10^{-2},$$

and

$$t = \frac{1}{3H_0\sqrt{\Omega_\Lambda}}\ln\frac{\sqrt{1+\frac{\Omega_m}{\Omega_\Lambda}a^{-3}}+1}{\sqrt{1+\frac{\Omega_m}{\Omega_\Lambda}a^{-3}}-1} \ \text{for } a \geq a_m, \tag{118.23}$$

where

$$a_{\text{eq}} = \frac{\Omega_r}{\Omega_m} \approx 3\times 10^{-4} \ll a_m \ll \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \approx 0.76 \ . \tag{118.24}$$

Using (118.13) and (118.23) allows us to make a table that gives H as a function of global time t. When the cosmological scale factor a is very small, we can make some approximations. These approximations are valid in the very early universe.

$$a^2 \approx 2H_0\sqrt{\Omega_r}t, \text{ that is, } t \approx 2.4\times 10^{19}a^2 \text{ seconds } \ \text{for } a \ll \Omega_r/\Omega_m. \tag{118.25}$$

$$H \approx \frac{1}{2t} \ \text{for } a \ll \Omega_r/\Omega_m. \tag{118.26}$$

For the inflation era (from 10^{-36} seconds to 10^{-34} seconds), we choose a constant value for the Hubble parameter H that will give 50, 55, or 60 e-foldings. Reference [390] estimates that there were about 50 to 60 e-foldings during inflation.

Table 118.2 gives the results of the calculations neglecting inflation for selected values of global time corresponding to values of the cosmological scale factor a. Table 118.1 gives the results of these calculations including inflation with 50, 55, or 60 e-foldings for the same values of global time and values of the cosmological scale factor a as in table 118.2.

118.7 Approximate Generalized Friedmann equation for small acceleration

If there were no acceleration, then we could use the Friedmann equation to calculate $\dot{a}$ in (118.6). However, with acceleration, the Friedmann equation, generalized to include vorticity, shear, and acceleration can be calculated from the Raychoudhury equation to give [386, Appendix F]

$$\dot{\ell} = \ell\sqrt{H(a)^2 + H_\omega^2 + H_\sigma^2 + H_a^2} \; , \tag{118.27}$$

where ℓ is a scale factor along lines of cosmic flow. In the presence of acceleration, a and ℓ differ. $H(a)$ [given by (118.13) in appendix 118.6] is the Hubble parameter without vorticity, shear, or acceleration,

$$H_\omega^2 \equiv \frac{4}{3\ell^2}\int \ell\omega^2 \, \mathrm{d}\ell \tag{118.28}$$

is the vorticity term, ω is vorticity,

$$H_\sigma^2 \equiv -\frac{4}{3\ell^2}\int \ell\sigma^2 \, \mathrm{d}\ell \tag{118.29}$$

is the shear term, σ is shear, and

$$H_a^2 \equiv \frac{2}{3\ell^2}\int \ell\dot{u}^a{}_{;a} \, \mathrm{d}\ell = \frac{2}{3\ell^2}\int \ell \, {}^{(3)}\nabla_a \dot{u}^a \, \mathrm{d}\ell + \frac{2}{3\ell^2}\int \ell\dot{u}_a\dot{u}^a \, \mathrm{d}\ell \to \frac{2}{3\ell^2}\int \ell\dot{u}_a\dot{u}^a \, \mathrm{d}\ell \tag{118.30}$$

is the acceleration term, where ${}^{(3)}\nabla_a$ is a 3-dimensional spatial gradient.

We can take acceleration to depend on the distance along flow lines ℓ as

$$\dot{u} \propto \ell^{-m} \; , \tag{118.31}$$

where the value of m depends on the assumptions we make.

118.8 Formulas for $C_I(a_f)$ including inflation

The details for the derivation of these formulas are in [414].

At the end of inflation, we have

$$C_I(a_N) \approx \frac{\alpha_1 w + \alpha_2}{16}\frac{e^{(2m_r+1)N}}{N^3 m_r^2} \; . \tag{118.32}$$

For $a_f \gg a_N$, some terms dominate over others, so that we have

$$C_I(a_f) \approx f(a_f)\frac{e^{2Nm_r}}{4N^2 m_r^2} \; , \tag{118.33}$$

where

$$\begin{aligned}
&f(a_f) = f_1(a_f) \text{ for } a_N \ll a_f \le a_{eq} \; , \\
&f(a_f) = f(a_{eq}) + f_2(a_f) f_3(a_f) \text{ for } a_{eq} \le a_f \le a_\Lambda , \\
&f(a_f) = f(a_\Lambda) - f_2(a_f)\pi\alpha_3 \left(\frac{a_f}{a_\Lambda}\right)^{2m_m - 3} \ln\frac{a_f}{a_\Lambda} \\
&\text{for } a_\Lambda \le a_f \le a_0 = 1 \; ,
\end{aligned} \tag{118.34}$$

$$f_1(a) \equiv \frac{\alpha_1 w + \alpha_2}{4} \left[\frac{e^N}{N} + \left(\frac{a}{a_N} \right)^{2m_r - 4} \right] , \tag{118.35}$$

$$f_2(a) \equiv 3 \frac{\Omega_m}{\Omega_r} \frac{a_N^4}{a^3} \left(\frac{a_{eq}}{a_N} \right)^{2m_r} \left(\frac{a}{a_{eq}} \right)^{2m_m} , \tag{118.36}$$

and

$$\begin{aligned} f_3(a) \equiv & -\frac{\Omega_\Lambda H_0^2}{3H(a)^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12} \right) \left[1 - \left(\frac{a_{eq}}{a} \right)^{9/2} \right] \\ & + \frac{\alpha_1 w + \alpha_2}{12} \left[1 - \left(\frac{a_{eq}}{a} \right)^{3/2} \right] . \end{aligned} \tag{118.37}$$

118.9 Formulas for $C_I(a_f)$ neglecting inflation

The details for the derivation of these formulas are in [414].

For $a_f \gg a_N$, keeping only the most significant terms gives

$$\begin{aligned}
& C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{27} \quad \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 1, \\
& C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{2} \left(\ln \frac{a_f}{a_N} \right)^2 \quad \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 2, \\
& C_I(a_f) \approx (\alpha_1 w + \alpha_2) \left[\frac{1}{2} \left(\frac{a_f}{a_N} \right)^2 - \frac{3}{4} \frac{a_f}{a_N} \left(\ln \frac{a_f}{a_N} \right)^2 \right] \\
& \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 3, \\
& C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{8} \left(\frac{a_f}{a_N} \right)^4 \quad \text{for } a_N \ll a_f \leq a_{eq} \text{ and for } m_r = 4, \\
& C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2 \Omega_r} \left\{ \frac{H_0^2 \Omega_\Lambda}{H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12} \right) \times \right. \\
& \left[-9 + 63 \left(\frac{a_{eq}}{a_f} \right)^{1/2} - \frac{1}{7} \left(67 - \frac{3}{7} + 12 \ln \frac{a_f}{a_{eq}} \right) \frac{a_{eq}}{a_f} - \frac{1}{7} \left(311 + \frac{3}{7} \right) \left(\frac{a_{eq}}{a_f} \right)^{9/2} \right] \\
& \left. + \frac{\alpha_2}{12} \left[27 + 36 \left(\frac{a_{eq}}{a_f} \right)^{1/2} + \left(326 - 60 \ln \frac{a_f}{a_{eq}} \right) \frac{a_{eq}}{a_f} - 389 \left(\frac{a_{eq}}{a_f} \right)^{3/2} \right] \right\} \\
& \text{for } a_{eq} \leq a_f \leq a_\Lambda \text{ and for } m_r = 1 \text{ and for } m_m = 2. \\
& C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2 \Omega_r} \left\{ \frac{H_0^2 \Omega_\Lambda}{H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12} \right) \times \right. \\
& \left[- \left(\ln \frac{a_{eq}}{a_N} \right)^2 - 4 \ln \frac{a_{eq}}{a_N} - 4 + 6 \left(\ln \frac{a_{eq}}{a_N} + 2 \right) \left(\frac{a_{eq}}{a_f} \right)^{1/2} \right. \\
& \left. - \frac{1}{7} \left(12 \ln \frac{a_{eq}}{a_N} + 57 - \frac{3}{7} + 12 \ln \frac{a_f}{a_{eq}} \right) \frac{a_{eq}}{a_f} + \left(\left(\ln \frac{a_{eq}}{a_N} \right)^2 - \frac{2}{7} \ln \frac{a_{eq}}{a_N} + \frac{4}{49} \right) \left(\frac{a_{eq}}{a_f} \right)^{9/2} \right] \\
& + \frac{\alpha_2}{12} \left[3 \left(\ln \frac{a_{eq}}{a_N} \right)^2 + 12 \ln \frac{a_{eq}}{a_N} + 12 + 12 \ln \frac{a_{eq}}{a_N} \left(\frac{a_{eq}}{a_f} \right)^{1/2} \right.
\end{aligned}$$

$$
+\left(-12\ln\frac{a_{eq}}{a_N}+84-60\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f}-\left(3\left(\ln\frac{a_{eq}}{a_N}\right)^2+12\ln\frac{a_{eq}}{a_N}+96\right)\left(\frac{a_{eq}}{a_f}\right)^{3/2}\Bigg]\Bigg\}
$$
for $a_{eq} \le a_f \le a_\Lambda$ and for $m_r = 2$ and for $m_m = 2$.
$$
C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_N^4}{H_0^2\Omega_r(m_r-2)^2}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times
$$
$$
\left\{-\frac{H_0^2\Omega_\Lambda}{H_f^2}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_2}{12}\right)\left[1-\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right]+\frac{\alpha_2}{4}\left[1-\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}
$$
for $a_{eq} \le a_f \le a_\Lambda$ and for $m_r > 2$ and for $m_m > 2$.
$$
C_I(a_f) \approx C_I(a_\Lambda) - 27\frac{H_f^2 a_f^4}{H_0^2\Omega_r}\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_2}{12}\right)\left[1-\left(\frac{a_\Lambda}{a_f}\right)^3\right]
$$
for $a_\Lambda \le a_f \le a_0 = 1$ and for $m_r = 1$ and for $m_m = 2$,
$$
C_I(a_f) \approx C_I(a_\Lambda) - 3\frac{H_f^2 a_f^4}{H_0^2\Omega_r}\left[\ln\frac{a_{eq}}{a_N}+2\right]^2\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_2}{12}\right)\left[1-\left(\frac{a_\Lambda}{a_f}\right)^3\right]
$$
for $a_\Lambda \le a_f \le a_0 = 1$ and for $m_r = 2$ and for $m_m = 2$,
$$
C_I(a_f) \approx C_I(a_\Lambda) - 3\frac{H_f^2 a_N^4}{H_0^2\Omega_r(m_r-2)^2}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times
$$
$$
\Omega_m\frac{H_0}{H_f}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_2}{12}\right)\left[1-\left(\frac{a_\Lambda}{a_f}\right)^3\right]
$$
for $a_\Lambda \le a_f \le a_0 = 1$ and for $m_r > 2$ and for $m_m > 2$,

(118.38)

where

$$
\begin{aligned}
C_I(a_{eq}) &\approx \frac{\alpha_1 w+\alpha_2}{27} \text{ for } m_r = 1,\\
C_I(a_{eq}) &\approx \frac{\alpha_1 w+\alpha_2}{2}\left(\ln\frac{a_{eq}}{a_N}\right)^2 \text{ for } m_r = 2,\\
C_I(a_{eq}) &\approx \frac{\alpha_1 w+\alpha_2}{2}\left(\frac{a_{eq}}{a_N}\right)^2 \text{ for } m_r = 3,\\
C_I(a_{eq}) &\approx \frac{\alpha_1 w+\alpha_2}{8}\left(\frac{a_{eq}}{a_N}\right)^4 \text{ for } m_r = 4,
\end{aligned}
\tag{118.39}
$$

and

$$
C_I(a_\Lambda) \approx C_I(a_{eq}) + 9\frac{H_\Lambda^2 a_\Lambda^4}{H_0^2\Omega_r}\left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_2}{12}\right)+\frac{\alpha_2}{4}\right\}
$$
for $m_r = 1$ and for $m_m = 2$,
$$
C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_\Lambda^4}{H_0^2\Omega_r}\left[\ln\frac{a_{eq}}{a_N}+2\right]^2\left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_2}{12}\right)+\frac{\alpha_2}{4}\right\}
$$
for $m_r = 2$ and for $m_m = 2$,
$$
C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_N^4}{H_0^2\Omega_r(m_r-2)^2}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a_\Lambda}{a_{eq}}\right)^{2m_m} \times
$$

$$\left\{-\frac{H_0^2\Omega_\Lambda}{H_\Lambda^2}\left(\frac{2\pi}{3}\alpha_3-\frac{\alpha_2}{12}\right)+\frac{\alpha_2}{4}\right\} \text{ for } m_r>2 \text{ and for } m_m>2. \tag{118.40}$$

Table 118.1: Maximum rms acceleration $\langle \dot{u}_f \rangle_{\max}$ as a function of global time t_f, m_r, and m_m for an inflation era with 50, 55, and 60 e-foldings. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. The last 3 columns are calculated using equations (118.12), (118.32), and (118.33).

t_f	m_r	m_m	$\langle \dot{u}_f \rangle$max for 50 e-foldings m s^{-2}	$\langle \dot{u}_f \rangle$max for 55 e-foldings m s^{-2}	$\langle \dot{u}_f \rangle$max for 60 e-foldings m s^{-2}
beginning of inflation 10^{-36} s					
end of inflation 10^{-34} s	1		9×10^{-46}	6×10^{-49}	3×10^{-52}
	2		3×10^{-67}	1×10^{-72}	6×10^{-76}
	3		9×10^{-89}	3×10^{-96}	9×10^{-104}
	4		2×10^{-110}	6×10^{-120}	9×10^{-130}
electroweak transition 2.4×10^{-11} s	1		1×10^{-64}	9×10^{-68}	6×10^{-71}
	2		6×10^{-86}	2×10^{-91}	9×10^{-97}
	3		2×10^{-107}	6×10^{-115}	1×10^{-122}
	4		3×10^{-129}	9×10^{-139}	2×10^{-148}
radiation era 0.24 s	1		4×10^{-77}	1×10^{-83}	4×10^{-87}
	2		2×10^{-102}	8×10^{-108}	2×10^{-113}
	3		2×10^{-124}	4×10^{-132}	1×10^{-139}
	4		1×10^{-155}	3×10^{-164}	5×10^{-173}
matter/ radiation equality 5×10^4 yr	1		4×10^{-92}	8×10^{-99}	4×10^{-114}
	2		1×10^{-117}	5×10^{-123}	2×10^{-140}
	3		1×10^{-145}	3×10^{-152}	1×10^{-158}
	4		1×10^{-183}	2×10^{-192}	4×10^{-201}
recombi- nation 3.8×10^5 yr	1	2	3×10^{-92}	2×10^{-97}	1×10^{-98}
	2	2	1×10^{-113}	5×10^{-119}	2×10^{-154}
	3	3	$\approx 10^{-141}$	$\approx 10^{-147}$	$\approx 10^{-141}$
	4	4	$\approx 10^{-179}$	$\approx 10^{-188}$	$\approx 10^{-197}$
matter/ dark energy equality 10×10^9 yr	1	2	8×10^{-101}	5×10^{-106}	3×10^{-107}
	2	2	3×10^{-122}	1×10^{-127}	6×10^{-133}
	3	3	$\approx 10^{-155}$	$\approx 10^{-162}$	$\approx 10^{-168}$
	4	4	$\approx 10^{-196}$	$\approx 10^{-205}$	$\approx 10^{-214}$
today 13.8×10^9 yr	1	2	5×10^{-101}	3×10^{-106}	2×10^{-107}
	2	2	2×10^{-122}	8×10^{-128}	3×10^{-133}
	3	3	$\approx 10^{-155}$	$\approx 10^{-162}$	$\approx 10^{-168}$
	4	4	$\approx 10^{-196}$	$\approx 10^{-205}$	$\approx 10^{-214}$

Table 118.2: Cosmological scale factor a_f, Hubble parameter H_f, and maximum rms acceleration $\langle \dot{u}_f \rangle \mathrm{max}$ as a function of global time t_f, m_r, and m_m neglecting inflation. Acceleration varies with the cosmological scale factor a as a^{-m_r} in the radiation era and as a^{-m_m} in the matter era. Everything but the last 3 columns is calculated using the formulas in appendix 118.6. The last column is calculated using equations (118.12) and (118.38). $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds. $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length.

t_f	a_f	H_f s^{-1}	$L^* \sqrt{\frac{H_f}{(r_0 a_f)^3}}$ $\mathrm{m\ s}^{-2}$	m_r	m_m	$\langle \dot{u}_f \rangle \mathrm{max}$ $\mathrm{m\ s}^{-2}$
end of inflation 10^{-34} s	$a_N =$ 2×10^{-20}	5×10^{33}	2.3×10^{-16}			
electroweak transition 2.4×10^{-11} s	10^{-15}	2×10^{10}	4.0×10^{-35}	1		2×10^{-34}
				2		6×10^{-36}
				3		1×10^{-39}
				4		3×10^{-44}
radiation era 0.24 s	10^{-10}	2	1.3×10^{-47}	1		7×10^{-47}
				2		8×10^{-49}
				3		4×10^{-57}
				4		1×10^{-66}
matter/ radiation equality 5×10^4 yr	$a_{\mathrm{eq}} =$ 3×10^{-4}	3×10^{-13}	9.5×10^{-63}	1		5×10^{-62}
				2		4×10^{-64}
				3		9×10^{-79}
				4		1×10^{-94}
recombi- nation 3.8×10^5 yr	$.9 \times 10^{-3}$	5×10^{-14}	7.5×10^{-63}	1	2	2×10^{-63}
				2	2	3×10^{-64}
				3	3	2×10^{-79}
				4	4	8×10^{-96}
matter/ dark energy equality 10×10^9 yr	$a_\Lambda = 0.76$	2.5×10^{-18}	2.1×10^{-71}	1	2	1×10^{-73}
				2	2	8×10^{-75}
				3	3	8×10^{-93}
				4	4	2×10^{-112}
today 13.8×10^9 yr	$a_0 = 1.0$	2.2×10^{-18}	1.3×10^{-71}	1	2	7×10^{-74}
				2	2	5×10^{-75}
				3	3	5×10^{-93}
				4	4	1×10^{-112}

Chapter 119

Quantum gravity and the relative acceleration of matter and inertial frames[1]

abstract

Although we do not yet have a generally accepted theory of quantum gravity, our best estimates for how the Lagrangian depends on various parameters allows us to make order-of-magnitude estimates in some cases. For example, knowing how the Lagrangian depends on the relative acceleration of matter and inertial frames allows us to explain (by phase interference) why measurements of the normalized scalar shear of the scalar mode in the Cosmic Microwave Background Radiation is less than 10^{-10}. This would correspond to a relative acceleration between about 10^{-19} m s^{-2} and 10^{-16} m s^{-2}, depending on multipole number, which would normally require finely tuned initial conditions to explain. A simple saddle-point approximation to a path-integral calculation is used to estimate the limits on the large-scale relative acceleration of matter and inertial frames as a function of cosmic time for inflation rates of 50, 55, and 60 e-foldings, and for values of the dependence of relative acceleration on cosmological scale factor a as a^{-m} for various values of m. Even as early as $a = 10^{-15}$ (about 2.4×10^{-11}s), relative acceleration would be limited to be less than about 6×10^{-64} m s^{-2} for inflation with 50 e-foldings and $m = 1$, with even tighter restrictions for 55 or 60 e-foldings or for larger m. But even neglecting inflation, relative acceleration would be restricted to be less than about 10^{-34} m s^{-2} at that time. Today, relative acceleration would be limited to be less than about 5×10^{-101} m s^{-2} for inflation with 50 e-foldings and $m = 1$ in the radiation era and $m = 2$ in the matter era. Even without inflation, relative acceleration would be limited to be less than about 7×10^{-74} m s^{-2} today. Assuming Big Bang cosmology, the lack of any measurable scalar shear in the CMB is strong evidence for the existence of quantum gravity.

119.1 Introduction

Measurements of the Cosmic Microwave Background Radiation show that the normalized scalar shear today $(\sigma_s/H)_0$ for the scalar mode is less than about 10^{-10} [369], where H is the Hubble parameter.

Relative acceleration of matter and inertial frames is also associated with the scalar mode [411, Chapter 29]. Using the formulas for the evolution of the scalar mode [411, Chapter 29] gives the

[1] 2 December 2022, 16:07, submitted to Foundations of Physics on 7 December 2022.

relative acceleration in terms of the normalized scalar shear as about

$$\frac{2\pi}{6} cH \left(\frac{\sigma_s}{H}\right) \frac{\ell}{a} , \tag{119.1}$$

where ℓ is the multipole number, with $2 < \ell < 2508$ [369], and c is the free-space speed of light. Taking $a = 1$ gives the relative acceleration of matter and inertial frames today to be less than about 1×10^{-19} m s^{-2} for $\ell = 2$ or less than about 2×10^{-16} m s^{-2} for $\ell = 2508$.

However, there are many solutions of Einstein's field equations for General Relativity that have large-scale relative acceleration of matter and inertial frames, e.g. [396, 397, 398, 399, 400, 401, 402, 403, 404]. Even with inflation, it would require very finely tuned initial conditions to explain such a small relative acceleration.

Quantum gravity might solve the problem, but, as is well-known, we do not have a generally accepted theory of quantum gravity. There are many difficulties in developing a theory of quantum gravity e.g. [373, 374], but it is generally believed that a theory of quantum gravity should exist e.g.[406]. Some ideas for developing a theory of quantum gravity are discussed in [62, 63, 64, 19, 375, 376, 377].

Fortunately, some calculations (including the present one) can be made without a final theory of quantum gravity. For example, we know enough about the properties of quantum gravity to show that almost any reasonable theory of quantum gravity can explain (by phase interference) why large-scale relative rotation of inertial frames and the matter distribution is very small [386, 415]. A similar calculation can explain why large-scale relative acceleration of matter and inertial frames is very small.

Here, we calculate the size of the region of constructive interference as a function of cosmic time. We show not only that phase interference leads to a very small relative acceleration, but even though inflation dominates the calculation, the allowed relative acceleration neglecting inflation would already have been very small a fraction of a second after the initial singularity.

Section 119.2 reviews path integrals in quantum cosmology. Section 119.3 gives the approximate action for small relative acceleration. Section 119.4 calculates a saddlepoint approximation to the path integral. Section 119.5 discusses the results.

Appendix 119.6 gives the background cosmology. Appendix 119.7 gives the approximate generalized Friedmann equation for small acceleration. Appendix 119.8 and appendix 119.9 give formulas for the function $C_I(a_f)$ that is used to calculate the action for small relative acceleration. Appendix 119.8 includes inflation, while appendix 119.9 neglects inflation. Although the derivation of the formulas in appendix 119.8 and appendix 119.9 is straightforward, and all of the information necessary to derive those formulas is included here, the derivation is left out because it is long and tedious.

The speed of light c and Newton's gravitational constant G are set to 1 throughout except when converting to conventional units.

119.2 Review of path integrals in quantum cosmology

Following Hartle and Hawking [124] and Jones [386, Appendix A], a path-integral calculation gives

$$\psi_f({}^{(3)}\mathcal{G}_f, \phi_f) = \int\int \exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right) \mathcal{D}^{(4)}\mathcal{G}\, \mathcal{D}\phi\, \psi_i({}^{(3)}\mathcal{G}_i, \phi_i)\, \mathcal{D}^{(3)}\mathcal{G}_i\, \mathcal{D}\phi_i \ , \tag{119.2}$$

where $\psi_i({}^{(3)}\mathcal{G}_i, \phi_i)$ is the amplitude that the 3-geometry was ${}^{(3)}\mathcal{G}_i$ on some initial space-like hypersurface and the matter fields on that 3-geometry were ϕ_i, $\psi_f({}^{(3)}\mathcal{G}_f, \phi_f)$ is the amplitude that the 3-geometry is ${}^{(3)}\mathcal{G}_f$ on some final space-like hypersurface and that the matter fields on that 3-geometry are ϕ_f, $\exp\left(iI[{}^{(4)}\mathcal{G}, \phi]/\hbar\right)$ is the contribution of the 4-geometry ${}^{(4)}\mathcal{G}$ and matter field ϕ

on that 4-geometry to the path integral, where $I[^{(4)}\mathcal{G}, \phi]$ is the action. A correct theory of quantum gravity would be necessary to specify the measures $\mathcal{D}^{(4)}\mathcal{G}$ and $\mathcal{D}\phi$.

To make accurate calculations of $\psi_f(^{(3)}\mathcal{G}_f, \phi_f)$ in (119.2) would require knowing the measures $\mathcal{D}^{(4)}\mathcal{G}$ and $\mathcal{D}\phi$ [332, 416]. However, because the effect of the action dominates over that of the measure, it is possible in some simple cases to make order-of-magnitude estimates for which values of some parameters in the action contribute significantly to the path integral without knowing the measure.

Although in (119.2), the integration is over all possible 4-geometries, not just classical 4-geometries, the main contribution to the integral (in most cases) comes from classical 4-geometries, e.g. [221, 220]. That allows us to restrict (119.2) to be an integration over classical 4-geometries.

The condition that there are not finely tuned initial conditions is equivalent to $\psi_i(^{(3)}\mathcal{G}_i, \phi_i)$ being a broad wave function. That allows us to neglect the effect of that initial wave function on the integration in the path integral in (119.2). In addition, we consider 4-geometries characterized by a parameter $\langle \dot{u}_f \rangle$ which we take to be the rms relative acceleration on the space-like hypersurface at $t = t_f$. Thus, we can rewrite (119.2) for our purposes as

$$\psi_f(^{(3)}\mathcal{G}_f, \phi_f) \propto \int_{-\infty}^{\infty} A(\langle \dot{u}_f \rangle) e^{iI(\langle \dot{u}_f \rangle)/\hbar} \, \mathrm{d}\langle \dot{u}_f \rangle \ , \tag{119.3}$$

where $A(\langle \dot{u}_f \rangle)$ is a slowly varying function of $\langle \dot{u}_f \rangle$, $I(\langle \dot{u}_f \rangle)$ is the action, and since for classical spacetimes, the acceleration is a known function of cosmological time, we can consider the action I to depend on the rms acceleration at any cosmological time we choose, say t_f, which we designate as $\langle \dot{u}_f \rangle$, where $\langle \dot{u}_f \rangle^2 \equiv \overline{\dot{u}_f^2}$, and the average is a spatial average over the volume within the past light cone.

Appendix 119.6 discusses the background cosmology, including the inflation, radiation, matter, and dark-energy eras because the Lagrangian would depend differently on the cosmological scale factor in the four eras.

119.3 Approximate action for small acceleration

The action in (119.3) is equal to a volume integral over spacetime plus a surface integral. The surface term is necessary to insure consistency if the action integral is broken into parts [183, 123]. For the present purposes, it is not necessary to consider the surface term.

In addition, for 4-geometries that are solutions of Einstein's field equations, the Lagrangian can be expressed as an effective Lagrangian that includes both the geometry Lagrangian and the matter Lagrangian, e.g. [386, Appendix B]

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda \ , \tag{119.4}$$

where p is pressure, ρ is density, Λ is the cosmological constant, and α_1, α_2, and α_3 are dimensionless constants of order unity. For example, using the matter Lagrangian in [161] gives $\alpha_1 = -3/2$, $\alpha_2 = +3/2$, and $\alpha_3 = 1/(8\pi)$. Or, using the matter Lagrangian in [334, 162] gives $\alpha_1 = -1/2$, $\alpha_2 = +1/2$, and $\alpha_3 = 1/(8\pi)$ [386, Appendix B]. Including the surface term can alter α_1, α_2, and α_3.

That allows us to write the action as

$$I = \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^4 x = \int_{t_i}^{t_f} \int (-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^3 x \mathrm{d}t \ , \tag{119.5}$$

where t_i is the initial time (which we take to be the beginning of the inflation era), and t_f is the final time, which we choose arbitrarily to calculate the action at any specified cosmic time.

We can convert the time integral in (119.5) to an integral over the cosmological scale factor a

$$I = \int_{a_i}^{a_f} \int \frac{(-g^{(4)})^{1/2} \tilde{L} \mathrm{d}^3 x \, \mathrm{d}\, a}{\dot{a}} \ , \tag{119.6}$$

where $\dot{a} = \mathrm{d}a/\mathrm{d}t$ is given by a generalization of the Friedmann equation that includes acceleration (appendix 119.7), a_i is the value of the cosmological scale factor at $t = t_i$, and a_f is the value of the cosmological scale factor at $t = t_f$.

Including relative acceleration in the calculation of the action (even approximately) is tedious, even though straightforward. That calculation includes taking into account the fact that if there is relative acceleration of the matter distribution and the inertial frame, then density and pressure will depend on the scale length ℓ along flow lines rather than depend on the cosmological scale factor a. The significance is that the cosmological scale factor a is a function of global time, but scale length ℓ along flow lines is not normal to surfaces of constant global time. We have $\ell = \int \mathrm{d}\, a/\cosh\theta$, where $\theta = \int \dot{u} \, \mathrm{d}\, t$. The formulas are then expanded to second order in $\dot{u}$ and θ. This causes the calculation to be much more complicated.

We can express the volume integral in (119.6) as a product of the spatial volume and the spatial average.

$$I = \int_{a_i}^{a_f} V(a) \overline{\left(\frac{\tilde{L}}{\dot{a}}\right)} \mathrm{d}\, a \ , \tag{119.7}$$

where we have used the knowledge that our universe is spatially flat, an overbar indicates a spatial average,

$$V(a) = \frac{4}{3} \pi a^3 r_0^3 \tag{119.8}$$

is the approximate spatial volume[2], and r_0 is the present radius of the cosmological horizon.

The result to lowest order in $\langle \dot{u}_f \rangle / H_f$ is

$$I(\langle \dot{u}_f \rangle) \approx I_0 + \hbar \left(\frac{\langle \dot{u}_f \rangle}{\dot{u}_m} \right)^2 f_I(\langle \dot{u}_f \rangle) \ , \tag{119.9}$$

where I_0 is the action for the standard cosmological model (FLRW cosmology [417, 418, 419, 420, 421, 422]),

$$\dot{u}_m = \left(\frac{\hbar H_f}{r_0^3 a_f^3} \right)^{1/2} = L^* \sqrt{\frac{H_f}{r_f^3}} = L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \ , \tag{119.10}$$

$r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds, $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length,

$$\begin{aligned} f_I(\langle \dot{u}_f \rangle) &\approx \left[C_I(a_f) + \tfrac{\langle \dot{u}_f \rangle^2 + \sigma_a^2 / \langle \dot{u}_f \rangle^2}{H_f^2} C_{II}(a_f) \right] \\ &\approx C_I(a_f) \text{ for small } \langle \dot{u}_f \rangle \end{aligned} \tag{119.11}$$

is some slowly varying dimensionless even function of $\langle \dot{u}_f \rangle$, σ_a^2 is the variance of $\dot{u}_f^2$, and $C_I(a_f)$ and $C_{II}(a_f)$ are dimensionless functions.

Notice that (119.9) reduces to I_0, the action for the standard cosmological model, when the rms acceleration $\langle \dot{u}_f \rangle$ is zero, and that (119.9) is an even function of $\langle \dot{u}_f \rangle$ if $f_I(\langle \dot{u}_f \rangle)$ is an even function of $\langle \dot{u}_f \rangle$.

[2]Since our universe is approximately spatially flat. Although there are arguments that V should be infinite because this is an open cosmology [351], considering causality leads to restricting the spatial part of the action to the past light cone.

119.4 Saddlepoint approximation

Comparing (119.3) and (119.9) shows that (119.3) has a saddlepoint at $\langle \dot{u}_f \rangle = 0$. If that saddlepoint is the only significant saddlepoint (and other criteria are satisfied), then the only significant contribution to the path integral (119.3) comes from values of the rms acceleration at the time t_f of

$$\langle \dot{u}_f \rangle = 0 \pm \dot{u}_m / \sqrt{| \; f_I(0) \; |} \; . \tag{119.12}$$

Either a stationary-phase path or a steepest-descent path could be used when making the saddlepoint approximation [134, 333, 219], but here, we use a stationary-phase path. Halliwell [332] gives an example of a more detailed path-integral calculation of quantum gravity.

The saddlepoint at $\langle \dot{u}_f \rangle = 0$ in (119.3) is isolated from other saddlepoints and any possible non-analytic points as shown by (119.11). The integral in (119.3) can be approximated by a saddlepoint integration to give

$$\begin{aligned}
&\psi_f \propto A(0)\dot{u}_m \sqrt{\pi} / \sqrt{f_I(0)} e^{i\pi/4} \\
&\text{for } \langle \dot{u}_f \rangle < \frac{\dot{u}_m}{\sqrt{| \; f_I(0) \; |}} \approx L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \frac{1}{\sqrt{| \; f_I(0) \; |}} \approx L^* \sqrt{\frac{H_f}{r_0^3 a_f^3}} \frac{1}{\sqrt{| \; C_I(a_f) \; |}}, \\
&\psi_f \approx 0, \text{ otherwise,}
\end{aligned} \tag{119.13}$$

where $r_0 \approx 46.5 \times 10^9$ light years is the present radius of the observable universe and $L^* \approx 2 \times 10^{-35}$ m is the Planck length.

Appendix 119.8 calculates $C_I(a_f)$ including inflation. This allows calculating estimates for allowed upper limits on the values for the rms relative acceleration $\langle \dot{u}_f \rangle$ from (119.13) given in table 119.1. Appendix 119.9 calculates $C_I(a_f)$ neglecting inflation. This allows calculating estimates for allowed upper limits on the values for the rms relative acceleration $\langle \dot{u}_f \rangle$ from (119.13) given in table 119.2.

The scalar mode perturbation falls off as a^{-3} [369]. The relative acceleration associated with the scalar mode perturbation has an additional factor of $1/a$ [411, Chapter 29], which leads to $m = 4$. However, we include also calculations for $m = 1$, $m = 2$, and $m = 3$ for completeness.

119.5 Discussion

Table 119.1 gives the limits on rms relative acceleration as a function of cosmic time from (119.13) including the effects of inflation with 50, 55, and 60 e-foldings. Calculations are included for $m = 1$ in the radiation era and $m = 2$ in the matter era and also for $m = 2$, 3, and 4 in all eras, where m gives the dependence of relative acceleration on cosmological scale factor a as a^{-m}. As listed in table 119.1, m_r is the value in the radiation era, and m_m is the value in the matter era.

Inflation dominates the calculation and gives a limit of relative acceleration at $a = 10^{-15}$ ($\approx 2.4 \times 10^{-11}$ s) of 10^{-64}, 10^{-85}, 10^{-107}, and 10^{-129} m s^{-2} for $m = 1$, 2, 3, and 4 for 50 e-foldings, with tighter restrictions for 55 and 60 e-foldings. The corresponding values at the present time are 10^{-100}, 10^{-122}, 10^{-155}, and 10^{-196} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

However, although inflation dominates the calculation, inflation is not necessary to restrict the relative acceleration to small values, as shown in table 119.2, which gives the limits on rms relative acceleration as a function of cosmic time from (119.13) neglecting inflation. Even if there were no inflation, relative acceleration would be restricted to be not much larger than about $L^* \sqrt{H/(r_0 a)^3}$, where $L^* \approx 2 \times 10^{-35}$ m is the Planck length, $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds is the approximate radius of the observable universe today, H is the Hubble parameter, which varies from about 2×10^{10} s^{-1} at $a = 10^{-15}$ to $\approx 2 \times 10^{-18}$ s^{-1} today.

As shown by table 119.2, even in the absence of inflation, relative acceleration at $a = 10^{-15}$ would have been limited to about 10^{-34}, 10^{-35}, 10^{-39}, and 10^{-44} m s^{-2} for $m = 1$, 2, 3, and 4. The corresponding values at the present time without inflation are 10^{-73}, 10^{-74}, 10^{-92}, and 10^{-112} m s^{-2}, where the first value is for $m = 1$ in the radiation era and $m = 2$ in the matter era.

Because quantum gravity (specifically, quantum cosmology) explains through phase interference why we should not observe relative rotation or relative acceleration between matter and inertial frames on a large scale, we could say that quantum gravity may provide the mechanism for implementing Mach's principle [152, 120, 102, 122, 15, 11, 16, 156, 109, 315, 159].

The lack of any measurable scalar shear in the CMB is strong evidence for the existence of quantum gravity.

119.6 Background cosmology

We start with the formula for the Hubble parameter neglecting vorticity, shear, and acceleration

$$\frac{1}{a}\frac{da}{dt} = H(a) = H_0\sqrt{\Omega_\Lambda + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_k}{a^2}} = \sqrt{\frac{\Lambda}{3} + \frac{8\pi\rho}{3} - \frac{k}{a^2}}\ , \tag{119.14}$$

where Λ is the cosmological constant, ρ is density, t is global time, $H_0 = 67.66$ km s^{-1} Mpc^{-1} = 6.92×10^{-11} yr^{-1} $\approx 2.193\times10^{-18}$ s^{-1} [409] is the present value of the Hubble parameter, $a = 1/(z{+}1)$ is the cosmological scale factor, whose present value is 1, z is the redshift factor, and $\Omega_\Lambda = 0.6889$ [409] is the dark energy density divided by the critical density today. Using $z_{\rm eq} = 3387$ [409] for the redshift at radiation/matter equality with $\Omega_m = 0.311$ [409] for the matter density today divided by the critical density gives $\Omega_r = 9.181 \times 10^{-5} \approx 9 \times 10^{-5}$ for the radiation energy density divided by the critical density today.[3] $\Omega_k = 0.0$ within measurement error [409]. Equation (119.14) is not valid during inflation, but that era is considered later.

For an equation of state, we take

$$p = w\rho\ , \tag{119.15}$$

where $w = 1/3$ in the radiation-dominated era, and $w = 0$ in the matter-dominated era. The variation of density ρ with cosmological scale factor a is given by [342, Table 6.1]

$$\rho = \rho_{eq}(a/a_{eq})^{-3(1+w)}\ , \tag{119.16}$$

where ρ_{eq} is the value of ρ at the boundary between the radiation era and the matter era where $a = a_{eq}$.

From (119.14), we have

$$\rho = \frac{3H_0^2}{8\pi}\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right)\ , \tag{119.17}$$

which gives a smooth transition between the radiation era and the matter era, instead of the abrupt transition given by (119.16). We can have a smooth transition for pressure, also, by taking w in (119.15) to be given by

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m}a\right)^{-n}\ , \tag{119.18}$$

where n is a positive integer. The larger n is, the sharper will be the transition. However, for calculating the action, the result does not depend strongly on how smooth or sharp is the transition.

[3] The values are from [409, Table 2], and include lensing and baryon acoustic oscillations.

Therefore, to keep the calculations simple, we take $n = 1$ to give

$$w = \frac{1}{3}\left(1 + \frac{\Omega_r}{\Omega_m} a\right)^{-1} . \tag{119.19}$$

Putting (119.19) and (119.17) in (119.15) gives

$$p = \frac{1}{3}\frac{3H_0^2}{8\pi}\frac{\Omega_r}{a^4} . \tag{119.20}$$

Putting (119.20) and (119.17) into (119.4) gives

$$\tilde{L} = \alpha_1 p + \alpha_2 \rho + \alpha_3 \Lambda = \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] , \tag{119.21}$$

with $w = 1/3$. We can approximate (119.21) in different eras.

$$\tilde{L} = \frac{3H_0^2}{8\pi}\left[\alpha_2\frac{\Omega_m}{a^3} + (\alpha_1 w + \alpha_2)\frac{\Omega_r}{a^4}\right] \text{ for } a \leq a_m \approx 10^{-2}$$

$$\tilde{L} = \frac{3H_0^2}{8\pi}\left[8\pi\alpha_3\Omega_\Lambda + \alpha_2\frac{\Omega_m}{a^3}\right] \text{ for } a \geq a_m \approx 10^{-2} \tag{119.22}$$

We can convert (119.14) into an integral to get

$$t = \frac{1}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_\Lambda a^2 + \Omega_m/a + \Omega_r/a^2}} . \tag{119.23}$$

Equation (119.23) is a well-defined integral to give the global time t as a function of the cosmological scale factor a. Although it is not easy to calculate in closed form, there is no region where more than two terms in the radical are significant. That allows a very good approximate evaluation of the integral in closed form. We have

$$t = \frac{2}{3H_0}\frac{\Omega_r^{3/2}}{\Omega_m^2}\left[2 - \left(2 - \frac{\Omega_m}{\Omega_r}a\right)\sqrt{1 + \frac{\Omega_m}{\Omega_r}a}\right] \text{ for } a \leq a_m \approx 10^{-2},$$

and

$$t = \frac{1}{3H_0\sqrt{\Omega_\Lambda}} \ln\frac{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda}a^{-3}} + 1}{\sqrt{1 + \frac{\Omega_m}{\Omega_\Lambda}a^{-3}} - 1} \text{ for } a \geq a_m, \tag{119.24}$$

where

$$a_{\text{eq}} = \frac{\Omega_r}{\Omega_m} \approx 3 \times 10^{-4} \ll a_m \ll \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \approx 0.76 . \tag{119.25}$$

Using (119.14) and (119.24) allows us to make a table that gives H as a function of global time t. When the cosmological scale factor a is very small, we can make some approximations. These approximations are valid in the very early universe.

$$a^2 \approx 2H_0\sqrt{\Omega_r}t, \text{ that is, } t \approx 2.4 \times 10^{19} a^2 \text{ seconds for } a \ll \Omega_r/\Omega_m. \tag{119.26}$$

$$H \approx \frac{1}{2t} \text{ for } a \ll \Omega_r/\Omega_m. \tag{119.27}$$

For the inflation era (from 10^{-36} seconds to 10^{-34} seconds), we choose a constant value for the Hubble parameter H that will give 50, 55, or 60 e-foldings. Reference [409] estimates that there were about 50 to 60 e-foldings during inflation.

Table 119.2 gives the results of the calculations neglecting inflation for selected values of global time corresponding to values of the cosmological scale factor a. Table 119.1 gives the results of these calculations including inflation with 50, 55, or 60 e-foldings for the same values of global time and values of the cosmological scale factor a as in table 119.2.

119.7 Approximate Generalized Friedmann equation for small acceleration

If there were no acceleration, then we could use the Friedmann equation to calculate $\dot{a}$ in (119.7). However, with acceleration, the Friedmann equation, generalized to include vorticity, shear, and acceleration can be calculated from the Raychoudhury equation to give [386, Appendix F]

$$\dot{\ell} = \ell\sqrt{H(a)^2 + H_\omega^2 + H_\sigma^2 + H_a^2} \ , \tag{119.28}$$

where ℓ is a scale factor along lines of cosmic flow. In the presence of acceleration, a and ℓ differ. $H(a)$ [given by (119.14) in appendix 119.6] is the Hubble parameter without vorticity, shear, or acceleration,

$$H_\omega^2 \equiv \frac{4}{3\ell^2}\int \ell\omega^2 \, \mathrm{d}\ell \tag{119.29}$$

is the vorticity term, ω is vorticity,

$$H_\sigma^2 \equiv -\frac{4}{3\ell^2}\int \ell\sigma^2 \, \mathrm{d}\ell \tag{119.30}$$

is the shear term, σ is shear, and

$$H_a^2 \equiv \frac{2}{3\ell^2}\int \ell\dot{u}^a{}_{;a} \, \mathrm{d}\ell = \frac{2}{3\ell^2}\int \ell \, {}^{(3)}\nabla_a \dot{u}^a \, \mathrm{d}\ell + \frac{2}{3\ell^2}\int \ell\dot{u}_a\dot{u}^a \, \mathrm{d}\ell \to \frac{2}{3\ell^2}\int \ell\dot{u}_a\dot{u}^a \, \mathrm{d}\ell \tag{119.31}$$

is the acceleration term, where ${}^{(3)}\nabla_a$ is a 3-dimensional spatial gradient.

We can take acceleration to depend on the distance along flow lines ℓ as

$$\dot{u} \propto \ell^{-m} \ , \tag{119.32}$$

where the value of m depends on the assumptions we make.

119.8 Formulas for $C_I(a_f)$ including inflation

At the end of inflation, we have

$$C_I(a_N) \approx \frac{\alpha_1 w + \alpha_2}{16}\frac{e^{(2m_r+1)N}}{N^3 m_r^2} \ . \tag{119.33}$$

For $a_f \gg a_N$, some terms dominate over others, so that we have

$$C_I(a_f) \approx f(a_f)\frac{e^{2Nm_r}}{4N^2 m_r^2} \ , \tag{119.34}$$

where

$$\begin{aligned} &f(a_f) = f_1(a_f) \text{ for } a_N \ll a_f \le a_{eq} \ , \\ &f(a_f) = f(a_{eq}) + f_2(a_f) f_3(a_f) \text{ for } a_{eq} \le a_f \le a_\Lambda , \\ &f(a_f) = f(a_\Lambda) - f_2(a_f)\pi\alpha_3 \left(\frac{a_f}{a_\Lambda}\right)^{2m_m-3} \ln\frac{a_f}{a_\Lambda} \\ &\text{for } a_\Lambda \le a_f \le a_0 = 1 \ , \end{aligned} \tag{119.35}$$

$$f_1(a) \equiv \frac{\alpha_1 w + \alpha_2}{4}\left[\frac{e^N}{N} + \left(\frac{a}{a_N}\right)^{2m_r-4}\right] \ , \tag{119.36}$$

$$f_2(a) \equiv 3\frac{\Omega_m}{\Omega_r}\frac{a_N^4}{a^3}\left(\frac{a_{eq}}{a_N}\right)^{2m_r}\left(\frac{a}{a_{eq}}\right)^{2m_m} , \tag{119.37}$$

and

$$f_3(a) \equiv -\frac{\Omega_\Lambda H_0^2}{3H(a)^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_1 w + \alpha_2}{12}\right)\left[1 - \left(\frac{a_{eq}}{a}\right)^{9/2}\right] + \frac{\alpha_1 w + \alpha_2}{12}\left[1 - \left(\frac{a_{eq}}{a}\right)^{3/2}\right] . \tag{119.38}$$

The calculations used for table 119.1 take the quantities $\alpha_1 w + \alpha_2$ and α_3 to be of order unity. If desired, other specific values could be used instead.

119.9 Formulas for $C_I(a_f)$ neglecting inflation

For $a_f \gg a_N$, keeping only the most significant terms gives

$$C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{27} \quad \text{for } a_N \ll a_f \le a_{eq} \text{ and for } m_r = 1,$$

$$C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{2}\left(\ln\frac{a_f}{a_N}\right)^2 \quad \text{for } a_N \ll a_f \le a_{eq} \text{ and for } m_r = 2,$$

$$C_I(a_f) \approx (\alpha_1 w + \alpha_2)\left[\frac{1}{2}\left(\frac{a_f}{a_N}\right)^2 - \frac{3}{4}\frac{a_f}{a_N}\left(\ln\frac{a_f}{a_N}\right)^2\right]$$

for $a_N \ll a_f \le a_{eq}$ and for $m_r = 3$,

$$C_I(a_f) \approx \frac{\alpha_1 w + \alpha_2}{8}\left(\frac{a_f}{a_N}\right)^4 \quad \text{for } a_N \ll a_f \le a_{eq} \text{ and for } m_r = 4,$$

$$C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2\Omega_r}\left\{\frac{H_0^2\Omega_\Lambda}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\times\right.$$
$$\left[-9 + 63\left(\frac{a_{eq}}{a_f}\right)^{1/2} - \frac{1}{7}\left(67 - \frac{3}{7} + 12\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - \frac{1}{7}\left(311 + \frac{3}{7}\right)\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right]$$
$$\left. + \frac{\alpha_2}{12}\left[27 + 36\left(\frac{a_{eq}}{a_f}\right)^{1/2} + \left(326 - 60\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - 389\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}$$

for $a_{eq} \le a_f \le a_\Lambda$ and for $m_r = 1$ and for $m_m = 2$.

$$C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_f^4}{H_0^2\Omega_r}\left\{\frac{H_0^2\Omega_\Lambda}{H_f^2}\left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right)\times\right.$$
$$\left[-\left(\ln\frac{a_{eq}}{a_N}\right)^2 - 4\ln\frac{a_{eq}}{a_N} - 4 + 6\left(\ln\frac{a_{eq}}{a_N} + 2\right)\left(\frac{a_{eq}}{a_f}\right)^{1/2}\right.$$
$$\left. -\frac{1}{7}\left(12\ln\frac{a_{eq}}{a_N} + 57 - \frac{3}{7} + 12\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} + \left(\left(\ln\frac{a_{eq}}{a_N}\right)^2 - \frac{2}{7}\ln\frac{a_{eq}}{a_N} + \frac{4}{49}\right)\left(\frac{a_{eq}}{a_f}\right)^{9/2}\right]$$
$$+\frac{\alpha_2}{12}\left[3\left(\ln\frac{a_{eq}}{a_N}\right)^2 + 12\ln\frac{a_{eq}}{a_N} + 12 + 12\ln\frac{a_{eq}}{a_N}\left(\frac{a_{eq}}{a_f}\right)^{1/2}\right.$$
$$\left.\left. + \left(-12\ln\frac{a_{eq}}{a_N} + 84 - 60\ln\frac{a_f}{a_{eq}}\right)\frac{a_{eq}}{a_f} - \left(3\left(\ln\frac{a_{eq}}{a_N}\right)^2 + 12\ln\frac{a_{eq}}{a_N} + 96\right)\left(\frac{a_{eq}}{a_f}\right)^{3/2}\right]\right\}$$

for $a_{eq} \leq a_f \leq a_\Lambda$ and for $m_r = 2$ and for $m_m = 2$.

$$C_I(a_f) \approx C_I(a_{eq}) + \frac{H_f^2 a_N^4}{H_0^2 \Omega_r (m_r - 2)^2} \left(\frac{a_{eq}}{a_N}\right)^{2m_r} \left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times$$

$$\left\{ -\frac{H_0^2 \Omega_\Lambda}{H_f^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) \left[1 - \left(\frac{a_{eq}}{a_f}\right)^{9/2}\right] + \frac{\alpha_2}{4} \left[1 - \left(\frac{a_{eq}}{a_f}\right)^{3/2}\right] \right\}$$

for $a_{eq} \leq a_f \leq a_\Lambda$ and for $m_r > 2$ and for $m_m > 2$.

$$C_I(a_f) \approx C_I(a_\Lambda) - 27 \frac{H_f^2 a_f^4}{H_0^2 \Omega_r} \Omega_m \frac{H_0}{H_f} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) \left[1 - \left(\frac{a_\Lambda}{a_f}\right)^3\right]$$

for $a_\Lambda \leq a_f \leq a_0 = 1$ and for $m_r = 1$ and for $m_m = 2$,

$$C_I(a_f) \approx C_I(a_\Lambda) - 3 \frac{H_f^2 a_f^4}{H_0^2 \Omega_r} \left[\ln \frac{a_{eq}}{a_N} + 2\right]^2 \Omega_m \frac{H_0}{H_f} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) \left[1 - \left(\frac{a_\Lambda}{a_f}\right)^3\right]$$

for $a_\Lambda \leq a_f \leq a_0 = 1$ and for $m_r = 2$ and for $m_m = 2$,

$$C_I(a_f) \approx C_I(a_\Lambda) - 3 \frac{H_f^2 a_N^4}{H_0^2 \Omega_r (m_r - 2)^2} \left(\frac{a_{eq}}{a_N}\right)^{2m_r} \left(\frac{a_f}{a_{eq}}\right)^{2m_m} \times$$

$$\Omega_m \frac{H_0}{H_f} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) \left[1 - \left(\frac{a_\Lambda}{a_f}\right)^3\right]$$

for $a_\Lambda \leq a_f \leq a_0 = 1$ and for $m_r > 2$ and for $m_m > 2$,

(119.39)

where

$$C_I(a_{eq}) \approx \frac{\alpha_1 w + \alpha_2}{27} \text{ for } m_r = 1,$$
$$C_I(a_{eq}) \approx \frac{\alpha_1 w + \alpha_2}{2} \left(\ln \frac{a_{eq}}{a_N}\right)^2 \text{ for } m_r = 2,$$
$$C_I(a_{eq}) \approx \frac{\alpha_1 w + \alpha_2}{2} \left(\frac{a_{eq}}{a_N}\right)^2 \text{ for } m_r = 3,$$
$$C_I(a_{eq}) \approx \frac{\alpha_1 w + \alpha_2}{8} \left(\frac{a_{eq}}{a_N}\right)^4 \text{ for } m_r = 4, \tag{119.40}$$

and

$$C_I(a_\Lambda) \approx C_I(a_{eq}) + 9 \frac{H_\Lambda^2 a_\Lambda^4}{H_0^2 \Omega_r} \left\{ -\frac{H_0^2 \Omega_\Lambda}{H_\Lambda^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) + \frac{\alpha_2}{4} \right\}$$

for $m_r = 1$ and for $m_m = 2$,

$$C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_\Lambda^4}{H_0^2 \Omega_r} \left[\ln \frac{a_{eq}}{a_N} + 2\right]^2 \left\{ -\frac{H_0^2 \Omega_\Lambda}{H_\Lambda^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) + \frac{\alpha_2}{4} \right\}$$

for $m_r = 2$ and for $m_m = 2$,

$$C_I(a_\Lambda) \approx C_I(a_{eq}) + \frac{H_\Lambda^2 a_N^4}{H_0^2 \Omega_r (m_r - 2)^2} \left(\frac{a_{eq}}{a_N}\right)^{2m_r} \left(\frac{a_\Lambda}{a_{eq}}\right)^{2m_m} \times$$

$$\left\{ -\frac{H_0^2 \Omega_\Lambda}{H_\Lambda^2} \left(\frac{2\pi}{3}\alpha_3 - \frac{\alpha_2}{12}\right) + \frac{\alpha_2}{4} \right\} \text{ for } m_r > 2 \text{ and for } m_m > 2.$$

(119.41)

The calculations used for table 119.2 take the quantities $\alpha_1 w + \alpha_2$ and α_3 to be of order unity. If desired, other specific values could be used instead.

Table 119.1: Maximum rms acceleration $\langle \dot{u}_f \rangle_{\max}$ as a function of global time t_f, m_r, and m_m for an inflation era with 50, 55, and 60 e-foldings.

t_f	m_r[a]	m_m[b]	$\langle \dot{u}_f \rangle$max[c] for 50 e-foldings m s^{-2}	$\langle \dot{u}_f \rangle$max for 55 e-foldings m s^{-2}	$\langle \dot{u}_f \rangle$max for 60 e-foldings m s^{-2}
start of inflation 10^{-36} s					
end	1		9×10^{-46}	6×10^{-49}	3×10^{-52}
of	2		3×10^{-67}	1×10^{-72}	6×10^{-76}
inflation	3		9×10^{-89}	3×10^{-96}	9×10^{-104}
10^{-34} s	4		2×10^{-110}	6×10^{-120}	9×10^{-130}
	1		1×10^{-64}	9×10^{-68}	6×10^{-71}
	2		6×10^{-86}	2×10^{-91}	9×10^{-97}
	3		2×10^{-107}	6×10^{-115}	1×10^{-122}
2.4×10^{-11} s	4		3×10^{-129}	9×10^{-139}	2×10^{-148}
	1		4×10^{-77}	1×10^{-83}	4×10^{-87}
radiation	2		2×10^{-102}	8×10^{-108}	2×10^{-113}
era	3		2×10^{-124}	4×10^{-132}	1×10^{-139}
0.24 s	4		1×10^{-155}	3×10^{-164}	5×10^{-173}
matter/	1		4×10^{-92}	8×10^{-99}	4×10^{-114}
radiation	2		1×10^{-117}	5×10^{-123}	2×10^{-140}
equality	3		1×10^{-145}	3×10^{-152}	1×10^{-158}
5×10^4 yr	4		1×10^{-183}	2×10^{-192}	4×10^{-201}
	1	2	3×10^{-92}	2×10^{-97}	1×10^{-98}
recombi-	2	2	1×10^{-113}	5×10^{-119}	2×10^{-154}
nation	3	3	$\approx 10^{-141}$	$\approx 10^{-147}$	$\approx 10^{-141}$
3.8×10^5 yr	4	4	$\approx 10^{-179}$	$\approx 10^{-188}$	$\approx 10^{-197}$
matter/	1	2	8×10^{-101}	5×10^{-106}	3×10^{-107}
dark energy	2	2	3×10^{-122}	1×10^{-127}	6×10^{-133}
equality	3	3	$\approx 10^{-155}$	$\approx 10^{-162}$	$\approx 10^{-168}$
10×10^9 yr	4	4	$\approx 10^{-196}$	$\approx 10^{-205}$	$\approx 10^{-214}$
	1	2	5×10^{-101}	3×10^{-106}	2×10^{-107}
today	2	2	2×10^{-122}	8×10^{-128}	3×10^{-133}
	3	3	$\approx 10^{-155}$	$\approx 10^{-162}$	$\approx 10^{-168}$
13.8×10^9 yr	4	4	$\approx 10^{-196}$	$\approx 10^{-205}$	$\approx 10^{-214}$

[a] Acceleration varies as a^{-m_r} in the radiation era.

[b] Acceleration varies as a^{-m_m} in the matter era.

[c] The last 3 columns are calculated using equations (119.13), (119.33), and (119.34).

Table 119.2: Cosmological scale factor a_f, Hubble parameter H_f, and maximum rms acceleration $\langle \dot{u}_f \rangle_{\max}$ as a function of global time t_f, m_r, and m_m neglecting inflation. $r_0 \approx 46.5 \times 10^9$ light years $\approx 1.5 \times 10^{18}$ light seconds. $L^* \approx 1.6 \times 10^{-35}$ m is the Planck length.

t_f	a_f	H_f s^{-1}	$L^* \sqrt{\frac{H_f}{(r_0 a_f)^3}}$ m s^{-2}	m_r[a]	m_m[b]	$\langle \dot{u}_f \rangle_{\max}$[c] m s^{-2}
end of inflation 10^{-34} s	$a_N = 2 \times 10^{-20}$	5×10^{33}	2.3×10^{-16}			
				1		2×10^{-34}
				2		6×10^{-36}
				3		1×10^{-39}
2.4×10^{-11} s	10^{-15}	2×10^{10}	4.0×10^{-35}	4		3×10^{-44}
				1		7×10^{-47}
radiation				2		8×10^{-49}
era				3		4×10^{-57}
0.24 s	10^{-10}	2	1.3×10^{-47}	4		1×10^{-66}
matter/				1		5×10^{-62}
radiation				2		4×10^{-64}
equality	$a_{eq} =$			3		9×10^{-79}
5×10^4 yr	3×10^{-4}	3×10^{-13}	9.5×10^{-63}	4		1×10^{-94}
				1	2	2×10^{-63}
recombi-				2	2	3×10^{-64}
nation				3	3	2×10^{-79}
3.8×10^5 yr	$.9 \times 10^{-3}$	5×10^{-14}	7.5×10^{-63}	4	4	8×10^{-96}
matter/				1	2	1×10^{-73}
dark energy				2	2	8×10^{-75}
equality				3	3	8×10^{-93}
10×10^9 yr	$a_\Lambda = 0.76$	2.5×10^{-18}	2.1×10^{-71}	4	4	2×10^{-112}
				1	2	7×10^{-74}
today				2	2	5×10^{-75}
				3	3	5×10^{-93}
13.8×10^9 yr	$a_0 = 1.0$	2.2×10^{-18}	1.3×10^{-71}	4	4	1×10^{-112}

[a] Acceleration varies as a^{-m_r} in the radiation era.

[b] Acceleration varies as a^{-m_m} in the matter era.

[c] The last column is calculated using equations (119.13) and (119.39).

Chapter 120

Comment on arXiv preprint "From maximum force to physics in 9 lines – and implications for quantum gravity" by Christoph Schiller[1]

abstract

The statement "So far, all experiments ever performed and all observations ever made can be described with general relativity and with the standard model of particle physics (with massive Dirac neutrinos)," in the preprint arXiv:2208.01038, "From maximum force to physics in 9 lines – and implications for quantum gravity" by Christoph Schiller ignores that on a large scale the relative rotation of matter and inertial frames is observed to be less than about 10^{-20} radians per year, which would require very finely tuned initial conditions to explain. However, nearly any reasonable theory of quantum gravity can explain that observation by phase interference

120.1 Introduction

The preprint arXiv:2208.01038 "From maximum force to physics in 9 lines – and implications for quantum gravity" by Christoph Schiller [424] makes the statement "So far, all experiments ever performed and all observations ever made can be described with general relativity and with the standard model of particle physics (with massive Dirac neutrinos)."

120.2 The rotation problem

An example of an observation that cannot be explained by General Relativity and the standard model is that on a large scale the relative rotation of matter and inertial frames is observed to be less than about 10^{-20} radians per year [164, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 369]. This is referred to as the "rotation problem," [298] and would require very finely tuned initial conditions to explain. Inflation by itself is not sufficient to solve the rotation problem.

[1] Copy sent to Christoph Schiller on 17 August 2022. His published version [423] on 27 November 2022 still has the same incorrect sentence.

120.3 Quantum gravity

However, nearly any reasonable theory of quantum gravity can explain the rotation problem by phase interference [386, 415]. That is, cosmologies that have large relative rotation cancel each other out by phase interference in a path-integral calculation. Only in a region around the saddlepoint at zero relative rotation rate is there constructive interference. A simple calculation shows that the region of constructive interference limits relative rotation to about 10^{-71} radians per year. A more detailed calculation gives the limit as about 10^{-73} radians per year. Including inflation limits the relative rotation even more. That the possible relative rotation of matter and inertial frames on a large scale is observed to be so small is strong evidence for the existence of quantum gravity.

Chapter 121

Conserved quantities[1]

abstract

We can treat all conserved quantities the same way we treat conservation of momentum and conservation of energy. That is, in terms of wave interference. It may require generalizing ray tracing in complex space to several dimensions. We may need octonians.

121.1 Introduction

When a plane wave is incident on a plane boundary, we get a reflected wave and a transmitted wave. The angle of the transmitted wave is related to the angle of the incident wave by Snell's law. Snell's law guarantees that the component of the wavenumber parallel to the boundary in the two media are equal. If they were not equal, there would be phase interference.

Since momentum and wavenumber are proportional, this can also be interpreted in terms of conservation of momentum. Further, this is related to Noether's theorem, in which every conservation law is related to a symmetry. Specifically, in this case, we have conservation of momentum in the direction along the surface separating the two media because the medium does not change in that direction, so there is translation symmetry in that direction.

The same applies to the reflected wave. If the medium of the incident wave are isotropic, then the angle of the reflected wave is the same as the angle of the incident wave, and again it goes back to conservation of momentum and Noether's theorem. If the medium of the incident wave is anisotropic, then the angle of reflection is related to the angle of incidence by Snell's law, since the effective refractive indexes for the reflected wave and the incident wave will be different.

Conservation of energy (when appropriate) can be treated similarly. It comes down to wave interference, Noether's theorem, and symmetry.

We can treat all conserved quantities the same way we treat conservation of momentum and conservation of energy. That is, in terms of wave interference. To show that may require some work.

Let's take Compton scattering as an example. In Compton scattering, we have a photon incident on a stationary electron. After the collision, the electron and the photon go off in different directions. Using conservation of momentum and conservation of energy allows us to calculate the change in wavelength of the photon as a function of the direction of the photon.

The Feynman diagrams for Compton scattering are given in Feynman's book, "Quantum Electrodynamics" [113, Figure 19-2]. In the first of these two diagrams, the electron absorbs a photon, then propagates with the combined momentum of both, then emits a photon. At each of the two

[1] 13 December 2022, 16:16

junctions, we have conservation of energy and momentum. In the second diagram, the electron first emits a photon, then propagates with the difference of their momenta, then absorbs a photon. Again, we have conservation of energy and momentum at each of the two junctions. The first of these two diagrams is also given in Feynman's book, "Theory of Fundamental Processes" [353, Chapter 25].

Feynman assumes conservation of energy and momentum at each junction, but it should be possible to show that by wave interference and path integrals.

Chapter 122

Quantum gravity: reconciling gravitation and quantum theory[1]

abstract

Reconciling gravitation and quantum theory requires some alteration to both General Relativity and quantum theory, at least with regard to interpretation.

A generalized derivation (for arbitrary Lagrangian and arbitrary measure) of a generalized quantum gravity evolution equation from a path-integral formulation connects the path-integral representation to the quantum gravity evolution equation.

Using the field-theoretical approach in General Relativity of Petrov and Pitts that uses two solutions of Einstein's field equations solves the problem of specifying matter distribution independent of the gravitational field.

The diffeomorphism problem can be solved by representing the geometry in terms of its unique parameters instead of by a metric.

Self-field quantum electrodynamics by Barut and Dowling gives an alternative to string theory to avoid representing particles by points.

122.1 Introduction

There are several considerations in trying to reconcile gravitation (General Relativity) and quantum theory. Some have to do with quantum theory itself and some have to do with General Relativity itself.

1. In the path-integral formulation of quantum gravity, it is necessary to specify the Lagrangian and the measure for weighting the paths.

2. In General relativity, it is difficult to specify the matter distribution independent of the gravitational field since gravitation is represented by geometry in General Relativity.

3. One of the known problems with formulating quantum gravity is the problem of diffeomorphisms. That is, the same geometry can be expressed in many ways by different metrics that are connected by a coordinate transformation.

4. The Schrödinger equation seems to represent particles as point particles. That is probably not quite correct.

[1] 27 December 2012, revised 12 June 2021, revised 12 October 2022

5. The problem of spin for fermions.
6. What is the source of quantum fluctuations?
7. Coarse graining versus fine graining.

122.2 From a path-integral formulation to a quantum gravity evolution equation

Help in getting the correct Lagrangian and the correct measure might come from deriving a general relation between the path-integral formulation (including a general Lagrangian and measure) and the quantum gravity evolution equation. See the paper by Halliwell [332] or the paper by Feng and Matzner [416]. Shestakova [425] also considers whether the Wheeler-deWitt equation is more fundamental than the Schrödinger equation.

122.3 Separating the gravitational field from geometry

The problem of separating the gravitational field from geometry can be solved by the 2-geometry representation of Petrov and Pitts [426], in which an arbitrary solution of Einstein's field equations is chosen as a background geometry, and Einstein's field equations for the actual geometry are written in terms of the background geometry. It may then be possible to calculate the correct Lagrangian specified in the background geometry.

122.4 Diffeomorphisms

A possible solution to the diffeomorphism problem might be to represent a geometry in terms of the parameters that define a geometry.

122.5 Point particles?

Self-field quantum electrodynamics by Barut and Dowling [427, 428] gives an alternative to string theory to avoid representing particles by points. Using $\rho = e\psi^*\psi$ as the charge density in the Schrödinger equation gives consistent and correct results.

122.6 Spin for Fermions

The spin problem can possibly be taken care of with the Slater determinant.

122.7 Quantum fluctuations

Quantum fluctuations are possibly intrinsic to the gravitational field and to all other fields.

122.8 Coarse graining versus fine graining

Maybe fine graining in quantum gravity is taken into account by calculating the gravitation from counting individual particles.

122.9 The Whole Elephant[2]

So far, we have been looking at different aspects of the elephant. Now, we have to look at the whole elephant. That is, we need to put it all together to reconcile gravitation and quantum theory. There are several aspects.

1. We somehow have to include renormalization or find a way for it to be unnecessary.
2. Somehow, spin has to be included.
3. Gravitation is expressed in General Relativity in terms of a metric field. There may be a better way.
4. We need to find out how to write a wave function for various spins.
5. We want to replace wave functions by the appropriate fields when applied to the wave function for a particle, but use quantum theory to give the amplitude for a process when that is appropriate.
6. The e^2/r potential in the Schrödinger equation treats the electron as a point object, but that is inconsistent with considering the electron as an extended object or as a field. Maybe this has something to do with renormalization, in which we renormalize not only mass and charge, but also the form of the potential.
7. Gravitation provides an inertial frame as a background for quantum theory.
8. Locally, gravitation and EM seem the same for classical physics, but maybe not for spin. Even not considering spin, it is only true for families or congruences of trajectories.
9. There are no particles, only fields.
10. Calculate τ and π fields from integrating over all matter in the universe.
11. If all mass come from a Higgs field, we need to consider only massless fields.

122.10 Gravitational Vector Potential[3]

In chapter 27, "the gravitational vector potential," I show that locally (without reference to sources), the geodesic equation is equivalent to a vector potential. There may be a difference in how spins are treated, and I need to look into that. However, it seems that the gravitational field and the EM field, with regard to how they affect quantum physics, can be treated the same. That is, I can treat both in the same way that EM is now treated in quantum physics, or I can treat both the same way gravitation is now treated in quantum physics. Unless there is a problem that EM has local sources, but inertial has negligible effect from local sources.

Actually, the gravitational vector potential is not a function of position in space-time only, but depends also on the family or congruence of trajectories. Wave functions in quantum theory also are for such families. There may be a connection, such that wave mechanics may unknowingly be formulated to depend on the gravitational vector potential. If so, it might be possible reformulate quantum theory to not depend on the gravitational vector potential.

[2] big.elephant.tex
[3] big.gvp.tex

To check this, it should be possible to write a wave function in terms of the gravitational vector potential. Actually, only the WKB approximation can be written in terms of the gravitational vector potential. This is

$$\psi_{\text{WKB}} = \left(\frac{A_0}{A}\right)^{1/2} \exp\left(i\int p_\mu dx^\mu\right) , \tag{122.1}$$

where A and A_0 are cross-sectional areas of a small bundle of rays and

$$p_\mu = \pi_\mu + eA_\mu = m(g_\mu + \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\frac{\partial\tau}{\partial x^\mu}) + eA_\mu \ , \tag{122.2}$$

where

$$g_\mu = \dot{x}_\mu - \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\frac{\partial\tau}{\partial x^\mu} \ . \tag{122.3}$$

So

$$p_\mu = q_\mu + \frac{m}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\frac{\partial\tau}{\partial x^\mu} \ , \tag{122.4}$$

where

$$q_\mu = mg_\mu + eA_\mu \ . \tag{122.5}$$

So,

$$\psi_{\text{WKB}} = \left(\frac{A_0}{A}\right)^{1/2} \exp\left(\frac{im}{2}\int g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\frac{\partial\tau}{\partial x^\mu}dx^\mu + i\int q_\mu dx^\mu\right) , \tag{122.6}$$

or,

$$\psi_{\text{WKB}} = \left(\frac{A_0}{A}\right)^{1/2} \exp\left(\frac{im}{2}\int g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta d\tau + i\int q_\mu dx^\mu\right) , \tag{122.7}$$

or,

$$\psi_{\text{WKB}} = \left(\frac{A_0}{A}\right)^{1/2} \exp\left(\frac{im}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\int d\tau + i\int q_\mu dx^\mu\right) , \tag{122.8}$$

or,

$$\psi_{\text{WKB}} = \left(\frac{A_0}{A}\right)^{1/2} \exp\left(-\frac{im\tau}{2} + i\int q_\mu dx^\mu\right) . \tag{122.9}$$

122.11 Principles[5]

1. Quantum theory and The Standard Model of particle physics are based on special relativity, which is only an approximation.

2. The Standard Model of cosmology is based on General Relativity, which is based on the Equivalence Principle, which is probably only an approximation.

So, both quantum theory and General Relativity are probably only approximate, so both must be altered in the reconciliation. But now we have no principles, so what do we replace them with?

Let us try the following:

1. Assume that all massive particles get their mass from the Higgs field. Therefore, we will take all particles (and their corresponding fields) to be massless.

2. To replace special relativity (the existence of inertial frames, which come from gravitational interactions with all of the matter in the universe), we assume all particles (fields) travel at infinite speed, or that distances do not exist.

[5] big.p.tex

122.12 Goals

1. Discover the un-normalized fields associated with the normalized wave functions of particles/fields.

2. Discover the dimensional gravitational potentials and fields associated with the non-dimensional metric tensor.

I think that $G_{\mu\nu}$ or $T_{\mu\nu}$ might work for the dimensional gravitational potential, since its covariant derivative gives the gravitational force. All of my previous discussion about the geodesic equation being of the wrong form can also just use $G_{\mu\nu}$ or $T_{\mu\nu}$.

No, the covariant derivative of each of those is zero.

$g_{\mu\nu}$ could be the potential for Γ, but it might have the wrong units. In any case, having it be the metric tensor, in which the elements are usually equal to about one is wrong.

Chapter 123

Epilogue - Where we are and where do we go from here?

abstract

Here, we consider where we are and where we go from here, both for quantum theory, for gravitational theory, and how they can be reconciled together in a consistent theory.

123.1 Introduction

In considering where we are, we consider what is the most minimalistic interpretation of quantum theory, which we do in two parts. The first part consists in summarizing the most minimalistic view of quantum theory. The second part consists in speculating on the most likely underlying behavior of particles and fields that is consistent with the minimalistic view of quantum theory, while being aware that this underlying behavior can never be tested.

Finally, with regard to where we are going, we speculate on how quantum theory might have to be altered to be consistent with gravitation.

With regard to where we are in gravitation, we note that gravitation is not just curvature because inertia is a gravitational force that exists in the absence of curvature, and that in General Relativity, there is nothing we can point to that can be considered to be a gravitational field, except in the frame of a particle, where the connection plays that role, but is not a tensor. See chapter 89.

123.2 Where we are with respect to quantum theory

Chapters 32, 33, 34, 35, 36, 38, 39, 40, 41, 43, and 76 consider the measurement process, including the problem of measurement in a quantum cosmology.

Chapter 80 considers the question of whether the wave function represents reality or knowledge.

Chapters 84 and 107 deal with the zero-point energy problem.

Chapter 90 proposes that the fluctuations calculated in quantum field theory are actual field fluctuations, and that the absolute square of wave functions, when integrated over some volume, give the probability that one quantum of energy for that field is actually in that volume.

Chapter 93 contrasts philosophies, interpretations, and models. It argues that "models" is a better description for what is usually called "interpretations." Chapter 88 points out that a wave function applies to an ensemble only. Chapter 94 argues that the wave function in ordinary (first-quantized) quantum mechanics is more like a weather forecast in that it is associated with the ensemble average of fields rather than actual fields. Chapter 37 talks about exclusive alternatives in quantum theory. Chapters 2, 6, 7, 8, 24, 25, and 26 consider the proper way to understand quantum physics.

As far as I am aware, there is no accepted formulation of the general quantum N-body problem. Until we have an acceptable formulation of the general quantum N-body problem, it will be difficult to formulate an acceptable theory of quantum gravity. Chapter 112 considers possible formulations of the general quantum N-body problem.

Actually, there is an exact formulation of the quantum N-body problem in terms of path integrals [21]. But, to include spin, each path needs to be expressed in terms of an infinite product of infinitesimal particle propagators [35]. Feynman, in his Nobel lecture [429], declared that when Dirac said that the kernel K was analogous to what you would get in classical mechanics if you took the exponential of i times the action, he must have meant equal or proportional. However, Dass [430] recently showed that Feynman was wrong. He showed that Dirac's path integrals were more accurate than Feynman's.

The discussion in chapters 32, 33, 34, 35, 36, 38, 39, 40, 41, 43, 76, 80, 84, 87, and 90 lead to the following summary.

123.2.1 A minimalistic view of present quantum theory

We can all agree that wave functions can be used to calculate probabilities for various occurrences, and that such probabilities agree with measurements on ensembles of identically prepared systems. However, such wave functions can represent only ensemble averages of fields, such as electric fields, magnetic fields, or other fields because of the existence of quantum fluctuations[1] that are superimposed on the ensemble averages associated with the wave functions.

Though not everyone will agree, it follows from the above that wave functions have a reality analogous to weather forecasts, even while accepting that there are differences between classical probabilities and quantum probabilities. It then follows that Schrödinger's cat does not represent a superposition of a dead cat with an alive cat, but rather simply the existence of two forecasts for two possible outcomes for an event.

We can all agree that calculations of wave functions gives energy levels consistent with absorption and emission spectra of atoms and molecules, including transition probabilities and widths of spectral lines.

123.2.2 Speculation on the underlying behavior of particles and fields

As physicists, we want to know a mechanism for things. We ask ourselves, "How can things happen like that?" In the case of quantum theory, we know there are limitations to our knowledge. There are aspects of particle and wave behavior that we know we can not find out from measurements. Still, our curiosity wants more understanding. Here, I will suggest a view that seems consistent with the above minimalistic view, and may represent what is really going on, even though acknowledging that there are no measurements that could verify that view.

Let us first consider photons, because Bosons are simpler. First consider an isolated atom in an excited state. The atom makes a transition to a state of lower energy and emits electromagnetic radiation. The wave function is a spherical wave that expands at the speed of light. However, the wave function represents an ensemble average, not the behavior of a single photon. Superimposed on the wave function are quantum fluctuations. The magnitude of these quantum fluctuations are on the order of one quantum, that is, one photon. In the present situation, that means roughly 100% fluctuation in the field. Most of these fluctuations probably occur within the expanding sphere that

[1]The existence of quantum fluctuations of all fields is well established, partly based on various uncertainty relations. See the discussion by Sakurai [239, pp. 32-36]. For example, if the occupation number is fixed, then the electric field strength is completely uncertain.[239, p. 33]

corresponds to the wave function, but there are probably fluctuations outside of that shell also. At some place at some time, this radiation might be absorbed by another atom or detector somewhere. We know how to calculate such probabilities. The usual view is that the wave function collapsed when that occurred. However, the wave function represents a forecast, not an actual field, so no difficulty there. However, what actually happened? I speculate that in the quantum fluctuations sometimes a full quantum of electromagnetic energy can be concentration in a region of space where there happens to be an atom or detector that can absorb that radiation, and then the radiation gets absorbed. Notice! This is speculation. We cannot test this hypothesis. The wave function did not collapse, but the radiation field was collapsing all of the time and jumping around.

The above discussion of a single atom in isolation emitting radiation is an ideal case that might occur in a laboratory, but is not what generally is observed by astronomers. A more realistic situation occurs on the surface of a star. Many atoms in exited states emit radiation. At a small region on the surface of that star, the radiation is not a spherical wave, but more like a hemisphere wave because the star gets in the way of radiation in one hemisphere. For the star as a whole, however, we have radiation as roughly spherical radiation. The total radiation field is the sum of radiation of single quanta from various atoms at various frequencies. Because photons are Bosons, the fields from individual radiated fields add together[2], so that the identity of individual radiated fields is lost. Therefore, we end up with a single radiation field with a spectrum of frequencies. However, because of quantum fluctuations, that total field fluctuates. The magnitude of that fluctuation at each frequency is one quantum, just as it was when only one quantum was emitted. However, in this case, the fluctuation relative to the average is much less. Again, we know how to calculate the probability of absorption, which is much larger now, since there are so many more quanta.

For Fermi-Dirac particles, the behavior is more complicated because individual quanta do not simply combine to give single field because of the Pauli exclusion principle, but otherwise, the behavior is probably similar, in that we have a wave function that represents an ensemble, and superimposed on that are quantum fluctuations, whose magnitude is on the order of a single quantum.

The above applies to present quantum theory. It does not try to consider how quantum theory might have to be modified.

Using the wave function ψ to calculate the amplitude for a process is OK, but the normalization needs to be different to use ψ as a field, like electric or magnetic fields.

123.3 Speculation on the true nature of quantum theory

Here, I speculate on the possibility discussed in the 2009 afterthoughts in chapter 27 and in chapter 95 that although quantum theory is not an illusion from trying to reconcile a tensor gravitational field with a vector electromagnetic field,[3] the <u>form</u> of the wave equation may be based (unknowingly) on the gravitational vector potential. It might be necessary to alter quantum theory in regions of large gravitational fields, such as near black holes.

Chapter 103 discusses possible limitations for the uncertainty relations, and suggests a solution to the zero-point energy problem.

Chapter 104 considers box normalization and the artificial nature of creation and annihilation operators.

[2]An emitted radiation from an atomic transition, for example, includes a spectrum of frequencies of a characteristic line width because atomic transitions take time (about 10^{-16} seconds for a typical atomic transition). The radiated field has a particular dependence on time and space, including polarization. The electric fields from all radiation fields can be added together to give a total electric field. Similarly for the magnetic field.

[3]See the various discussions about the gravitational vector potential.

Chapters 87 and 108 suggest that an electron wave function is an electroweak wave whose components actually include electric and magnetic fields as an electromagnetic wave. I speculate the wave function for any particle has associated with it some or all of the following: an electric field, a magnetic field, a gravitational field, a weak nuclear field, and a strong nuclear field. The electric field of an electron, for example, would not just be the e/r^2 classical electric field, nor even the $e\psi^*\psi$ average quantum electric field, but a periodic in space and time electric field associated with the wave function.

Chapters 114 and 117 present a realistic quantum model as the fluctuating field interpretation of quantum mechanics. In this model, the wave function represents the properties of each particle, not just an ensemble. The quantum fluctuations of each particle are such as to, in effect, cause each particle to be its own ensemble. The quantum fluctuations cause each particle to explore the full potentials of the wave function. There are no hidden variables. Fluctuations do not depend on some hidden initial value. Of course, it would require an actual ensemble of particles to compare with a prediction. This model is identical to standard quantum theory. There are no experiments that can distinguish them.

Chapter 112 considers possible formulations of the general quantum N-body problem. It is necessary to find a general formulation of the quantum N-body problem.

123.4 Where we are on gravitation

At present, we consider that gravitation is geometry, and that it forms the background upon which the rest of physics, including quantum field theory resides. It is still an open question about whether gravitation must be quantized, but there are some arguments supporting that view [194, Adler, 2010], including an argument that we could, in principle, violate the uncertainty principle if gravitation were not quantized by measuring the gravitational field of a particle with infinite precision. If gravitation is quantized, there are probably quantum fluctuations of the gravitational field (geometry).

The most compelling observational evidence for the existence of quantum gravity is the observed lack of relative rotation of inertial frames and the matter distribution on a large scale (chapter 106) and the observed lack of relative acceleration of inertial frames and the matter distribution on a large scale (chapter 119).

Chapters 5, 22, 24, and 25 consider geometry and inertia. Chapter 92 calculates the inertia of light. More specifically, it tries to write Maxwell's equations in a curved background as a set of linear first-order differential equations in such a way that the inertial and gravitational effects are explicit rather than implicit. Chapter 46 talks about the effective rest mass of a photon in a plasma. Chapter 78 considers the conjectured spacetime structure of the universe. Chapter 79 points out situations in which particles can exceed the speed of light. Chapter 83 considers the origin of geometry.

Representing gravitation as a refractive index in a propagation medium instead of as geometry is a representation that is more intuitive. Chapter 111 considers the refractive index for a particle wave in a gravitational field.

In the same way that using a refractive index to represent propagation of a wave in a medium is coarse graining while fine graining considers scattering of the wave by each atom in the medium, representing gravitation by a refractive index or by geometry is coarse graining. Fine graining would consider the inductive gravitational interaction of a wave with each gravitating body in the universe. Chapter 105 discusses how a discrete version of General Relativity might be a way to express gravitation in terms of fine graining.

123.5 Where do we go from here with gravitation?

Having gravitation (geometry) as the background for everything else is analogous to treating the hydrogen atom in terms of a fixed nucleus surrounded by a moving electron. As we now know, looking instead at the wave function of the relative distance between the proton and the electron leads not only to a more accurate description, but it puts the electron and the proton on an equal footing.

A step in the right direction would be to consider gravitation (geometry) and the rest of physics (a piece at a time) on an equal footing. That is, the geometry would not be a background, but instead, we would look for a formulation in which the geometry and each piece of physics were on an equal footing. I am not sure quite how to do that, but that would be only a step in the right direction.

The next step would be to somehow consider each body and each field that contributes to geometry (especially inertia) on an equal footing with the body or particle or field experiencing the geometry (especially inertia). Would it be possible to reformulate the hydrogen atom in such terms? We already have a formulation in terms of the relative coordinate between the proton and electron (but on a fixed background inertial frame). Can we extend that formulation to include relative coordinates between the proton and all other bodies in the universe and between the electron and all other bodies in the universe? That is, a general N-body formulation.

Chapter 86 gives a complete integral form for Maxwell's equations, and tries to give a complete integral form for Einstein's field equations.

Chapter 89 tries to represent General Relativity in terms of gravitational fields rather than potentials (the metric).

Chapter 4 considers using other variables for gravitation.

Chapter 81 considers fluctuations due to gravitational waves.

Chapters 51, 52, 53, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 72, 73, 109, and 110 consider how to separate geometry from gravitation.

Mach's principle is often misunderstood. In fact, in any group of scientists who profess to believe in some form of Mach's principle, there will be as many opinions about what Mach's principle means as there are scientists in the group. Chapter 30 talks about Machian cosmologies as wave packets. Chapter 42 talks about the principles in Mach's principle. One of Mach's main points is that only relative coordinates can be determined, and therefore, only relative coordinates should enter physical law. However, it is difficult to formulate General Relativity in relative coordinates. Chapter 69 considers symmetries in relative coordinates. Chapter 70 gives the essence of Mach's ideas.

Rather than consider gravitation as geometry, it is more appropriate to consider that gravitation alters the medium for the propagation of light and other fields. The electromagnetic interaction does this also.

123.6 Quantum Gravity

The next step would be to convert the above formulation into a quantum formulation.

Chapters 21 and 23 consider Mach's ideas, and how they might apply in quantum theory.

Chapter 44 asks whether quantum theory applies to the whole universe.

Chapters 28, 29, 31, 49, 50, 65, 66, 67, 74, 75, 85, 102, 106, 113, and 120 try to answer the question of why we do not observe a relative rotation of our local inertial frame and the distant stars.

Chapters 115, 116, 118, and 119 try to answer the question of why we do not observe a relative acceleration of our local inertial frame and the distant stars.

Chapters 71 and 82 consider sparse universes and their significance in understanding how the distribution of matter in the universe might affect local quantum theory.

Chapter 112 considers possible formulations of the general quantum N-body problem. It is necessary to find a general formulation of the quantum N-body problem to have a theory of quantum gravity.

123.7 To publish

I need to submit chapter 119 to a journal, probably Foundations of Physics.

In addition, I should finish writing the paper on zero-point energy, based on Chapters 84 and 107 and submit it to a journal.

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